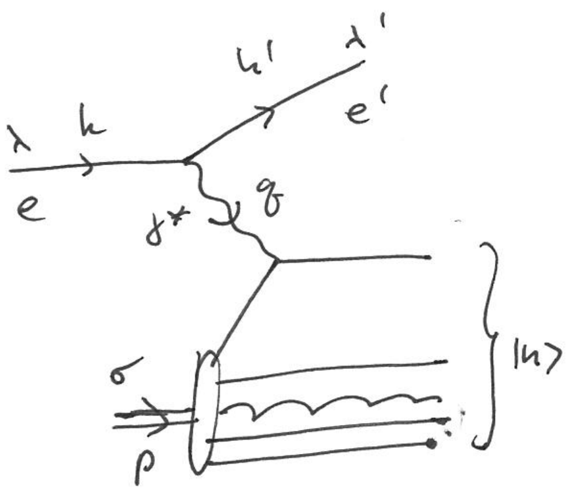
Last time | Parton Model and DIS

DIS



$$\boxed{\begin{aligned} Q^2 &\equiv -q^2 \\ x &\equiv \frac{Q^2}{2p\cdot q} \end{aligned}}$$ ~ Bjorken x

$$y = \frac{p\cdot q}{p\cdot k} = \frac{\mathcal{E}-\mathcal{E}'}{\mathcal{E}}$$

↑ rest frame of proton

$$x = \frac{Q^2}{\hat{s}+Q^2}, \quad \hat{s} = (p+q)^2 \qquad x = \frac{Q^2}{2m_p\nu}$$

rest frame of proton:

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{\alpha_{EM}^2}{Q^4 EE'}\, l_{\mu\nu}\, W^{\mu\nu}}$$

$$l_{\mu\nu} = \frac{1}{2}\sum_{\lambda,\lambda'} \left[\bar{u}_{\lambda'}(k')\gamma_\mu u_\lambda(k)\right]^* \left[\bar{u}_{\lambda'}(k')\gamma_\nu u_\lambda(k)\right]$$

$$W_{\mu\nu} = \frac{1}{4\pi m_p}\frac{1}{2}\sum_{\sigma,n} \langle n|j^\mu(0)|p,\sigma\rangle^* \langle n|j^\nu(0)|p,\sigma\rangle \cdot (2\pi)^4\delta^4(q+p-p_n)$$

$l_{\mu\nu}$ = leptonic tensor, $W^{\mu\nu}$ = hadronic tensor

Therefore

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{\alpha_{EM}^2}{Q^4 \quad \varepsilon\cdot\varepsilon'} \quad l_{\mu\nu} \, W^{\mu\nu}}$$

(rest frame of proton)

$$l_{\mu\nu} = \frac{1}{2} \sum_{\lambda,\lambda'} \left[\bar{u}_{\lambda'\alpha}(k') (\gamma_\mu)_{\alpha\beta} \, u_{\lambda\beta}(k) \right]^* \bar{u}_{\lambda'\alpha'}(k') \cdot$$

see next page

$$\cdot (\gamma_\nu)_{\alpha'\beta'} \, u_{\lambda\beta'}(k) = \frac{1}{2} \sum_{\lambda,\lambda'} \bar{u}_\lambda(k) \, \gamma_\mu \, u_{\lambda'}(k') \cdot$$

$$\cdot \bar{u}_{\lambda'}(k') \, \gamma_\nu \, u_\lambda(k) = \frac{1}{2} \mathrm{Tr}\left[\gamma_\mu \, \gamma\cdot k' \, \gamma_\nu \, \gamma\cdot k \right]$$

as $\sum_{\lambda'} u_{\lambda'}(k') \bar{u}_{\lambda'}(k') = \gamma\cdot k' + m_e \simeq \gamma\cdot k'$

Using $\mathrm{Tr}\left[\gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu\right] = 4\left[g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\beta\mu} - \right.$

$\left. - g_{\alpha\mu} g_{\beta\nu} \right]$

we get

$$\boxed{l_{\mu\nu} = 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} \, k\cdot k' \right)}$$

$$W_{\mu\nu} = \frac{1}{4\pi m} \frac{1}{2} \sum_{\sigma,n} \langle n | j^\mu(0) | p,\sigma\rangle^* \langle n | j^\nu(0) | p,\sigma\rangle$$

$$\cdot (2\pi)^4 \, \delta^4(q + p - p_n) = \frac{1}{4\pi m} \frac{1}{2} \sum_{\sigma,n} \int d^4x \, e^{iq\cdot x} \cdot$$

$$\langle p,\sigma | j^\mu(x) | n\rangle \langle n | j^\nu(0) | p,\sigma\rangle$$

$$\overset{\shortparallel}{e^{i\hat{p}\cdot x} \, j^\mu(0) \, e^{-i\hat{p}\cdot x}} \quad \text{(Heisenberg picture)}$$

$$\left[\bar{u}_{\lambda'}(k')\gamma^\mu u_\lambda(k)\right]^* = \left[u^\dagger_{\lambda'}\gamma^0\gamma^\mu u_\lambda\right]^\dagger =$$

as this is a scalar

$$= u^\dagger_\lambda \underbrace{(\gamma^0)^2}_{\mathbb{1}} \gamma^{\dagger\mu}\gamma^{\dagger 0} u_{\lambda'} = \bar{u}_\lambda(k)\gamma^0\gamma^{\dagger\mu}\gamma^{\dagger 0}u_{\lambda'}$$

now, $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Rightarrow \gamma^{\dagger 0} = \gamma^0 \Rightarrow \gamma^0\gamma^{\dagger\mu}\gamma^{\dagger 0} = \gamma^0\gamma^{\dagger\mu}\gamma^0$

