Last time | **Light Cone Wave Function (cont'd)**

We discussed how a general QFT state can be decomposed in the basis of Fock states:

$$|\psi\rangle = c_0 |0\rangle + \int \frac{d^3k}{(2\pi)^3 2E_k} \, c_{\vec{k}} \, |\vec{k}\rangle + \frac{1}{2!} \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \, c_{\vec{k}_1, \vec{k}_2} \, |\vec{k}_1, \vec{k}_2\rangle + \ldots$$

single-particle wave function

two-particle wave function

$\Rightarrow$ n-particle wave function is

$$\Psi_{\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n} = \langle \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n | \psi \rangle .$$

[illegible]

whenever we have a general (n) state, we can be decomposed in the basis of Fock states

$$|\psi\rangle = c_0|0\rangle + \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(k)\,|k\rangle + \frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,\tilde{\psi}(k_1,k_2)\,|k_1 k_2\rangle + \dots$$

single particle wave function

two particle wave function

$\Rightarrow$ n-particle wave function is

$$\Psi_{k_1,k_2,\dots,k_n} = \langle k_1, k_2, \dots, k_n|\psi\rangle$$

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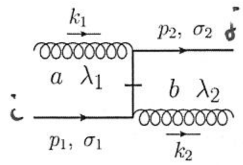
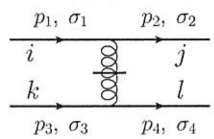
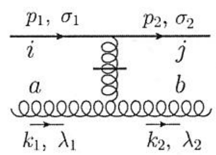
line crossing them.



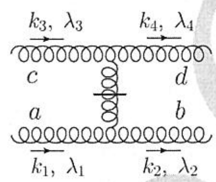
$$= g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \not{\epsilon}_{\lambda_1}(k_1)\, \frac{\gamma^+}{2(p_1^+ - k_2^+)}\, \not{\epsilon}^*_{\lambda_2}(k_2) \times (t^a\, t^b)_{ji}\, u_{\sigma_1\, i}(p_1), \tag{1.60}$$



$$= g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \gamma^+\, (t^a)_{ji}\, u_{\sigma_1\, i}(p_1) \times \bar{u}_{\sigma_4\, l}(p_4)\, \gamma^+\, (t^a)_{lk}\, u_{\sigma_3\, k}(p_3)\, \frac{1}{(p_1^+ - p_2^+)^2}, \tag{1.61}$$



$$= -\, g^2\, \bar{u}_{\sigma_2\, j}(p_2)\, \gamma^+\, (t^c)_{ji}\, u_{\sigma_1\, i}(p_1) \times \frac{k_1^+ + k_2^+}{(k_1^+ - k_2^+)^2}\, i f^{abc} \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}, \tag{1.62}$$



$$= g^2 f^{abe} f^{cde} \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}\, \epsilon^*_{\lambda_4} \cdot \epsilon_{\lambda_3} \times \frac{(k_1^+ + k_2^+)\,(k_3^+ + k_4^+)}{(k_1^+ - k_2^+)^2}. \tag{1.63}$$

6. For each independent momentum k^μ integrate with the measure

$$\int \frac{dk^+\, d^2k_\perp}{2(2\pi)^3}. \tag{1.64}$$

Sum over all internal quark and gluon polarizations and colors.

Again, standard parts of the rules, common to both LCPT and Feynman diagram calculations, such as symmetry factors and a factor -1 for fermion loops and for fermion lines beginning and ending at the initial state, are assumed implicitly.

The rules of LCPT are supplemented by tables of Dirac matrix elements in appendix section A.1. These tables are very useful in the evaluation of LCPT vertices.

