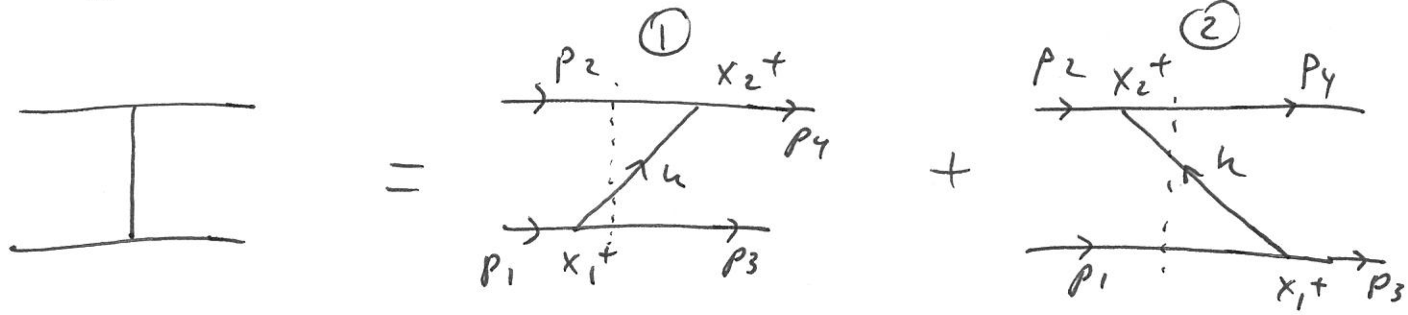
Last time | Light Cone Perturbation Theory (LCPT).

We started discussing LCPT as a x^+-ordered theory. We showed that



$x_1^+ < x_2^+$ $\qquad\qquad$ $x_1^+ > x_2^+$

$$① = \frac{\theta(k^+)}{k^+} \underbrace{\frac{2i}{p_1^- - p_3^- - \frac{k_\perp^2+m^2}{k^+} + i\varepsilon}}_{\sum p^-_{inc} - \sum p^-_{interm} + i\varepsilon} \underbrace{4\pi\,\delta(p_1^- + p_2^- - p_3^- - p_4^-)}_{\text{"energy" conservation}}$$

$$② = \frac{\theta(-k^+)}{-k^+} \overbrace{\frac{2i}{p_2^- - p_4^- - \frac{k_\perp^2+m^2}{-k^+} + i\varepsilon}} \overbrace{4\pi\,\delta(p_1^- + p_2^- - p_3^- - p_4^-)}$$

$\Rightarrow$ intermediate states give energy denominators

$\Rightarrow$ all particles are on mass shell: $k^- = \frac{k_\perp^2 + m^2}{k^+}$

$\Rightarrow$ Energy (k^-) is conserved only in the initial / final states, but not in the intermediate states.

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1.3.1 QCD LCPT rules

1. Draw all diagrams for a given process at the desired order in the coupling constant, including all possible orderings of the interaction vertices in the light cone time x^+. Assign a four-momentum k^μ to each line such that it is on mass shell, so that $k^2 = m^2$ with m the mass of the particle. Each vertex conserves only the k^+ and $\vec{k}_\perp$ components of the four-momentum. Hence for each line the four-momentum has components as follows:

$$k^\mu = \left(k^+, \frac{\vec{k}_\perp^2 + m^2}{k^+}, \vec{k}_\perp^2\right). \tag{1.49}$$

2. With quarks associate on-mass-shell spinors in the Lepage and Brodsky (1980) convention:

$$u_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ + m\gamma^0 + \gamma^0 \vec{\gamma}_\perp \cdot \vec{p}_\perp\right) \chi(\sigma), \tag{1.50}$$

$$v_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ - m\gamma^0 + \gamma^0 \vec{\gamma}_\perp \cdot \vec{p}_\perp\right) \chi(-\sigma), \tag{1.51}$$

with

$$\chi(+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{1.52}$$

Gluon lines come with a polarization vector $\epsilon_\lambda^\mu(k)$. In the $A^+ = 0$ gauge this vector is given by

