Off-forward anomalous dimensions for the transvesity operators

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Introduction

Hard exclusive processes

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(Generalised) parton distributions

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Leading-twist operators

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Renormalization of the twist-2 operators

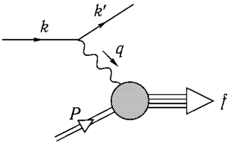


Figure: Deep Inelastic Scattering. (Pic. from the Peskin-Schroeder book)

Björken limit

$$Q^2 = -q_\mu q^\mu \quad x = \frac{Q^2}{2P \cdot q}. \tag{1}$$

Considering limit $Q^2 \to \infty$, while x is fixed.

Hadronic tensor

Hadronic part of the interactions is described by the tensor

$$W_{\mu\nu} = \int d^4x e^{iq\cdot x} \langle P| T\{j_\mu(x) j_\nu(0)\} |P\rangle \tag{2}$$

Wilson's OPE

For the arbitrary operators A and B there is the following expansion

$$T\{A(x)B(0)\} = \sum_i C_i(x) \mathcal{O}_i(0). \tag{3}$$

Scaling

Operators with a twist $\tau = d - s = 2n + 2$ contribute as $\left(M^2/Q^2\right)^n$, where d is a scaling dimension and s is a spin of operator $\mathcal{O}_i$.

Twist-2 operators

Local twist-2 operators

We introduce the family of operators

$$\mathcal{O}_{m,k}(x) = \bar{q}(x)\left(\overleftarrow{D}\cdot n\right)^{m} \not{\Gamma} \left(\overrightarrow{D}\cdot n\right)^{k} q(x), \tag{4}$$

where D_μ is the covariant derivative, n^μ is a light-like vector $(n^2 = 0)$ and $\not{\Gamma}$ can represent different Dirac structures quark fields $q(x)$ and $\bar{q}(x)$ are assumed to be different flavour.

Composite operators mix under renormalization

$$[\mathcal{O}_k] = \sum_j Z_{kj}\mathcal{O}_j. \tag{5}$$

Anomalous dimension matrix

We introduce the anomalous dimension matrix

$$\gamma_{kj} = -\frac{d\ln Z_{kj}}{d\ln\mu} \tag{6}$$

Light-ray operators

Light-ray (generating) non-local operator

$$\mathcal{O}(x; z_1, z_2) = \bar{q}(x + z_1 n)\not{n}[z_1 n, z_2 n]q(x + z_2 n), \tag{7}$$

where $z_1, z_2 \in \mathbb{R}$, $n^2 = 0$ is a light-like vector and $[z_1 n, z_2 n]$ is a Wilson line.

$$[z_1 n, z_2 n] = \mathrm{Pexp}\left(igz_{12}\int_0^1 du\, n^\mu A_\mu(z_{21}^u n)\right), \tag{8}$$

where $z_{12}^u = z_1\bar{u} + z_2 u$, $\bar{u} = 1 - u$ and $z_{12} = z_1 - z_2$.

Renormalization of light-ray operators

$$[\mathcal{O}]\,(z_1, z_2) = Z\mathcal{O}(x = 0, z_1, z_2), \tag{9}$$

where Z is an integral operator, which acts on the sample function f in the form

$$Zf(z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, z(\alpha, \beta) f(z_{12}^\alpha, z_{21}^\beta) \tag{10}$$

Renormalization

RG-equation

Renormalization group equation for the light-ray operators takes the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathbb{H}(a)\right)[\mathcal{O}](z_1, z_2) = 0, \tag{11}$$

where $a = \alpha_s/4\pi$ and $\mathbb{H}(a)$ is a so-called **evolution kernel**

$$\mathbb{H} = -\mu\frac{d\mathbb{Z}}{d\mu}\mathbb{Z}^{-1}, \tag{12}$$

where $\mathbb{Z} = ZZ_q^{-2}$.