1.3.2 Light cone wave function

An important quantity in LCPT, which is hard to construct in the standard Feynman diagram language, is the light cone wave function. Its definition is similar to that of the wave function

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in quantum mechanics. In our presentation of the light cone wave function we will follow Brodsky, Pauli, and Pinsky (1998). Imagine that we have a hadron state $|\Psi\rangle$. In general this is a superposition of different Fock states

$$\left|n_G, n_q\right\rangle \equiv \left|n_G, \{k_i^+, \vec{k}_{i\perp}, \lambda_i, a_i\}; n_q, \{p_j^+, \vec{p}_{j\perp}, \sigma_j, \alpha_j, f_j\}\right\rangle, \tag{1.65}$$

where a particular Fock state has n_G gluons and n_q quarks (and antiquarks). The gluon momenta are labeled k_i^+, $\vec{k}_{i\perp}$, with polarizations λ_i and gluon color indices a_i where $i = 1, \ldots, n_G$. (As usual in LCPT $k_i^- = \vec{k}_{i\perp}^2/k_i^+$, as all particles are on mass shell.) The quark momenta are labeled p_j^+, $\vec{p}_{j\perp}$, with helicities σ_j, colors α_j, and flavors f_j where $j = 1, \ldots, n_q$.

The Fock states form a complete basis such that

$$\sum_{n_G,n_q} \int d\Omega_{n_G+n_q} \, |n_G, n_q\rangle\langle n_G, n_q| = \mathbf{1}, \tag{1.66}$$

where the phase-space integral is defined by

$$\int d\Omega_{n_G+n_q} = \frac{2P^+(2\pi)^3}{S_n} \int \prod_{i=1}^{n_G} \sum_{\lambda_i,a_i} \frac{dk_i^+\, d^2k_{i\perp}}{2k_i^+\,(2\pi)^3} \prod_{j=1}^{n_q} \sum_{\sigma_j,\alpha_j,f_j} \frac{dp_j^+\, d^2p_{j\perp}}{2p_j^+\,(2\pi)^3}$$
$$\times\, \delta\left(P^+ - \sum_{l_1=1}^{n_G} k_{l_1}^+ - \sum_{l_2=1}^{n_q} p_{l_2}^+\right) \delta^2\left(\vec{P}_\perp - \sum_{m_1=1}^{n_G} \vec{k}_{m_1\perp} - \sum_{m_2=1}^{n_q} \vec{p}_{m_2\perp}\right) \tag{1.67}$$

with symmetry factor $S_n = n_G!\, n_Q!\, n_{\bar{Q}}!$. Here n_Q and $n_{\bar{Q}}$ are respectively the numbers of quarks and antiquarks in the wave-function, so that $n_q = n_Q + n_{\bar{Q}}$. The delta functions in Eq. (1.67) represent the conservation of the "plus" and transverse components of the momenta, according to rule 1 of LCPT. The incoming hadron has longitudinal momentum P^+ and transverse momentum $\vec{P}_\perp$. We assume that each Fock state is normalized to 1, so that $\langle n_G, n_q|n_G, n_q\rangle = 1$.

Using Eq. (1.66) we can write

$$|\Psi\rangle = \sum_{n_G,n_q} \int d\Omega_{n_G+n_q} \, \left|n_G, n_q\right\rangle\left\langle n_G, n_q\right|\Psi\rangle. \tag{1.68}$$

The quantity

$$\Psi(n_G, n_q) = \left\langle n_G, n_q\middle|\Psi\right\rangle \tag{1.69}$$

is called *the light cone wave function*. It is a multi-particle wave function, describing a Fock state in the hadron with n_G gluons and n_q quarks.

Note that requiring that the state $|\Psi\rangle$ is normalized to unity, $\langle\Psi|\Psi\rangle = 1$, and using Eq. (1.68) we can write

$$1 = \left\langle\Psi\middle|\Psi\right\rangle = \sum_{n_G,n_q} \int d\Omega_{n_G+n_q} \, \left|\Psi(n_G, n_q)\right|^2. \tag{1.70}$$

$$\Rightarrow \int d\Omega_{n_G+n_q} \, |\Psi(n_G, n_q)|^2 \leq 1 .$$

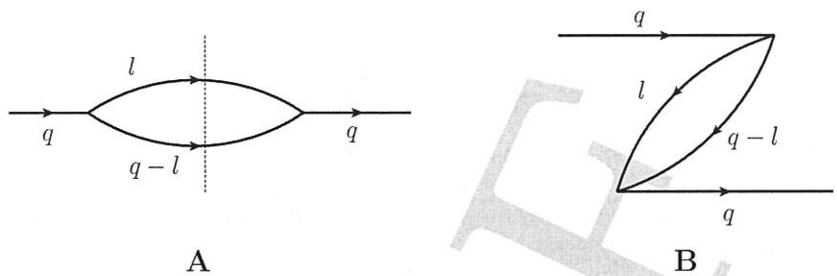

Fig. 1.2. Light cone perturbation theory diagrams in the ϕ^3-theory corresponding to the Feynman diagram in Fig. 1.1. Time flows to the right. The arrows indicate the momentum direction. The vertical dotted line indicates an intermediate state.