$$\epsilon_\lambda^\mu(k) = \left(0, \frac{2\,\vec{\epsilon}_\perp^{\,\lambda} \cdot \vec{k}_\perp}{k^+}, \vec{\epsilon}_\perp^{\,\lambda}\right) \tag{1.53}$$

with transverse polarization vector

$$\vec{\epsilon}_\perp^{\,\lambda} = -\frac{1}{\sqrt{2}} (\lambda, i), \tag{1.54}$$

where $\lambda = \pm 1$. Equation (1.53) follows from requiring that $\epsilon_\lambda^+ = 0$ and $\epsilon_\lambda(k) \cdot k = 0$.

3. For each intermediate state there is a factor equal to the light cone energy denominator

$$\frac{1}{\sum_{inc} k^- - \sum_{interm} k^- + i\epsilon} \tag{1.55}$$

where the sums run respectively over all incoming particles present in the initial state in the diagram ("inc") and over all the particles in the intermediate state at hand ("interm"). According to rule 1 above, for each particle we have $k^- = (\vec{k}_\perp^2 + m^2)/k^+$. Since the k^- momentum component is not conserved at the vertices the intermediate states are not on the "energy shell" and the light cone denominator in (1.55) is nonzero. Note that the light

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cone energy is conserved for the whole scattering process: $\sum_{inc} k^-$ is equal to $\sum_{out} k^-$, where "out" stands for a all outgoing particles.[3]

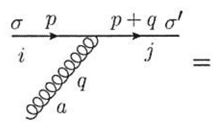
4. Include a factor

$$\frac{\theta(k^+)}{k^+} \tag{1.56}$$

for each internal line, where k^+ flows in the future light cone time direction.

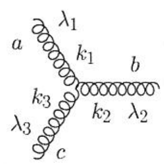
5. For vertices include factors as follows (we assume that the light cone time flows from left to right).

Quark–gluon vertex (i and j are quark color indices):



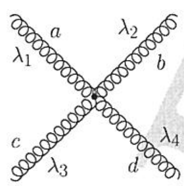
$$= -g\bar{u}_{\sigma' j}(p+q)\,\not\epsilon_\lambda(q)\,(t^a)_{ji}\,u_{\sigma i}(p). \tag{1.57}$$

Three-gluon vertex (all momenta flow into the vertex; asterisks denote complex conjugation):



$$\begin{aligned} = \; -igf^{abc}\,[&(k_1-k_3)\cdot\epsilon^*_{\lambda_2}(k_2)\,\epsilon_{\lambda_1}(k_1)\cdot\epsilon_{\lambda_3}(k_3) \\ &+(k_2-k_1)\cdot\epsilon_{\lambda_3}(k_3)\,\epsilon_{\lambda_1}(k_1)\cdot\epsilon^*_{\lambda_2}(k_2) \\ &+(k_3-k_2)\cdot\epsilon_{\lambda_1}(k_1)\,\epsilon_{\lambda_3}(k_3)\cdot\epsilon^*_{\lambda_2}(k_2)]. \end{aligned} \tag{1.58}$$

Four-gluon vertex:



$$\begin{aligned} = \; g^2\big[&f^{abe}f^{cde}\,(\epsilon_{\lambda_1}\cdot\epsilon_{\lambda_3}\,\epsilon^*_{\lambda_2}\cdot\epsilon^*_{\lambda_4}-\epsilon_{\lambda_1}\cdot\epsilon^*_{\lambda_4}\,\epsilon_{\lambda_3}\cdot\epsilon^*_{\lambda_2}) \\ &+f^{ace}f^{bde}(\epsilon_{\lambda_1}\cdot\epsilon^*_{\lambda_2}\,\epsilon_{\lambda_3}\cdot\epsilon^*_{\lambda_4}-\epsilon_{\lambda_1}\cdot\epsilon^*_{\lambda_4}\,\epsilon_{\lambda_3}\cdot\epsilon^*_{\lambda_2}) \\ &+f^{ade}f^{bce}(\epsilon_{\lambda_1}\cdot\epsilon^*_{\lambda_2}\,\epsilon_{\lambda_3}\cdot\epsilon^*_{\lambda_4}-\epsilon_{\lambda_1}\cdot\epsilon_{\lambda_3}\,\epsilon^*_{\lambda_2}\cdot\epsilon^*_{\lambda_4})\big]. \end{aligned} \tag{1.59}$$