Evolution kernel vs Anomalous dimension matrix

$$\mathbb{H}(a) \Leftrightarrow \gamma_{kj}. \tag{13}$$

Back to the local

Gegenbauer basis

Local twist-2 operators can be expressed in the Gegenbauer basis as follows

$$\mathcal{O}^G_{n,k} = (\partial_{z_1} + \partial_{z_2})^k C_n^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right) \mathcal{O}(z_1, z_2)\Big|_{z_1=z_2=0}, \tag{14}$$

where $C_N^\nu(x)$ is the Gegenbauer polynomial.

RG-equation then takes the form

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)\left[\mathcal{O}^G_{n,k}\right] = -\sum_{n'=0}^{n} \gamma^G_{n,n'}\left[\mathcal{O}^G_{n',k}\right]. \tag{15}$$

Anomalous dimension matrix

- $\gamma^G_{n,n}$ — forward anomalous dimensions, contribute to the $\langle P|\mathcal{O}(z_1, z_2)|P\rangle$
- $\gamma^G_{n,n'}$ — **off-forward** anomalous dimensions, contribute to the $\langle P'|\mathcal{O}(z_1, z_2)|P\rangle$

We want to use notation $x_+ = n^\mu x_\mu$, $x_- = \bar{n}^\mu x_\mu$, where $\bar{n}^2 = 0$ and $n \cdot \bar{n} = 1$, and $x_\perp$ is a transverse projection.

Different Dirac structure

- $\not{\Gamma} = \gamma_+$ – Three-loop result in [V. M. Braun, A. N. Manashov, S. Moch, M. Strohmaier'2017];
- $\not{\Gamma} = \gamma_+\gamma_5$ – Three-loop result in [V. M. Braun, A. N. Manashov, S. Moch, M. Strohmaier'2021];
- $\not{\Gamma} = \sigma_{\perp +}$ – **Three-loop result** in this work,

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

Usage

Transversity operators are connected to the polarized processes:

- Semi-inclusive DIS;
- Polarized Drell-Yan.

Conformal symmetry

The QCD in 4 physical dimensions is definetely **NOT** a conformal theory.

Critical point

At the critical point $\beta(a^*) = 0$ usual Poincare symmetry is enhanced by the conformal symmetry. The point $a = a^*$ is obtained by the $\epsilon \neq 0$ in $d = 4 - 2\epsilon$.

Generators of the conformal group act on the primary field $\Phi(x)$ as follows

$$\begin{aligned} i[\mathbf{P}^\mu, \Phi(x)] &= \partial^\mu \Phi(x); \\ i[\mathbf{M}^{\mu\nu}, \Phi(x)] &= (x^\mu \partial^\nu - x^\nu \partial^\mu - \Sigma^{\mu\nu})\Phi(x); \\ i[\mathbf{D}, \Phi(x)] &= (x \cdot \partial + d_\Phi)\Phi(x); \\ i[\mathbf{K}^\mu, \Phi(x)] &= (2x^\mu (x \cdot \partial) - x^2 \partial^\mu + 2d_\Phi x^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x), \end{aligned} \tag{16}$$

where $\Sigma_{\mu\nu}$ is a spin part of the rotation generator, which acts different for the fields from the different representation of Lorentz group, and d_Φ is a scaling dimension of the field $\Phi(x)$.

Independent kernels

- Physical QCD: $\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \ldots;$
- Critical point QCD: $\mathbb{H}(a^*) = a^*\mathbb{H}^{(1)} + (a^*)^2\mathbb{H}^{(2)} + \ldots.$

We are interested in the transformations which map light-like direction into itself.