for the intermediate state (denoted by the dotted line in Fig. 1.2A), according to LCPT rule 3, and integrating over the internal momentum l with the integration measure

$$\int \frac{dl^+ \, d^2 l_\perp}{2(2\pi)^3}, \tag{1.79}$$

as prescribed by LCPT rule 6. In LCPT each vertex gives a factor λ (a modification of rule 5 for ϕ^3-theory) and one has to include the symmetry factor $1/2!$ as well. (Scalar particles obviously have no polarization. Neither do they have instantaneous terms.)

We have demonstrated that starting from the Feynman diagram amplitude expression (1.72) we can reduce it to the result that one would obtain by the rules of LCPT. Hence the two approaches in the end give identical expressions for the amplitudes, as expected.

A few words of caution are in order here. In principle the Feynman diagram in Fig. 1.1 corresponds to the two LCPT diagrams A and B shown in Fig. 1.2, which correspond to two different orderings of the vertices (see LCPT rule 1). The two graphs A and B in fact correspond to cases (i) and (ii) considered after Eq. (1.75). Our argument above was simplified by the fact that diagram B in Fig. 1.2 is zero as, according to the LCPT rules, it comes with a factor $\theta(-l^+)\,\theta(l^+ - q^+)$, which is zero for $q^+ > 0$. The physical meaning of this is quite clear: one cannot generate three particles with positive plus momenta out of nothing (see the lower vertex in Fig. 1.2B). Conversely, three particles with positive plus momenta cannot combine to give nothing (see the upper vertex in Fig. 1.2B). Because of this simplification, we have a one-to-one correspondence between the Feynman diagram in Fig. 1.1 and the LCPT diagram in Fig. 1.2A. In general, each Feynman diagram corresponds to a sum of all the LCPT diagrams with the same topology, including all possible time-orderings and instantaneous terms. A general derivation of an LCPT diagram starting from a Feynman diagram does not simply involve integration over the minus components of the internal momenta; one has to assign each vertex an x^+-coordinate and Fourier transform the diagram (by integrating over the minus momenta) into x^+ coordinate space. One then

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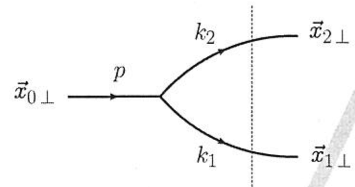

Fig. 1.3. Light cone wave function for a scalar particle splitting into two. The vertical dotted line denotes an intermediate state.

has to integrate over all the x^+-coordinates of the vertices, imposing different orderings: each ordering will lead to a different LCPT diagram.

1.4.2 A sample light cone wave function

Let us calculate, using the rules of LCPT, a sample light cone wave function. The calculation will be instructive, as the wave function we will calculate is similar to certain light cone wave functions that we will use throughout the book. In this calculation we will also illustrate in more detail what is actually meant by the light cone wave function definition (1.69) and will set up the rules for wave function calculations.

The sample wave function is depicted in Fig. 1.3. Again we are working in ϕ^3 real scalar field theory, with the Lagrangian (1.71). The wave function describes a single incoming particle splitting into two. For the scalar field theory only rules 1, 3, 4, and 6 from Sec. 1.3 apply. On top of these rules there is a factor equal to the coupling λ coming from the vertex. In calculating light cone wave functions one has to treat the "outgoing" state on the right of the diagram (the state denoted by the dotted line in Fig. 1.3) as an intermediate state. The reason is that, in describing a scattering process, the light cone wave function is thought of as a part of a larger diagram in which this "outgoing" state in fact undergoes subsequent interactions with other particles and therefore is truly an intermediate state. Our definition of the boost-invariant integration measure (1.67) dictates a slight modification of LCPT rule 4 as well, when calculating light cone wave functions: we treat the incoming lines (the external lines on the left, e.g. line p in Fig. 1.3) as "internal" and include a factor $1/p^+$ for them, while the outgoing lines (the lines on the right, e.g. lines k_1 and k_2 in Fig. 1.3) will be treated as "external" and so will not bring in such factors.