In addition to the above vertices, which are (up to some trivial factors due to a different convention) identical to the same vertices in the Feynman rules, there are instantaneous terms in the light cone Hamiltonian giving the four vertices below. Again, light cone time flows to the right while the momentum flow direction is indicated by arrows. Instantaneous quark and gluon lines are denoted by regular quark and gluon lines with a short

[3] This light cone energy conservation condition does not apply to light cone wave functions, to be discussed shortly, as they represent only part of the scattering process.

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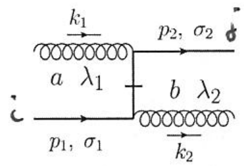
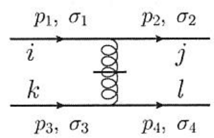
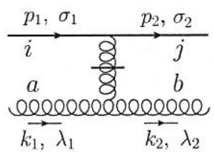
line crossing them.



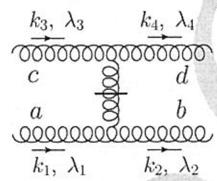
$$= g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \not\epsilon_{\lambda_1}(k_1)\, \frac{\gamma^+}{2(p_1^+ - k_2^+)}\, \not\epsilon^*_{\lambda_2}(k_2) \times (t^a\, t^b)_{ji}\, u_{\sigma_1\, i}(p_1), \tag{1.60}$$



$$= g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \gamma^+\, (t^a)_{ji}\, u_{\sigma_1\, i}(p_1) \times \bar{u}_{\sigma_4\, l}(p_4)\, \gamma^+\, (t^a)_{lk}\, u_{\sigma_3\, k}(p_3)\, \frac{1}{(p_1^+ - p_2^+)^2}, \tag{1.61}$$



$$= -\, g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \gamma^+\, (t^c)_{ji}\, u_{\sigma_1\, i}(p_1) \times \frac{k_1^+ + k_2^+}{(k_1^+ - k_2^+)^2}\, i f^{abc}\, \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}, \tag{1.62}$$



$$= g^2 f^{abe} f^{cde}\, \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}\, \epsilon^*_{\lambda_4} \cdot \epsilon_{\lambda_3} \times \frac{(k_1^+ + k_2^+)\,(k_3^+ + k_4^+)}{(k_1^+ - k_2^+)^2}. \tag{1.63}$$

6. For each independent momentum k^μ integrate with the measure

$$\int \frac{dk^+\, d^2k_\perp}{2(2\pi)^3}. \tag{1.64}$$

Sum over all internal quark and gluon polarizations and colors.

Again, standard parts of the rules, common to both LCPT and Feynman diagram calculations, such as symmetry factors and a factor -1 for fermion loops and for fermion lines beginning and ending at the initial state, are assumed implicitly.

The rules of LCPT are supplemented by tables of Dirac matrix elements in appendix section A.1. These tables are very useful in the evaluation of LCPT vertices.

1.3.2 Light cone wave function

An important quantity in LCPT, which is hard to construct in the standard Feynman diagram language, is the light cone wave function. Its definition is similar to that of the wave function

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in quantum mechanics. In our presentation of the light cone wave function we will follow Brodsky, Pauli, and Pinsky (1998). Imagine that we have a hadron state $|\Psi\rangle$. In general this is a superposition of different Fock states

$$\big|n_G, n_q\big\rangle \equiv \big|n_G, \{k_i^+, \vec{k}_{i\perp}, \lambda_i, a_i\}; n_q, \{p_j^+, \vec{p}_{j\perp}, \sigma_j, \alpha_j, f_j\}\big\rangle, \tag{1.65}$$

where a particular Fock state has n_G gluons and n_q quarks (and antiquarks). The gluon momenta are labeled k_i^+, $\vec{k}_{i\perp}$, with polarizations λ_i and gluon color indices a_i where $i = 1, \ldots, n_G$. (As usual in LCPT $k_i^- = \vec{k}_{i\perp}^2/k_i^+$, as all particles are on mass shell.) The quark momenta are labeled p_j^+, $\vec{p}_{j\perp}$, with helicities σ_j, colors α_j, and flavors f_j where $j = 1, \ldots, n_q$.

