Parton form-factors for heavy-light decays at three loops in leading-color

Sudeepan Datta

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Three loop QCD corrections to the heavy-light form factors in the color-planar limit

2308.12169 [hep-ph]

S. Datta, N. Rana, V. Ravindran, R. Sarkar

Journal of High Energy Physics, 2023(12), Dec-2023

QCD MASTER CLASS

Saint-Jacut-de-la-Mer, France

1. The physics context
    - Top physics frontier
    - B physics frontier
    - Formal aspects
2. Three loop results for the UV renormalised HLFF
    - UV renormalisation
    - Universal IR structure
3. Asymptotic behavior of the HLFF

Top physics frontier

A key object of study at the LHC: m_t

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Use pp-collision decay products to reconstruct the top

Indirect measurements

Obtain top’s mass from cross-section measurements

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$m_t = 171.77 \pm 0.37$ GeV

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$m_t^{pole} = 170.5 \pm 0.8$ GeV

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Important problem

Systematic interpretation of direct measurements

See - Corella (2019), Hoang (2020), Myllymäki (2024)

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- Computed through the *optical-theorem*: $t \to Wb \to t$
  One loop higher due to 'stitching' results in a self-energy (also called 'propagator-type') graph.

- Γ_t suppressed by 9 % at NLO (QCD)
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  and by a further 2 % at NNLO (QCD)
  - Gao, Li, Zhu (2013), Brucherseifer, Caola, Melnikov (2013), Chen, Li, Wang, Wang (2022)

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- **State-of-the-art**:
  Analytic results for N^{3}LO (QCD) leading-color corrections, with numerical estimates of the sub-leading color-factors
  – Chen, Li, Li, Wang, Wang, Wu (2023)
  High-precision numerical results for N^{3}LO (QCD) full-color corrections
  – Chen, Chen, Guan, Ma (2023)

B physics frontier

At LHCb: $B \to X_u l \bar{\nu}_l\,, \quad B \to X_c l \bar{\nu}_l\,, \quad B \to X_s \gamma \;\; \ldots$

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Local OPE

Non- Local OPE

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Local OPE

$$\Gamma(B \to X_u l\bar{\nu}_l) = \Gamma_0\left[1 + C_F \sum_{n\geq 1}\left(\frac{\alpha_s}{\pi}\right)^n X_n\right] + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{m_b^2}\right)$$

Non- Local OPE

$$d\Gamma(B \to X_u l\bar{\nu}_l) \sim H \cdot J \otimes S + \mathcal{O}\left(\frac{1}{m_b}\right)$$

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State of the art:

Fermionic contributions to X_3

Fael, Usovitsch (2023)

Non- Local OPE

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State of the art:

Three-loop hard coefficients recently calculated for QCD-SCET matching for S, PS, V, AV & T currents

Fael, Huber, Lange, Müller, Schönwald, Steinhauser (2024)

Formal aspects

Factorisation $\rightarrow$ Exponentiation $\rightarrow$ Resummation

Formal aspects

Factorisation → Exponentiation → Resummation

Massless partons

Massive partons (small-mass limit)

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Soft and collinear divergences exponentiate order-by-order and exhibit universal behavior.

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Massless QCD corrections do exponentiate. Use factorisation theorems in this limit to obtain massive amplitudes from massless ones.

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Mitov, Moch (2006)
Becher, Melnikov (2007)
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Use SCET and non-Abelian exponentiation to show that IR poles do factorise.

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Massive partons (general scenario)

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Becher, Neubert (2009)

- On the structure of infrared singularities of gauge-theory amplitudes (0903.1126 [hep-ph])
- Infrared singularities of QCD amplitudes with massive partons (0904.1021 [hep-ph])

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Asymptotic behavior

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Integro-differential (K-G) equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln \hat{F}_I\left(\frac{Q^2}{\mu^2}, \frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right) = \frac{1}{2}\left[K_I\left(\frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right) + G_I\left(\frac{Q^2}{\mu^2}, \frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right)\right]$$

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   - UV renormalisation, Ward id
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- **External currents**: vector, axial-vector, scalar, pseudo-scalar
- **Process**: top-decay dominant channel, ie. $t(P) \rightarrow b(p) + W^*(q)$, $q = P - p$
- **Amplitude**: $\bar{b}_c(p)\,\Gamma^{\mu}_{cd}\,t_d(P)$
- Express Γ^{μ}_{cd} in terms of 3 independent **form factors**
- $\Gamma^{\mu}_{cd} = -\,i\,\delta_{cd}\,[G_1\,\gamma^{\mu}\,(1-\gamma^5) + \frac{G_2}{2\,m_t}\,(1+\gamma_5)\,(P^{\mu}+p^{\mu}) + \frac{G_3}{2\,m_t}\,(1+\gamma_5)\,(P^{\mu}-p^{\mu})]$
- **Goal**: Compute G_1, G_2 and G_3