To summarize, when calculating the light cone wave function using LCPT one should follow the rules stated in Sec. 1.3, with the following modifications.

(i) The outgoing state on the right of a diagram is treated as an internal state and brings in an energy denominator according to LCPT rule 3.
(ii) At the same time the outgoing external lines on the right of the diagram bring in only factors $\theta(k^+)$, in modification of LCPT rule 4. (As usual, light cone time flows to the right.)

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(iii) The incoming external lines on the left of a diagram bring in factors $1/p^+$, i.e., LCPT rule 4 is extended to apply to those lines. (We will drop $\theta(p^+ > 0)$ as incoming lines always have positive p^+ momentum.)

According to the above-stated rules, the light cone wave function depicted in Fig. 1.3 is

$$\Psi(k_1, k_2) = \frac{1}{p^+} \frac{\lambda}{p^- - k_1^- - k_2^-} \, \theta(k_1^+)\,\theta(k_2^+)$$
$$= \frac{1}{p^+} \frac{\lambda}{\frac{\vec{p}_\perp^{\,2} + m^2}{p^+} - \frac{\vec{k}_{1\perp}^2 + m^2}{k_1^+} - \frac{\vec{k}_{2\perp}^2 + m^2}{k_2^+}}, \tag{1.80}$$

where we have omitted the regulator $i\epsilon$ for simplicity (in fact we will not need it below). Before we simplify this expression, let us note that, as can be seen from Eq. (1.70), the probability of finding such a configuration in a general "dressed" state $|\Psi\rangle$ of the incoming particle is

$$\int d\Omega_2 \left|\Psi(k_1, k_2)\right|^2, \tag{1.81}$$

where, as follows from Eq. (1.67), the phase-space integral for two identical particles is given by

$$\int d\Omega_2 = \frac{2p^+ (2\pi)^3}{2!} \int \frac{dk_1^+ \, d^2k_{1\perp}}{2k_1^+ (2\pi)^3} \frac{dk_2^+ \, d^2k_{2\perp}}{2k_2^+ (2\pi)^3} \, \delta(p^+ - k_1^+ - k_2^+)$$
$$\times \, \delta^2\left(\vec{p}_\perp - \vec{k}_{1\perp} - \vec{k}_{2\perp}\right)$$
$$= \frac{1}{2!} \int \frac{dk_1^+ \, d^2k_{1\perp}}{2k_1^+ (2\pi)^3} \frac{p^+}{p^+ - k_1^+}. \tag{1.82}$$

We see that $k_2^+ = p^+ - k_1^+$ and $\vec{k}_{2\perp} = \vec{p}_\perp - \vec{k}_{1\perp}$. Using these to replace k_2^+ and $\vec{k}_{2\perp}$ in Eq. (1.80) and doing some algebra yields

$$\Psi(k_1, p - k_1) = -\frac{\lambda z_1 (1 - z_1) \, \theta(z_1)\,\theta(1 - z_1)}{(\vec{k}_{1\perp} - z_1 \vec{p}_\perp)^2 + m^2 \left[1 - z_1(1 - z_1)\right]}, \tag{1.83}$$

where

$$z_1 = \frac{k_1^+}{p^+} \tag{1.84}$$

is the longitudinal fraction of the original particle's momentum p carried by the particle k_1, which will be identified as a Feynman-x variable in the next chapter. Equation (1.83) gives us the momentum-space two-particle light cone wave function at the lowest order in λ.