The Fock states form a complete basis such that

$$\sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, |n_G, n_q\rangle\langle n_G, n_q| = \mathbf{1}, \tag{1.66}$$

where the phase-space integral is defined by

$$\int d\Omega_{n_G+n_q} = \frac{2P^+ (2\pi)^3}{S_n} \int \prod_{i=1}^{n_G} \sum_{\lambda_i, a_i} \frac{dk_i^+ \, d^2k_{i\perp}}{2k_i^+ \, (2\pi)^3} \prod_{j=1}^{n_q} \sum_{\sigma_j, \alpha_j, f_j} \frac{dp_j^+ \, d^2p_{j\perp}}{2p_j^+ \, (2\pi)^3}$$
$$\times \, \delta\left(P^+ - \sum_{l_1=1}^{n_G} k_{l_1}^+ - \sum_{l_2=1}^{n_q} p_{l_2}^+\right) \delta^2\left(\vec{P}_\perp - \sum_{m_1=1}^{n_G} \vec{k}_{m_1\perp} - \sum_{m_2=1}^{n_q} \vec{p}_{m_2\perp}\right) \tag{1.67}$$

with symmetry factor $S_n = n_G! \, n_Q! \, n_{\bar{Q}}!$. Here n_Q and $n_{\bar{Q}}$ are respectively the numbers of quarks and antiquarks in the wave-function, so that $n_q = n_Q + n_{\bar{Q}}$. The delta functions in Eq. (1.67) represent the conservation of the "plus" and transverse components of the momenta, according to rule 1 of LCPT. The incoming hadron has longitudinal momentum P^+ and transverse momentum $\vec{P}_\perp$. We assume that each Fock state is normalized to 1, so that $\langle n_G, n_q | n_G, n_q\rangle = 1$.

Using Eq. (1.66) we can write

$$|\Psi\rangle = \sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, \big|n_G, n_q\big\rangle\big\langle n_G, n_q\big|\Psi\big\rangle. \tag{1.68}$$

The quantity

$$\Psi(n_G, n_q) = \big\langle n_G, n_q\big|\Psi\big\rangle \tag{1.69}$$

is called *the light cone wave function*. It is a multi-particle wave function, describing a Fock state in the hadron with n_G gluons and n_q quarks.

Note that requiring that the state $|\Psi\rangle$ is normalized to unity, $\langle\Psi|\Psi\rangle = 1$, and using Eq. (1.68) we can write

$$1 = \big\langle\Psi\big|\Psi\big\rangle = \sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, \big|\Psi(n_G, n_q)\big|^2. \tag{1.70}$$

Table A.1. *Dirac matrix elements constructed from u-spinors only. Reprinted table with permission from Lepage and Brodsky (1980). Copyright 1980 by the American Physical Society*

Matrix element	Value
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^+\frac{u_\sigma(p)}{\sqrt{p^+}}$	$2\delta_{\sigma\sigma'}$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\frac{2}{p^+p'^+}(\vec{p}_\perp\cdot\vec{p}'_\perp - i\sigma\vec{p}_\perp\times\vec{p}'_\perp + m^2)$ $-\delta_{\sigma,-\sigma}\sigma\frac{2m}{p^+p'^+}\left[(p'^1+i\sigma p'^2)-(p^1+i\sigma p^2)\right]$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\left(\frac{p'^i_\perp - i\sigma\epsilon^{ij}p'^j_\perp}{p'^+}+\frac{p^i_\perp + i\sigma\epsilon^{ij}p^j_\perp}{p^+}\right)$ $-\delta_{\sigma,-\sigma}\sigma m\left(\frac{p'^+-p^+}{p'^+p^+}\right)(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}m\frac{p^++p'^+}{p^+p'^+} - \delta_{\sigma,-\sigma}\sigma\left(\frac{p'^1+i\sigma p'^2}{p'^+}-\frac{p^1+i\sigma p^2}{p^+}\right)$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$4\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\,4\,\frac{p'^i_\perp - i\epsilon^{ij}\sigma p'^j_\perp}{p'^+} + \delta_{\sigma,-\sigma'}\,\sigma\frac{4m}{p'^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\,4\,\frac{p^i_\perp + i\epsilon^{ij}\sigma p^j_\perp}{p^+} - \delta_{\sigma,-\sigma'}\,\sigma\frac{4m}{p^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma_\perp^j\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}2(\delta^{ij}+i\sigma\epsilon^{ij})$