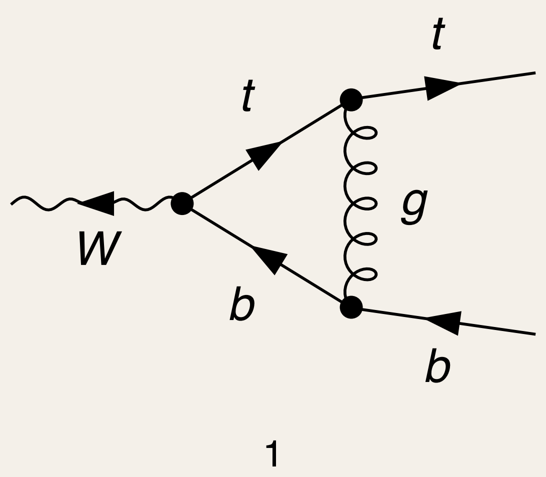
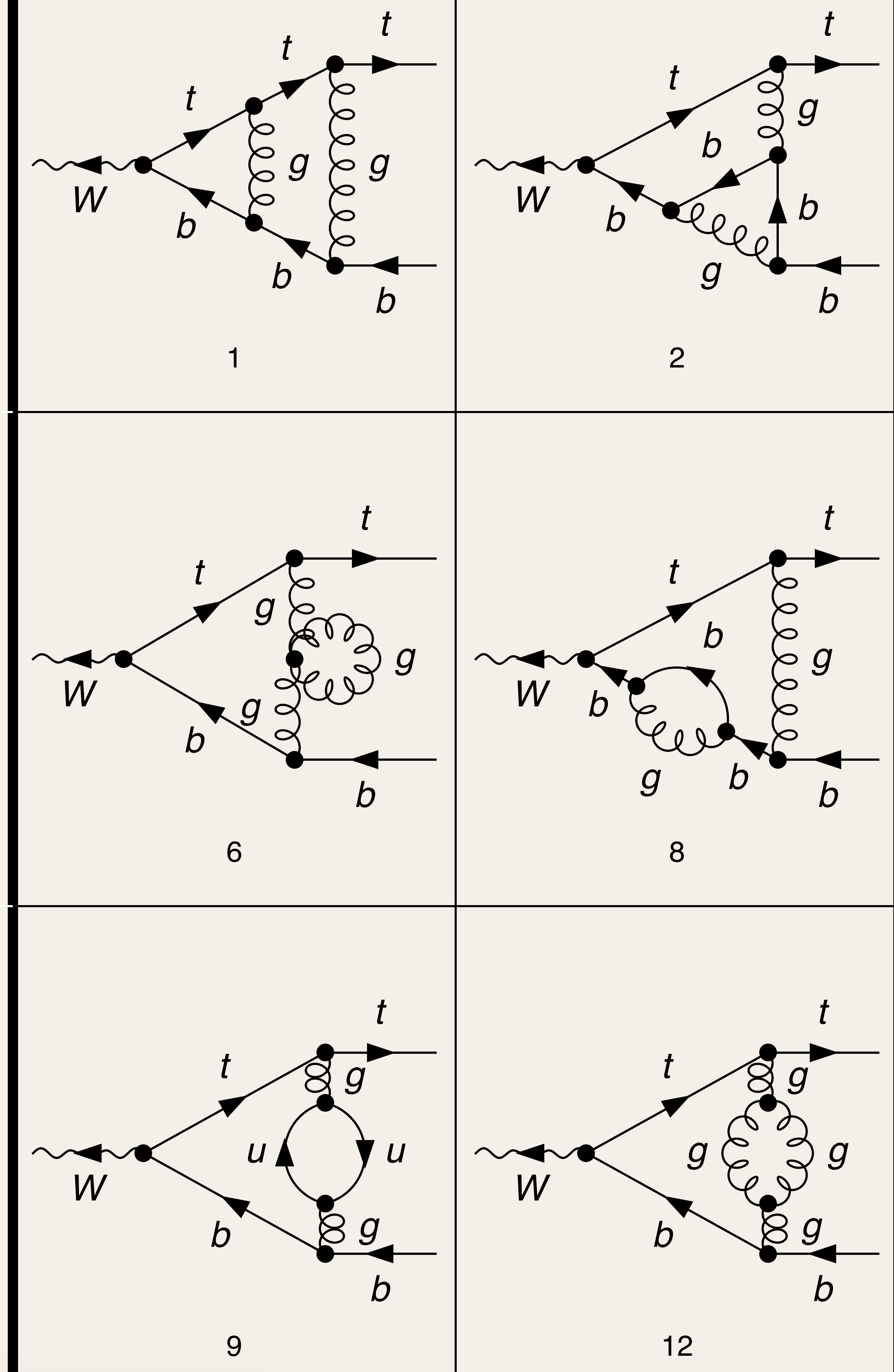
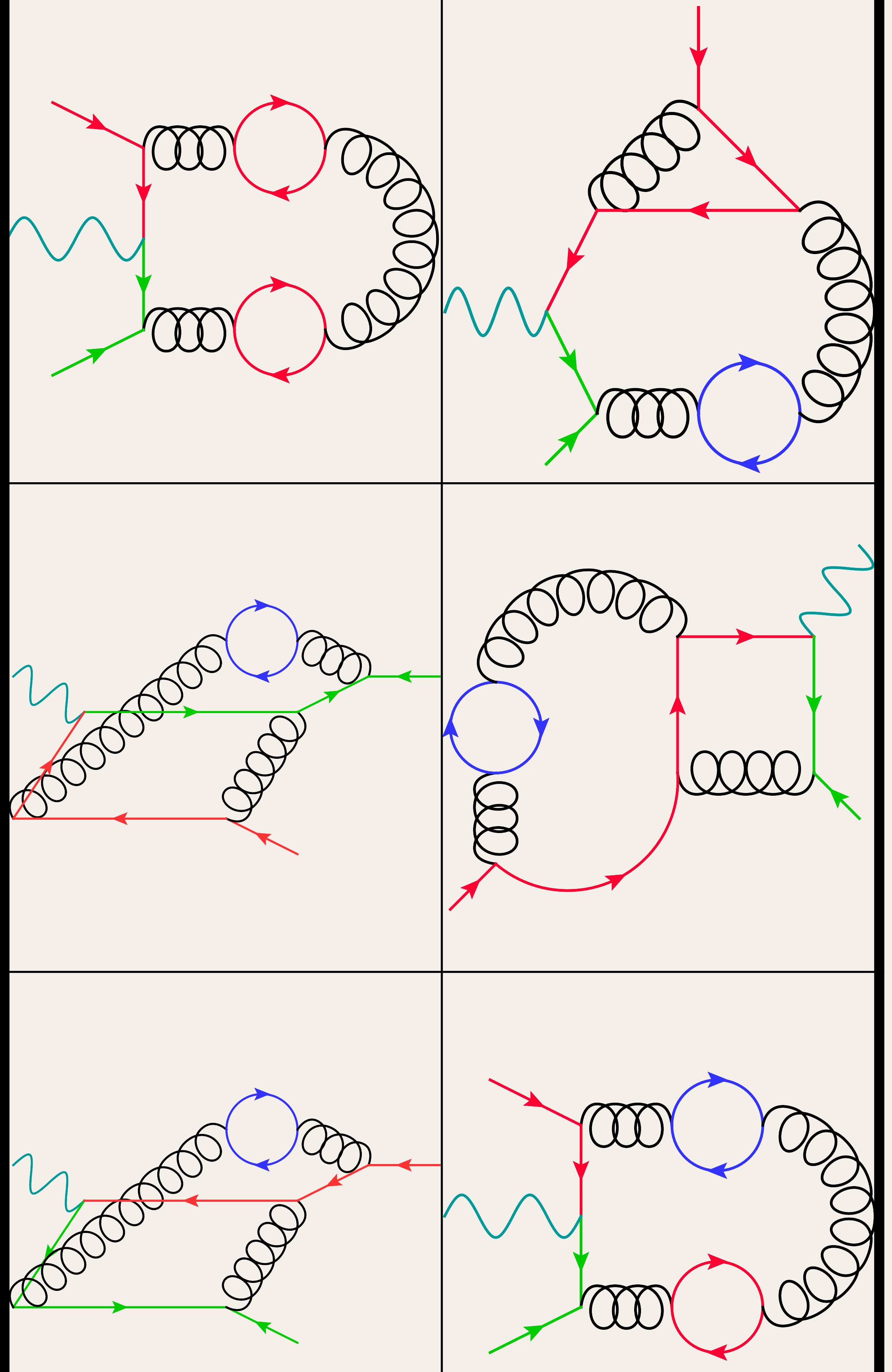
One-loop diagram	Two-loop diagrams (6 out of a total of 13 shown)	Three-loop diagrams (6 out of a total of 263 shown)
		