Substituting the wave function (1.83) into Eq. (1.81) and using Eq. (1.82) for the phase-space integration measure, one obtains the probability for one particle to fluctuate into two

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particles:

$$\frac{\lambda^2}{2!}\int \frac{dz_1\, d^2k_{1\perp}}{2(2\pi)^3} \frac{z_1(1-z_1)}{\left\{(\vec{k}_{1\perp} - z_1\vec{p}_\perp)^2 + m^2\left[1 - z_1(1-z_1)\right]\right\}^2} \sim \frac{\lambda^2}{m^2}. \tag{1.85}$$

Thus the probability of the configuration in Fig. 1.3 is proportional to the coupling constant squared. As the coupling in ϕ^3-theory has the dimension of the mass, the factor m^2 in the denominator of Eq. (1.85) makes the expression dimensionless. We note in passing that the effective dimensionless coupling constant for the perturbative expansion of ϕ^3-theory is λ/m.

It is also instructive to Fourier-transform the wave function (1.83) into transverse coordinate space. The transverse coordinates of the lines are shown in Fig. 1.3. The Fourier transform is accomplished by integrating over the independent transverse momenta, assigning a factor $e^{i\vec{k}_\perp\cdot\vec{x}_\perp}$ for each line, with k the net outgoing momentum carried by the line. For the two-particle wave function (1.83) we have

$$\begin{aligned}
&\Psi(\vec{x}_{1\perp}, \vec{x}_{2\perp}, \vec{x}_{0\perp}, z_1) \\
&\quad = \int \frac{d^2k_{1\perp}\, d^2p_\perp}{(2\pi)^4}\, e^{i\vec{k}_{1\perp}\cdot\vec{x}_{1\perp} + i\vec{k}_{2\perp}\cdot\vec{x}_{2\perp} - i\vec{p}_\perp\cdot\vec{x}_{0\perp}}\, \Psi(k_1, p - k_1) \\
&\quad = \int \frac{d^2k_{1\perp}\, d^2p_\perp}{(2\pi)^4}\, e^{i\vec{k}_{1\perp}\cdot(\vec{x}_{1\perp} - \vec{x}_{2\perp}) - i\vec{p}_\perp\cdot(\vec{x}_{0\perp} - \vec{x}_{2\perp})}\, \Psi(k_1, p - k_1).
\end{aligned} \tag{1.86}$$

Substituting Eq. (1.83) into Eq. (1.86) and integrating yields (see Eq. (A.11) in appendix section A.2)

$$\begin{aligned}
\Psi(\vec{x}_{1\perp}, \vec{x}_{2\perp}, \vec{x}_{0\perp}, z_1) = &-\frac{\lambda}{2\pi}\, z_1(1-z_1)\, K_0\Big(|\vec{x}_{12}|\, m\sqrt{1 - z_1(1-z_1)}\Big)\, \theta(z_1)\,\theta(1-z_1) \\
&\times \delta^2(\vec{x}_{0\perp} - z_1\vec{x}_{1\perp} - (1-z_1)\vec{x}_{2\perp}),
\end{aligned} \tag{1.87}$$

where $\vec{x}_{ij} \equiv \vec{x}_{i\perp} - \vec{x}_{j\perp}$. Equation (1.87) gives us the $1 \to 2$ splitting wave function shown in Fig. 1.3 in coordinate space. Even though this wave function has been obtained for the scalar ϕ^3-theory case it has a feature valid for theories with higher spin: it contains a delta function insuring that $\vec{x}_{0\perp} = z_1\vec{x}_{1\perp} + (1-z_1)\vec{x}_{2\perp}$. This means that the transverse coordinate positions of the two produced particles are indeed related to each other (Kopeliovich, Tarasov, and Schafer 1999): both the original particle and the two new particles lie on one straight line in transverse coordinate space, and $x_{02} : x_{01} = z_1 : (1 - z_1)$ where $x_{ij} = |\vec{x}_{ij}|$. The transverse coordinate space structure of the wave function (1.87) is illustrated in Fig. 1.4. The same constraint on the transverse plane locations of the produced particles applies to the splittings of particles in quantum electrodynamics (QED) as in QCD.

1.5 Asymptotic freedom

A remarkable property of QCD, known as *asymptotic freedom*, is the fact that the running QCD coupling tends to be small at short distances (corresponding to large values of the

$$\int \frac{d^2q_\perp}{q_\perp^2 + m^2}\, e^{i\underline{q}\cdot\underline{x}} = 2\pi\, K_0(m x_\perp)$$