Collinear subgroup of the conformal group

Generators of the collinear subgroup of the full conformal group:

$$\begin{aligned} \mathbf{L}_+ &= -in^\mu \mathbf{P}_\mu; \\ \mathbf{L}_- &= \frac{i}{2}\bar{n}^\mu \mathbf{K}_\mu; \\ \mathbf{L}_0 &= \frac{i}{2}\left(\mathbf{D} + n^\mu \bar{n}^\nu \mathbf{M}_{\mu\nu}\right). \end{aligned} \tag{17}$$

Commutation relations

Collinear subgroup is isomorphic to the $SL(2,\mathbb{R})$ with the commatation relation

$$[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp, \quad [\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0. \tag{18}$$

Action on the light-ray operators can be represented as a differential operators acting on the z-variables

$$\left[S_0^{(0)}, S_\pm^{(0)}\right] = \pm S_\pm^{(0)}, \quad \left[S_+^{(0)}, S_-^{(0)}\right] = 2S_0^{(0)}. \tag{19}$$

Canonical generators

$$\begin{aligned} S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2}; \\ S_0^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2; \\ S_+^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2). \end{aligned} \tag{20}$$

For the one-loop evolution kernel

$$\left[\mathbb{H}^{(1)}, S_\alpha\right] = 0, \tag{21}$$

where $\alpha = +, -, 0$.

Unfortunately, symmetry doesn't hold in the higher loop orders.

Deformed generators

We can adjust generators of the collinear subgroup to restore the exact conformal symmetry

$$[S_0, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_0. \tag{22}$$

Deformed generators

$$\begin{aligned} S_- &= S_-^{(0)}; \\ S_0 &= S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a^*); \\ S_+ &= S_+^{(0)} + \underbrace{(z_1 + z_2)\left(-\epsilon + \frac{1}{2}\mathbb{H}(a^*)\right) + (z_1 - z_2)\Delta_+(a^*)}_{\Delta S_+}. \end{aligned} \tag{23}$$

Term $\Delta_+(a*)$ is called a **conformal anomaly** and can be calculated only perturbatively.

$$\Delta_+(a^*) = (a^*)^2\Delta_+^{(1)} + (a^*)^2\Delta_+^{(1)} + \ldots \tag{24}$$

Conformal constraints

In the representation of deformed generators conformal symmetry of the full evolution kernel the is restored

$$[S_\alpha, \mathbb{H}(a^*)] = 0. \tag{25}$$

Conformal constraint for the kernel

$$\begin{aligned}
\left[S_+^{(0)}, \mathbb{H}^{(1)}\right] &= 0; \\
\left[S_+^{(0)}, \mathbb{H}^{(2)}\right] &= \left[\mathbb{H}^{(1)}, \Delta S_+^{(1)}\right]; \\
\left[S_+^{(0)}, \mathbb{H}^{(3)}\right] &= \left[\mathbb{H}^{(1)}, \Delta S_+^{(2)}\right] + \left[\mathbb{H}^{(2)}, \Delta S_+^{(1)}\right].
\end{aligned} \tag{26}$$

Conformal constraints

In the representation of deformed generators conformal symmetry of the full evolution kernel the is restored

$$[S_\alpha, \mathbb{H}(a^*)] = 0. \tag{25}$$

Conformal constraint for the kernel

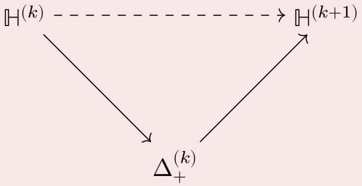
$$\begin{aligned}
\left[S_+^{(0)}, \mathbb{H}^{(1)}\right] &= 0; \\
\left[S_+^{(0)}, \mathbb{H}^{(2)}\right] &= \left[\mathbb{H}^{(1)}, \Delta S_+^{(1)}\right]; \\
\left[S_+^{(0)}, \mathbb{H}^{(3)}\right] &= \left[\mathbb{H}^{(1)}, \Delta S_+^{(2)}\right] + \left[\mathbb{H}^{(2)}, \Delta S_+^{(1)}\right].
\end{aligned} \tag{26}$$