Diagram generation

QGRAF, FeynArts

Color/Dirac/Lorentz algebra

FORM, FeynCalc

IBP reduction

LiteRed, Kira

- Only a single integral-family suffices:

$$I_\nu(d,x) = \int \prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d} \prod_{j=1}^{12} \frac{1}{D_j^{\nu_j}}; \;\; \nu = \prod_{j=1}^{12} \nu_j, \quad x = \frac{q^2}{m_t^2}$$

- D_j - s are defined as follows:

$$\{\mathscr{D}_1 - m_t^2, \mathscr{D}_2 - m_t^2, \mathscr{D}_3 - m_t^2, \mathscr{D}_{12}, \mathscr{D}_{23}, \mathscr{D}_{13}, \mathscr{D}_{1;1}, \mathscr{D}_{2;1}, \mathscr{D}_{3;1}, \mathscr{D}_{1;12}, \mathscr{D}_{2;12}, \mathscr{D}_{3;12}\}$$

where,

$$\mathscr{D}_i = k_i^2, \mathscr{D}_{ij} = (k_i - k_j)^2, \mathscr{D}_{i;1} = (k_i - P)^2, \mathscr{D}_{i;12} = (k_i - P + p)^2$$

- After reduction to MIs - **70 MIs** obtained.

#	sector	master integrals
3	7	$I_{111000000000}$
4	29	$I_{101110000000}$
	78	$I_{011100100000}$
	92	$I_{001110100000}$
	519	$I_{111000000100}$
	526	$I_{011100000100}$, $I_{(-1)11100000100}$
	540	$I_{001110000100}$, $I_{(-1)01110000100}$
5	110	$I_{011101100000}$
	244	$I_{001011110000}$
	247	$I_{111011110000}$
	541	$I_{101110000100}$
	558	$I_{011101000100}$, $I_{(-1)11101000100}$
	653	$I_{101100010100}$
	661	$I_{101010010100}$
	668	$I_{001110010100}$
	684	$I_{001101010100}$, $I_{(-1)01101010100}$
	689	$I_{100011010100}$
	692	$I_{001011010100}$, $I_{(-1)01011010100}$
	1543	$I_{111000000110}$
	1557	$I_{101010000110}$, $I_{1(-1)1010000110}$
	1588	$I_{001011000110}$, $I_{(-1)01011000110}$
8		
9	1918	$I_{011111101110}$, $I_{(-1)11111101110}$

#	sector	master integrals
6	655	$I_{111100010100}$, $I_{111100(-1)10100}$
	669	$I_{101110010100}$, $I_{1(-1)1110010100}$, $I_{10111(-1)010100}$, $I_{101110(-1)10100}$, $I_{1011100101(-1)0}$
	686	$I_{011101010100}$, $I_{(-1)11101010100}$, $I_{0111(-1)1010100}$, $I_{011101(-1)10100}$
	691	$I_{110011010100}$, $I_{11(-1)011010100}$
	693	$I_{101011010100}$, $I_{1(-1)1011010100}$
	694	$I_{011011010100}$, $I_{(-1)11011010100}$
	700	$I_{001111010100}$, $I_{(-1)01111010100}$
	937	$I_{100101011100}$
	1587	$I_{110011000110}$
	1811	$I_{110010001110}$
	1841	$I_{100011001110}$
	3591	$I_{111000000111}$
7	695	$I_{111011010100}$, $I_{111(-1)11010100}$, $I_{111011(-1)10100}$, $I_{1110110101(-1)0}$
	939	$I_{110101011100}$, $I_{11(-1)101011100}$
	1591	$I_{111011000110}$, $I_{111(-1)11000110}$
	1654	$I_{011011100110}$, $I_{011(-1)11100110}$
	1815	$I_{111010001110}$, $I_{11101(-1)001110}$
	1821	$I_{101110001110}$, $I_{10111(-1)001110}$
	1845	$I_{101011001110}$, $I_{1(-1)1011001110}$

Table 1. List of the master integrals. # indicates the number of propagators.

JHEP12(2023)001

- Canonical bases not used - make use of factorisation to first order for the univariate system to solve analytically
- $\partial_x \vec{I} = M_{70\times 70}\, \vec{I}$, arrange M in upper block-triangular form.
- Compute MIs block-wise starting from the last (easiest) one. Successive order-by-order solution in ϵ for each block starting with the leading singular term.
- The spanning alphabet: $\left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{2-x}\right\}$
- Function space: HPLs and generalised HPLs