where Λ is the IR cutoff on the integration. Taking the transverse gradient of Eq. (A.9) yields

$$\int d^2q_\perp e^{i\vec{q}_\perp\cdot\vec{x}_\perp}\frac{\vec{q}_\perp}{q_\perp^2} = 2\pi i\frac{\vec{x}_\perp}{x_\perp^2}. \tag{A.10}$$

Here is a variation of Eq. (A.9), for a massive Green function:

$$\int \frac{d^2q_\perp}{q_\perp^2+m^2}e^{i\vec{q}_\perp\cdot\vec{x}_\perp} = 2\pi K_0(m x_\perp). \tag{A.11}$$

Table A.2. *Dirac matrix elements constructed from u- and v- spinors. Reprinted table with permission from Lepage and Brodsky (1980). Copyright 1980 by the American Physical Society*

Matrix element	Value
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^+\frac{u_\sigma(p)}{\sqrt{p^+}}$	$2\delta_{\sigma,-\sigma'}$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\frac{2}{p^+p'^+}(\vec{p}_\perp\cdot\vec{p}'_\perp - i\sigma\vec{p}_\perp\times\vec{p}'_\perp - m^2)$ $-\delta_{\sigma\sigma'}\sigma\frac{2m}{p^+p'^+}\left[(p'^1+i\sigma p'^2)+(p^1+i\sigma p^2)\right]$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^i_\perp\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\left(\frac{p'^i_\perp - i\sigma\epsilon^{ij}p'^j_\perp}{p'^+}+\frac{p^i_\perp+i\sigma\epsilon^{ij}p^j_\perp}{p^+}\right)$ $-\delta_{\sigma\sigma'}\sigma m\left(\frac{p'^+ + p^+}{p'^+p^+}\right)(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma}m\frac{p'^+ - p'^+}{p^+p'^+} - \delta_{\sigma\sigma'}\sigma\left(\frac{p'^1+i\sigma p'^2}{p'^+}-\frac{p^1+i\sigma p^2}{p^+}\right)$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$4\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma^i_\perp\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,4\frac{p'^i_\perp - i\epsilon^{ij}\sigma p'^j_\perp}{p'^+} + \delta_{\sigma\sigma'}\sigma\frac{4m}{p'^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^i_\perp\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,4\frac{p^i_\perp + i\epsilon^{ij}\sigma p^j_\perp}{p^+} - \delta_{\sigma\sigma'}\sigma\frac{4m}{p^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^i_\perp\gamma^+\gamma^j_\perp\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,2(\delta^{ij}+i\sigma\epsilon^{ij})$

Equations (A.10) and (A.9) can be used to show that

$$\int d^2y_\perp \frac{\vec{y}_\perp\cdot(\vec{y}_\perp+\vec{x}_\perp)}{y_\perp^2(\vec{y}_\perp+\vec{x}_\perp)^2} = 2\pi\ln\frac{1}{x_\perp\Lambda}. \tag{A.12}$$

Several angular integrals are useful too:

$$\int_0^{2\pi}\frac{d\varphi_q}{(\vec{q}_\perp-\vec{l}_\perp)^2} = \frac{2\pi}{|l_\perp^2-q_\perp^2|}, \tag{A.13}$$