Three-loop order

Two-loop order

Conformal constraints

The main idea



Note

We can only restore a non-invariant part of the kernel $\left[\mathbb{H}^{\text{non-inv}}, S_\alpha^{(0)}\right] \neq 0$ and invariant part can be restored using the forward anomalous dimensions

$$\mathbb{H} z_{12}^{N-1} = \gamma_N z_{12}^{N-1}. \tag{27}$$

Modification of QCD action

Conformal anomaly can be calculated in the framework of the adjusted QCD action

$$S_{QCD} \mapsto S_\omega = S_{QCD} + \delta^\omega S = S_{QCD} - 2\omega \int d^d y (\bar{n}y) \left(\frac{1}{4} F^2 + \frac{1}{2\xi} (\partial A)^2 \right). \tag{28}$$

The renoramlization operator then takes the form

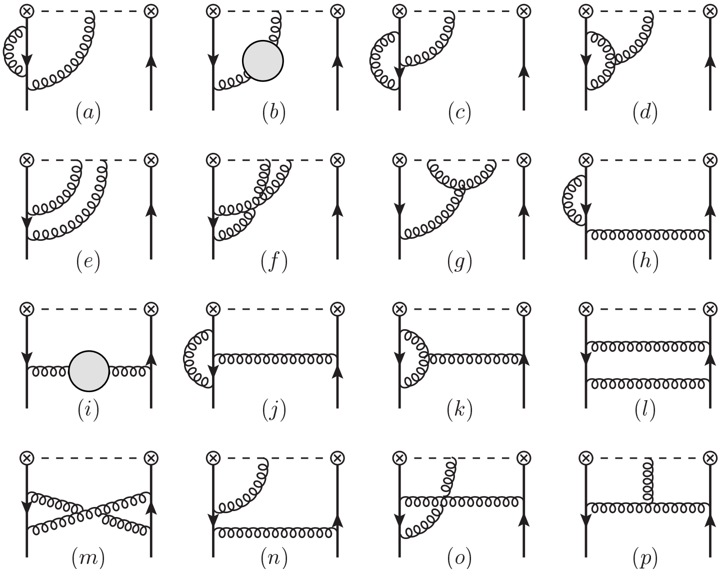
$$Z \mapsto Z_\omega = Z + \omega(n\bar{n})\widetilde{Z}, \qquad \widetilde{Z} = \frac{1}{\epsilon}\widetilde{Z}_1(a) + \frac{1}{\epsilon^2}\widetilde{Z}_2 + \dots . \tag{29}$$

Conformal anomaly and residues

Connection between conformal anomaly and renormalization operator has the form

$$\widetilde{Z}_1(a) = z_{12}\Delta_+(a) + \frac{1}{2}\left[\mathbb{H}(a) - 2\gamma_q(a)\right](z_1 + z_2). \tag{30}$$

Two-loop diagrams for the evolution kernel



Two-loop conformal anomaly

The kernel $\Delta_+^{(2)}$ can be written in the following form

$$[\Delta_+^{(2)} f](z_1, z_2) = \int_0^1 du \int_0^1 dt\, \varkappa(t) \Big[f(z_{12}^{ut}, z_2) - f(z_1, z_{21}^{ut}) \Big] \\ + \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Big[\omega(\alpha,\beta) + \overline{\omega}(\alpha,\beta)\mathbb{P}_{12} \Big] \Big[f(z_{12}^{\alpha}, z_{21}^{\beta}) - f(z_{12}^{\beta}, z_{21}^{\alpha}) \Big]. \quad (31)$$

Two-loop conformal anomaly

$$\varkappa(t) = C_F^2 \varkappa_P(t) + \frac{C_F}{N_C} \varkappa_{FA}(t) + C_F \beta_0 \varkappa_{bF}(t), \tag{32}$$