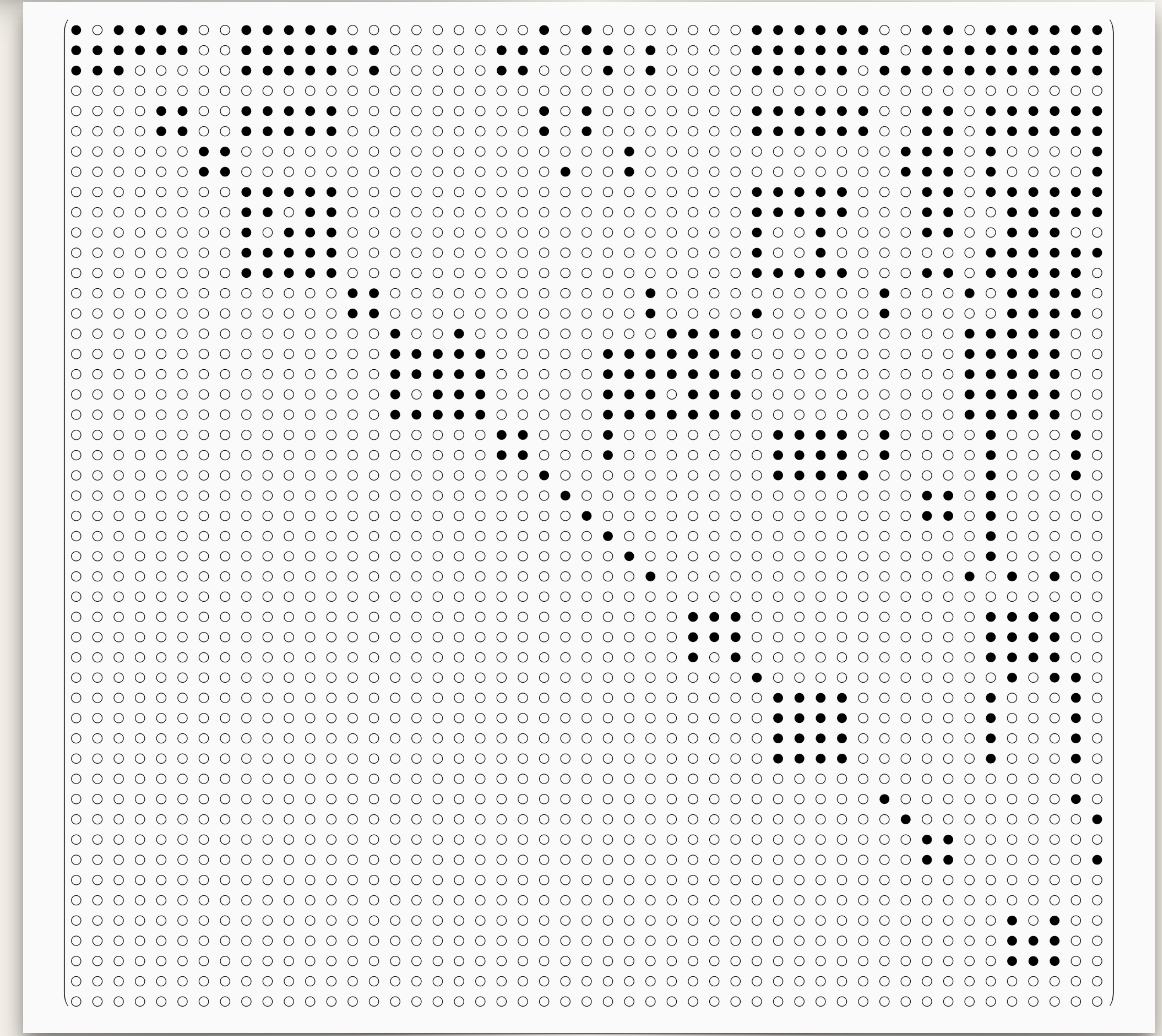
Differential Equations

Sigma, OreSys, HarmonicSums, PolyLogTools

Boundary Conditions

Analytic: AMBRE2.1.1, MBConicHulls, HypExp2

Numeric: AMFlow, FIESTA, PSLQ



Pic credit - N. Rana, ACAT 2019

$$\partial_x J_n(x,\epsilon) = \mathscr{C}_{nm}(x,\epsilon)\, J_m(x,\epsilon) + \mathscr{R}_n(x,\epsilon)$$

Let the leading singularity be at ϵ^{-p}, then, expanding in ϵ :

$$J_n(x,\epsilon) = \sum_{k=-p}^{\infty} J_n^{(k)}(x)\,\epsilon^k$$

$$\mathscr{C}_n(x,\epsilon) = \sum_{k=0}^{\infty} \mathscr{C}_n^{(k)}(x)\,\epsilon^k$$

$$\mathscr{R}_n(x,\epsilon) = \sum_{k=-p}^{\infty} \mathscr{R}_n^{(k)}(x)\,\epsilon^k$$

$$\partial_x J_n^{(k)}(x) = \mathscr{C}_{nm}^{(0)}(x)\, J_m^{(k)}(x) + \sum_{j=1}^{k+p} \mathscr{C}_{nm}^{(p)}(x)\, J_m^{(k-j)}(x) + \mathscr{R}_n^{(k)}(x)$$

- Ablinger, Blümlein, Marquard, Rana, Schneider (2018)
- Blümlein, Marquard, Rana, Schneider (2019)

- No canonical bases used - **no** *uniform transcendentality*.
- But since the DE system is first-order factorisable, no complicated higher transcendental constants such as eMZVs.
- PSLQ needs the full set of transcendental constants **till** weight $2L + k$ to obtain the ϵ^k-coefficient for the boundary integrals in terms of these constants. However, watch out for *ugly* fractions, and prune the set if necessary.
- Also watch out for unstable behaviour relative to the numerical precision used for the fitting.
- Else, require higher precision for the numerical result.

UV renormalisation

- Dim-reg to regularise the bare form factors - γ_5 treated using **CDR-scheme**, ie. $\{\gamma_\mu, \gamma_5\} = 0$ and $\gamma_5^2 = 1$.
- UV renormalisation in mixed scheme: Z_m , $Z_{2,t}$, $Z_{2,b}$ in **OS** scheme; Z_{α_s} in $\overline{\text{MS}}$ scheme ($n_h \neq 0$). All Z_i -s can be expanded in α_s: $\boldsymbol{Z_i = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n Z_i^{(n)}}$.
- Relevant results for Z_i -s mostly available in literature.
- Relate renormalised form factors G_i to bare $\hat{G}_i$ -s: $\boldsymbol{G_i = Z_{2,t}^{\frac{1}{2}} Z_{2,b}^{\frac{1}{2}} \left(\hat{G}_i + \hat{G}_{ct,i}\right)}$; $\hat{G}_{ct,i}$ denotes appropriate CT-contributions from lower orders in α_s.

Ward id

- The following Ward-identity holds: $q_\mu \Gamma^\mu - m_W \Gamma_{PS} = 0$; Γ_{PS} denotes the scattering amplitude for $t \to b\omega^-$, ω^- is the negatively charged pseudo-Goldstone boson.
- Can further express Γ_{PS} using a form factor S : $\Gamma_{PS} = \frac{m_t}{m_W} S\,(1+\gamma_5)$.
- S is computed till 3-loops and renormalised as well.
- At the level of form factors, the Ward identity takes the following form: $2\,G_1^{(n)} + G_2^{(n)} + x\,G_3^{(n)} - 2\,S^{(n)} = 0$.
- Our results for $n = 3$ satisfy the above identity - very important self-consistency check!

IR subtraction

The IR divergences factorise. Becher, Neubert (2009)

$$G_i(\alpha_s, x, \epsilon) = Z(\alpha_s, x, \epsilon, \bar{\mu})\, G_i^{\text{fin}}(\alpha_s, x, \epsilon, \bar{\mu})$$

where, $G_i^{\text{fin}}(\bar{\mu})$ is finite as $\epsilon \to 0$; $\bar{\mu}$: scale for this IR factorisation. Z is process-independent.

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2. The anomalous dimensions are computed in massless QCD (n_l flavors).

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   Problem: The form-factors are considered in full-QCD ($n_f = n_l + n_h = n_l + 1$ flavors).

IR subtraction

Problem: The form-factors are considered in full-QCD $(n_f = n_l + n_h = n_l + 1$ flavors).

Solution: Use **QCD decoupling relations**.

Now let's put everything together.

1. Write an RGE for $\bar{Z}$, the n_l - counterpart for what Z in the full (n_f) - theory:

$$\frac{d}{d\ln\bar{\mu}}\ln\bar{Z}(\alpha_s, x, \epsilon, \bar{\mu}) = -\Gamma(\alpha_s, x, \bar{\mu})$$

2. Expand both $\bar{Z}$ and Γ in α_s:

$$\bar{Z} = \sum_{n=0}^{\infty}\left(\frac{\bar{\alpha}_s}{4\pi}\right)^n \bar{Z}^{(n)}, \quad \Gamma = \sum_{n=0}^{\infty}\left(\frac{\bar{\alpha}_s}{4\pi}\right)^{n+1}\Gamma_n$$

IR subtraction

The anomalous dimension for the HLFF:

$$\Gamma = \gamma^t(\bar{\alpha}_s) + \gamma^b(\bar{\alpha}_s) - \gamma^{\text{cusp}}(\bar{\alpha}_s) \ln\left(\frac{\bar{\mu}}{m_t(1-x)}\right)$$

1. γ^t known till 3-loops: Korchemsky, Radyushkin ('87, '92); Kidonakis ('09); Grozin et al.('15); ...
2. γ^b known till 4-loops: Moch et al.('05); Baikov et al.('09); Manteuffel et al. ('20); Agarwal et al. ('21) ...
3. γ^{cusp} known till 4-loops: Henn et al. ('20); ...