$$\begin{aligned}
\varkappa_{bF}(t) =& -2\frac{\bar{t}}{t}\left(\ln \bar{t} + \frac{5}{3}\right), \\
\varkappa_{FA}(t) =& \frac{2\bar{t}}{t}\left\{(2+t)\left[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(t)\right] - (2-t)\left(\frac{t}{\bar{t}}\ln t + \ln \bar{t}\right)\right. \\
& \left. - \frac{\pi^2}{6}t - \frac{4}{3} - \frac{t}{2}\left(1 - \frac{t}{\bar{t}}\right)\right\}, \\
\varkappa_P(t) =& 4\bar{t}\left[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(1)\right] + 4\left(\frac{t^2}{\bar{t}} - \frac{2\bar{t}}{t}\right)\left[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\right] - 2t\ln t \ln \bar{t} \\
& - \frac{\bar{t}}{t}(2-t)\ln^2 \bar{t} + \frac{t^2}{\bar{t}}\ln^2 t - 2\left(1 + \frac{1}{t}\right)\ln \bar{t} - 2\left(1 + \frac{1}{\bar{t}}\right)\ln t \\
& - \frac{16}{3}\frac{\bar{t}}{t} - 1 - 5t\,.
\end{aligned} \tag{33}$$

Two-loop conformal anomaly

$$\overline{\omega}(\alpha,\beta) = \frac{C_F}{N_C}\overline{\omega}_{NP}(\alpha,\beta), \tag{34}$$

$$\overline{\omega}_{NP}(\alpha,\beta) = -2\left\{\frac{\alpha}{\bar{\alpha}}\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar{\beta}}\right) - \mathrm{Li}_2(\alpha)\right] - \alpha\bar{\tau}\ln\bar{\tau} - \frac{1}{\bar{\alpha}}\ln\bar{\alpha}\ln\bar{\beta} - \frac{\beta}{\bar{\beta}}\ln\bar{\alpha} - \frac{1}{2}\beta\right\}. \tag{35}$$

$$\omega(\alpha,\beta) = C_F^2\omega_P(\alpha,\beta) + \frac{C_F}{N_C}\omega_{NP}(\alpha,\beta), \tag{36}$$

$$\begin{aligned}
\omega_P(\alpha,\beta) &= \frac{4}{\alpha}\left[\mathrm{Li}_2(\bar{\alpha}) - \zeta_2 + \frac{1}{4}\bar{\alpha}\ln^2\bar{\alpha} + \frac{1}{2}(\beta-2)\ln\bar{\alpha}\right] \\
&\quad + \frac{4}{\bar{\alpha}}\left[\mathrm{Li}_2(\alpha) - \zeta_2 + \frac{1}{4}\alpha\ln^2\alpha + \frac{1}{2}(\bar{\beta}-2)\ln\alpha\right], \\
\omega_{NP}(\alpha,\beta) &= 2\left\{\frac{\bar{\alpha}}{\alpha}\left[\mathrm{Li}_2\left(\frac{\beta}{\bar{\alpha}}\right) - \mathrm{Li}_2(\beta) - \mathrm{Li}_2(\alpha) + \mathrm{Li}_2(\bar{\alpha}) - \zeta_2\right] - \ln\alpha - \frac{1}{\alpha}\ln\bar{\alpha}\right. \\
&\quad \left. + \alpha\left(\frac{\bar{\tau}}{\tau}\ln\bar{\tau} + \frac{1}{2}\right)\right\}.
\end{aligned} \tag{37}$$

Form of the invariant part

Invariant part of the evolution kernel $\mathbb{H}_{\text{inv}}$ is highly constrained

$$\mathbb{H}_{\text{inv}}(a) = \Gamma_{\text{cusp}}(a)\widehat{\mathcal{H}} + \mathcal{A}(a) + \mathcal{H}(a), \tag{38}$$

where $\Gamma_{\text{cusp}}(a)$ is a cusp anomalous dimension, $\mathcal{A}(a)$ is a constant and operators have the form

$$\left[\widehat{\mathcal{H}} f\right](z_1, z_2) = \int_0^1 \frac{d\alpha}{\alpha}\left(2f(z_1, z_2) - \bar{\alpha}\big(f(z_{12}^{\alpha}, z_2) + f(z_1, z_{21}^{\alpha})\big)\right). \tag{39}$$

and

$$\left[\mathcal{H}(a) f\right](z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \left(h(\tau) + \overline{h}(\tau)\mathbb{P}_{12}\right) f(z_{12}^{\alpha}, z_{21}^{\beta}). \tag{40}$$

Consequences of $SL(2,\mathbb{R})$ invariance

The crucial point is that $h(\tau)$ and $\overline{h}(\tau)$ are functions of only one variable $\tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$.