IR subtraction

Finally,

$$\ln \bar{Z} = \left(\frac{\bar{\alpha}_s}{4\pi}\right)\left[\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right] + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^2\left[-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right]$$

$$+ \left(\frac{\bar{\alpha}_s}{4\pi}\right)^3\left[\frac{11\beta_0^2\Gamma_0'}{72\epsilon^4} - \frac{5\beta_0\Gamma_1' + 8\beta_1\Gamma_0' - 12\beta_0^2\Gamma_0}{72\epsilon^3} + \frac{\Gamma_2' - 6\beta_0\Gamma_1 - 6\beta_1\Gamma_0}{36\epsilon^2} + \frac{\Gamma_2}{6\epsilon}\right] + \mathcal{O}(\alpha_s^4)$$

where, $\Gamma_n' = \frac{\partial}{\partial \bar{\mu}}\Gamma_n$

Now, use the decoupling relation to obtain Z from $\bar{Z}$: $\bar{\alpha}_s = \zeta_{\alpha_s}\alpha_s$

where, the decoupling constant ζ_{α_s} is known till 4-loops. Schröder, Steinhauser ('05)

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Typically, factorisation theorems → evolution equations → resummation of *something*

eg.,

1. factorisation of singular cutoff dependence into universal Z-factors → Callan-Symanzik evolution equations → resummation of logs in μ_R,

2. collinear factorisation for hadronic collisions → DGLAP evolution equations → resummation of collinear logs, …

   - generalisable to a *soft-collinear* factorisation of scattering amplitudes
   - leads to the K-G evolution equations shown earlier
   - resummation of Sudakov logs and IR divergences

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The K-G equation for form-factors with massive-partons:

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln \hat{F}_I\left(\frac{Q^2}{\mu^2}, \frac{m_t^2}{\mu^2}, \frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right) = \frac{1}{2}\left[K_I\left(\frac{m_t^2}{\mu^2}, \frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right) + G_I\left(\frac{Q^2}{\mu^2}, \frac{\mu_R^2}{\mu^2}, \alpha_s, \epsilon\right)\right]$$

- I labels the external current coupling to the heavy-light fermion pair

- $\hat{F}_I$ has contributions from universal logs and IR structures

- K_I is process-independent; has mass-dependence

- G_I has the process-dependence through the hard-scale Q^2

$$\mu^2 \frac{d}{d\mu^2} G_I\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) = -\lim_{m_t \to 0} \mu^2 \frac{d}{d\mu^2} K_I\left(\frac{m_t^2}{\mu^2}, \alpha_s, \epsilon\right) = \gamma^{\text{cusp}}(\alpha_s)$$

- where we have set the soft-collinear factorisation scale $\mu = \mu_R$
- with boundary conditions set at $K_I(\alpha_s(m_t^2), 1, \epsilon) \equiv \mathscr{K}_I$ and $G_I(\alpha_s(Q^2), 1, \epsilon) \equiv \mathscr{G}_I$

$$K_I = \mathscr{K}_I - \int_{\frac{m_t^2}{\mu^2}}^{1} \frac{d\lambda}{\lambda} \gamma^{\text{cusp}}(\alpha_s(\lambda\mu^2)); \; G_I = \mathscr{G}_I + \int_{\frac{Q^2}{\mu^2}}^{1} \frac{d\lambda}{\lambda} \gamma^{\text{cusp}}(\alpha_s(\lambda\mu^2))$$

For the HQFF@$\mathcal{O}(\alpha_s^3)$, the solutions for $\hat{F}_I$ have been computed. Blümlein, Marquard, Rana ('18)
NOTE: these solutions are devoid of massive internal fermion-loops.

Solutions for the HLFF@$\mathcal{O}(\alpha_s^3)$ should be same as the HQFF@$\mathcal{O}(\alpha_s^3)$, upto a reinterpretation of $\mathscr{K}_I$

Solutions for the HLFF@$\mathcal{O}(\alpha_s^3)$ should be same as the HQFF@$\mathcal{O}(\alpha_s^3)$, upto a reinterpretation of $\mathcal{K}_I$

Since $\mathcal{K}_I$ encodes the universality of the IR singularities, we expect it to have **equal** contributions from its counterparts for the purely massless and massive form-factors:

$$\mathcal{K}_I = \frac{1}{2}\left(\mathcal{K}_{I,0} + \mathcal{K}_{I,m_t}\right)$$

$\hat{F}_I$ -s are related to $\tilde{F}_I$ -s (asymptotic limits of F_I -s) through matching-coefficients $\mathcal{C}_I$ -s

$$\tilde{F}_I\left(\frac{Q^2}{\mu^2}, \frac{m_t^2}{\mu^2}, \alpha_s, \epsilon\right) = \mathcal{C}_I \hat{F}_I\left(\frac{Q^2}{\mu^2}, \frac{m_t^2}{\mu^2}, \alpha_s, \epsilon\right)$$

- In summary, everything available in literature to compute the HLFF matching coefficients $\mathcal{C}_I$ -s

With all these, we have successfully predicted the following quantities:

1. For the 3-loop HLFF, complete log-contributions (series in $L = \ln\left(-\frac{q^2}{m_t^2}\right)$) to the finite part, in the asymptotic limit (eg., $\tilde{G}_1^{(3,0)}$ and $S^{(3,0)}$).
2. For the 4-loop HLFF:
    - ϵ^{-3} at full-color
    - ϵ^{-2} for full-n_l
    - ϵ^{-1} with all orders in L
    - Finite term till L^2

1. As discussed earlier, the matching coefficients $\mathscr{C}_I$ are known only partially - the full n_l and color-planar contributions. Non-log contributions to the finite HLFF-s in the asymptotic limit have been obtained for these color-structures.

2. We have found perfect agreement between the color-planar predictions and our results, after expanding our results in the large-x limit.

   Thus, yet another strong **consistency-check**!

Summary

1. Computed HLFF@$\mathcal{O}(\alpha_s^3)$ in the color-planar limit.

2. Multiple consistency checks - analytic vs numeric, Ward, asymptotic limit …

3. Essential for phenomenology, particularly B-physics.

4. Results have been independently confirmed in Fael, Huber, Lange, Müller, Schönwald, Steinhauser (2024).

5. Next steps: completing calculations for other color-structures (in-progress)…

Thanks !

Backup material

PSLQ tips

- **Example 1**:

  ϵ-coefficient for boundary integral J_2: number with 100 digits precision.

  For this case, the naive choice for the PSLQ basis works -

  $$\{1, \zeta_2, \zeta_3, \zeta_2^2, \zeta_2\zeta_3, \zeta_5, \zeta_2^3, \zeta_3^2, \zeta_2^2\zeta_3, \zeta_2\zeta_5, \zeta_7\}$$

  result -

  $$\frac{48}{5}\zeta_2^2 + \frac{2122}{35}\zeta_2^3 + 32\zeta_2\zeta_3 + \frac{681}{10}\zeta_2^2\zeta_3 + 12\zeta_3^2 + 8\zeta_5 - \frac{67}{2}\zeta_2\zeta_5 + \frac{8529}{16}\zeta_7$$

PSLQ tips

- **Example 2**:

ϵ-coefficient for boundary integral J_1: number with 100 digits precision.

For this case, the naive choice for the PSLQ basis does not work.

incorrect result (at 83 to 94 digits of fitting-precision) -

$$\frac{169638071}{2242112} + \frac{3794333\,\pi^2}{4484224} + \frac{35759009\,\pi^4}{40358016} - \frac{3328583\,\pi^6}{242148096} + \frac{4080315\,\zeta_3}{2242112} + \dots$$

incorrect result (at 98 and 99 digits of fitting-precision) -

$$-\frac{157188793}{53038334} + \frac{4722082655\,\pi^2}{318230004} - \frac{477444277\,\pi^4}{1909380024} + \frac{1100756927\,\pi^6}{3818760048} - \frac{572414197\,\zeta_3}{53038334} + \dots$$

PSLQ tips

- **Example 2**:

  ϵ-coefficient for boundary integral J_1: number with 100 digits precision.

  Prune the PSLQ basis -

  $$\{1, \zeta_2, \zeta_3, \zeta_2^2, \zeta_2\zeta_3, \zeta_5, \zeta_2^3, \zeta_3^2, \zeta_2^2\zeta_3, \zeta_2\zeta_5, \zeta_7\}$$

  correct result -

  $$\frac{39151}{90} + \frac{1211}{10}\zeta_2 + \frac{80939}{1440}\zeta_2^2 + \frac{71737}{720}\zeta_3 + \frac{20587}{1440}\zeta_2\zeta_3 + \frac{93589}{1200}\zeta_5$$