Forward anomalous dimensions can be divided into invariant and non-invariant part as well

$$\gamma(N) = \gamma_{\text{inv}}(N) + \gamma_{\text{non-inv}}(N) \tag{41}$$

Structure of the invariant part

Invariant part also has particular structure

$$\gamma_{\text{inv}}(N) = 2\Gamma_{\text{cusp}}(a)S_1(N) + \mathcal{A}(a) + m(N), \tag{42}$$

where $S_1(N)$ is a Harmonic sum.

Using **forward** anomalous dimensions

We use the result for the $\gamma^{(3)}(N)$ [V. N. Velizhanin'2012]

$$\gamma^{(3)}(N) \ \rightarrow m^{(3)}(N) \rightarrow h^{(3)}(\tau), \overline{h}^{(3)}(\tau). \tag{43}$$

$$
\begin{aligned}
h^{(3)}(\tau) = & -C_F n_f^2 \frac{16}{9} + C_F^2 n_f \left(\frac{352}{9} - \frac{8}{3}\mathrm{H}_0 + \frac{16}{3}\frac{\bar{\tau}}{\tau}(\mathrm{H}_2 - \mathrm{H}_{10}) \right) \\
& + \frac{C_F n_f}{N_c} \left(8 - \frac{8}{3}\mathrm{H}_1 - \frac{4}{3}\mathrm{H}_0 + \frac{\bar{\tau}}{\tau}\left(8\mathrm{H}_2 - \frac{8}{3}\mathrm{H}_{10} + \frac{16}{3}\mathrm{H}_{11} + \frac{160}{9}\mathrm{H}_1 \right) \right) \\
& + C_F^3 \left(-\frac{1936}{9} + \frac{88}{3}\mathrm{H}_0 + 32\frac{\bar{\tau}}{\tau}\left(\mathrm{H}_3 + \mathrm{H}_{12} - \mathrm{H}_{110} - \mathrm{H}_{20} - \frac{1}{3}\mathrm{H}_2 + \frac{1}{3}\mathrm{H}_{10} + \frac{1}{2}\mathrm{H}_1 \right) \right) \\
& + \frac{C_F^2}{N_c} \Bigg(-\frac{152}{3} - 96\zeta_3 - \left(\frac{8}{3} - 48\zeta_2 \right)\mathrm{H}_0 + \frac{76}{3}\mathrm{H}_1 - 32\mathrm{H}_{10} + 4\mathrm{H}_2 - 48\mathrm{H}_{20} - 16\mathrm{H}_{11} \\
& - 24\mathrm{H}_{21} + \frac{\tau}{\bar{\tau}}\Big(-24\zeta_2 - 48\zeta_3 + 64\mathrm{H}_0 \Big) + \frac{\tau+1}{\bar{\tau}}\Big(-(32 - 16\zeta_2)\mathrm{H}_0 \\
& + 12\mathrm{H}_2 - 16\mathrm{H}_{20} - 8\mathrm{H}_{21} \Big) + \frac{\bar{\tau}}{\tau}\Big(-\left(\frac{2000}{9} + 16\zeta_2 \right)\mathrm{H}_1 + \frac{32}{3}\mathrm{H}_{10} - \frac{208}{3}\mathrm{H}_2 \\
& - 64\mathrm{H}_{20} - \frac{32}{3}\mathrm{H}_{11} - 32\mathrm{H}_{110} + 64\mathrm{H}_3 + 80\mathrm{H}_{12} + 64\mathrm{H}_{21} + 96\mathrm{H}_{111} \Big) \Bigg) \\
& + \frac{C_F}{N_c^2} \Bigg(\frac{544}{9} + 16\zeta_2 - 96\zeta_3 - \left(\frac{68}{3} - 36\zeta_2 \right)\mathrm{H}_0 + \frac{68}{3}\mathrm{H}_1 - 24\mathrm{H}_{10} + 4\mathrm{H}_2 - 36\mathrm{H}_{20} \\
& + \frac{\tau}{\bar{\tau}}\Big(-8\zeta_2 - 48\zeta_3 + 48\mathrm{H}_0 \Big) + \frac{\tau+1}{\bar{\tau}}\Big((-24 + 12\zeta_2)\mathrm{H}_0 + 4\mathrm{H}_2 - 12\mathrm{H}_{20} \Big) \\
& + \frac{\bar{\tau}}{\tau}\Big(-\left(\frac{1072}{9} + 16\zeta_2 \right)\mathrm{H}_1 + \frac{44}{3}\mathrm{H}_{10} - 44\mathrm{H}_2 - 32\mathrm{H}_{20} - \frac{16}{3}\mathrm{H}_{11} - 16\mathrm{H}_{110} \\
& + 32\mathrm{H}_3 + 32\mathrm{H}_{12} + 48\mathrm{H}_{21} + 32\mathrm{H}_{111} \Big) \Bigg).
\end{aligned}
\tag{44}
$$

Results for the invariant part

$$
\begin{aligned}
\overline{h}^{(3)}(\tau) = &-\frac{C_F n_f}{N_c}\left(\frac{104}{9}+\frac{8}{3}\mathrm{H}_0+\frac{8}{9}\Big(23-20\tau\Big)\mathrm{H}_1+\frac{16}{3}\bar{\tau}\Big(\mathrm{H}_{11}+\mathrm{H}_{10}\Big)\right)\\
&+\frac{C_F^2}{N_c}\Bigg(\frac{1480}{9}-40\zeta_2-48\zeta_3+\left(\frac{28}{3}+24\zeta\right)\mathrm{H}_0+\frac{76}{3}\mathrm{H}_1+16\mathrm{H}_{10}-4\mathrm{H}_2-24\mathrm{H}_{20}\\
&-16\mathrm{H}_{11}+24\mathrm{H}_{21}+\frac{\tau}{\bar{\tau}}\Big(-24\zeta_2+48\zeta_3-32\mathrm{H}_0\Big)+\frac{\tau+1}{\bar{\tau}}\Bigg(\Big(16-8\zeta_2\Big)\mathrm{H}_0+12\mathrm{H}_2\\
&+8\mathrm{H}_{20}-8\mathrm{H}_{21}\Bigg)+\bar{\tau}\Bigg(-24+48\zeta_2+48\zeta_3-16\zeta_2\mathrm{H}_0+\left(\frac{2144}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{104}{3}\mathrm{H}_{10}\\
&-24\mathrm{H}_2+16\mathrm{H}_{20}+\frac{32}{3}\mathrm{H}_{11}-16\mathrm{H}_{110}-32\mathrm{H}_{12}-32\mathrm{H}_{21}-96\mathrm{H}_{111}\Bigg)\Bigg)\\
&+\frac{C_F}{N_c^2}\Bigg(\frac{1028}{9}-24\zeta_2-48\zeta_3+\left(\frac{44}{3}+36\zeta_2\right)\mathrm{H}_0+\frac{68}{3}\mathrm{H}_1+24\mathrm{H}_{10}-4\mathrm{H}_2-36\mathrm{H}_{20}\\
&+\frac{\tau}{\bar{\tau}}\Big(-8\zeta_2+48\zeta_3-48\mathrm{H}_0\Big)+\frac{\tau+1}{\bar{\tau}}\Bigg(\Big(24-12\zeta_2\Big)\mathrm{H}_0+4\mathrm{H}_2+12\mathrm{H}_{20}\Bigg)\\
&+\bar{\tau}\Bigg(-24+24\zeta_2+48\zeta_3-32\zeta_2\mathrm{H}_0+\left(\frac{1072}{3}+16\zeta_2\right)\mathrm{H}_1+\frac{88}{3}\mathrm{H}_{10}\\
&-24\mathrm{H}_2+32\mathrm{H}_{20}+\frac{16}{3}\mathrm{H}_{11}-32\mathrm{H}_{110}+16\mathrm{H}_{12}+16\mathrm{H}_{21}-32\mathrm{H}_{111}\Bigg)\Bigg).
\end{aligned}
\tag{45}
$$

Final result in the matrix form

Light-ray $\longrightarrow$ local

We can extract local operators with the formula

$$\mathcal{O}_{nk}(0) = (\partial_{z_1} + \partial_{z_2})^k C_n^{(3/2)} \left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) [\mathcal{O}](z_1, z_2) \Big|_{z_1 = z_2 = 0}, \tag{46}$$

and the RG equation takes the form

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} \right) O_{nk} = - \sum_{n'=0}^{n} \gamma_{nn'} O_{n'k}. \tag{47}$$

where $n, n' = 0, 1, ...$

Let us separate the diagonal and off-diagonal parts for the convenience

$$\gamma_{\text{off}}^{(3)} = \gamma_1^{(3)} + n_f \gamma_{n_f}^{(3)} + n_f^2 \gamma_{n_f^2}^{(3)}. \tag{48}$$

We consider $N_c = 3$ and $0 \leq n, n' \leq 5$

$$\gamma_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{44992}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1316680}{2187} & 0 & 0 & 0 & 0 \\ \frac{1977808}{10125} & 0 & \frac{54669748}{91125} & 0 & 0 & 0 \\ 0 & \frac{68848018}{273375} & 0 & \frac{443231668}{759375} & 0 & 0 \end{pmatrix} \tag{49}$$

and

$$\gamma_{n_f}^{(3)} = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{21008}{243} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{200060}{2187} & 0 & 0 & 0 & 0 \\ \frac{998842}{30375} & 0 & \frac{898436}{10125} & 0 & 0 & 0 \\ 0 & \frac{745418}{18225} & 0 & \frac{4266496}{50625} & 0 & 0 \end{pmatrix}, \tag{50}$$

$$\gamma_{n_f^2}^{(3)} = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{160}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{520}{243} & 0 & 0 & 0 & 0 \\ \frac{1012}{2025} & 0 & \frac{4088}{2025} & 0 & 0 & 0 \\ 0 & \frac{3268}{3645} & 0 & \frac{416}{225} & 0 & 0 \end{pmatrix}. \tag{51}$$

Result for the matrix

The diagonal elements have the form

$$\begin{aligned}
\gamma_{00}^{(3)} &= \frac{105110}{81} - \frac{1856}{27}\zeta_3 - \left(\frac{10480}{81} + \frac{320}{9}\zeta_3\right) n_f - \frac{8}{9} n_f^2 \\
\gamma_{11}^{(3)} &= \frac{19162}{9} - \left(\frac{5608}{27} + \frac{320}{3}\zeta_3\right) n_f - \frac{184}{81} n_f^2 \\
\gamma_{22}^{(3)} &= \frac{17770162}{6561} + \frac{1280}{81}\zeta_3 - \left(\frac{552308}{2187} + \frac{4160}{27}\zeta_3\right) n_f - \frac{2408}{729} n_f^2 \\
\gamma_{33}^{(3)} &= \frac{206734549}{65610} + \frac{560}{27}\zeta_3 - \left(\frac{3126367}{10935} + \frac{5120}{27}\zeta_3\right) n_f - \frac{14722}{3645} n_f^2 \\
\gamma_{44}^{(3)} &= \frac{144207743479}{41006250} + \frac{9424}{405}\zeta_3 - \left(\frac{428108447}{1366875} + \frac{5888}{27}\zeta_3\right) n_f - \frac{418594}{91125} n_f^2 \\
\gamma_{55}^{(3)} &= \frac{183119500163}{47840625} + \frac{3328}{135}\zeta_3 - \left(\frac{1073824028}{3189375} + \frac{2176}{9}\zeta_3\right) n_f - \frac{3209758}{637875} n_f^2 .
\end{aligned} \tag{52}$$