Let's find $\gamma^0\gamma^{\dagger\mu}\gamma^0$:

$\mu = 0 \Rightarrow \gamma^0\gamma^{\dagger 0}\gamma^0 = \gamma^0$

$\mu = i \Rightarrow \gamma^0\gamma^{\dagger i}\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^i(\gamma^0)$

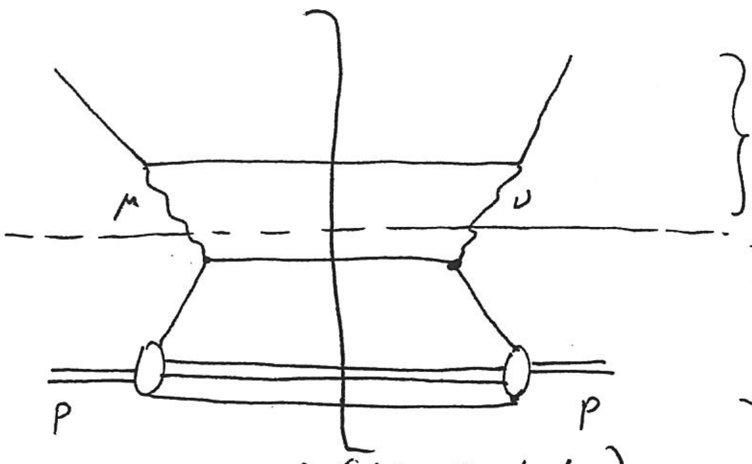
$= \gamma^i$

$$\Rightarrow \boxed{\gamma^0\gamma^{\dagger\mu}\gamma^0 = \gamma^\mu}$$

$\Rightarrow$ get $\bar{u}_\lambda(k)\gamma^\mu u_{\lambda'}(k')$ as desired.

$$W_{\mu\nu} = \frac{1}{4\pi m_p} \int d^4x \, e^{iq\cdot x} \langle p | j_\mu(x) \, j_\nu(0) | p \rangle$$

(averaged over σ)
↖ proton polarization



$l_{\mu\nu}$ (EM part)

$W_{\mu\nu}$ (hadronic part)

cut (final state)

$$W_{\mu\nu}(p,q) = A(x,Q^2)\, p_\mu p_\nu + B(x,Q^2)\, q_\mu q_\nu + C(x,Q^2)\, g_{\mu\nu} +$$

$$+ D(x,Q^2)\,(p_\mu q_\nu + p_\nu q_\mu) + E(x,Q^2)\,(p_\mu q_\nu - p_\nu q_\mu) +$$

$$+ F(x,Q^2)\, \varepsilon_{\mu\nu\rho\sigma}\, p^\rho q^\sigma$$

$F = 0$ in $\gamma^* p$, $\gamma^* A$ (F comes from γ_5's, appears in ν DIS or for polarized protons.)

(1) $q_\mu W^{\mu\nu} = 0$ (current conservation)

$q_\mu \to \partial_\mu \Rightarrow \partial_\mu j^\mu(x) = 0$

$$A\, p_\nu (p\cdot q) + B\, q_\nu \cdot q^2 + C\, q_\nu + D\,(p\cdot q\, q_\nu + q^2 p_\nu)$$

$$+ E\,(p\cdot q\, q_\nu - q^2 p_\nu) = 0$$

(2) $q_\nu W^{\mu\nu} = 0 = A\, p_\mu (p\cdot q) + B\, q^2 q_\mu + C q_\mu + D(p\cdot q\, q_\mu +$

$+ q^2 p_\mu) + E\,(p_\mu q^2 - p\cdot q\, q_\mu) = 0$

Plugging in the amplitude squared we get

$$\boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} = \frac{\pi\alpha_{EM}^2}{m^2}\left(\frac{\varepsilon_{\omega'}}{\varepsilon_\omega}\right)^2\left[\frac{\varepsilon_\omega}{\varepsilon_{\omega'}} + \frac{\varepsilon_{\omega'}}{\varepsilon_\omega} - \sin^2\theta\right]}$$

Klein-Nishina formula (1929).

Low energy limit $\varepsilon_\omega \to 0 \Rightarrow \varepsilon_{\omega'} \simeq \varepsilon_\omega \Rightarrow$

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} \simeq \frac{\pi\alpha_{EM}^2}{m^2}\left[1+\cos^2\theta\right]$$

Low-energy limit of Compton scattering

$$\sigma_{tot}^{Lab} \simeq \frac{8}{3}\frac{\pi\alpha_{EM}^2}{m^2}$$

Thomson scattering x-section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{EM}^2}{m^2}(\varepsilon_1\cdot\varepsilon_2)^2$$

polarizations before & after.

Also,

$$\frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{1}{(4\pi)^2}\frac{\pi}{s(s-m_e^2)}\langle|M|^2\rangle$$

$$\Rightarrow \frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{4\pi\alpha_{EM}^2}{s(s-m^2)}\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2}+\frac{s-m^2}{u-m^2}\right) + 2m^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) + 2m^4\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)^2\right\}.$$

The Optical Theorem and Cutkosky Rules.

Start from the S-matrix: $S^+ S = \mathbb{1}$. (Unitarity).

$$S = \mathbb{1} + i\mathbb{T} \Rightarrow (\mathbb{1} - iT^+)(1 + iT) = 1$$

$$\Rightarrow \quad i(T - T^+) + T^+T = 0$$

$$\Rightarrow \boxed{-i(T - T^+) = T^+T}$$ unitarity condition for T-matrix.

Sandwitch all this between states $|k_1, k_2\rangle$:

$$-i\langle k_1', k_2' | T - T^+ | k_1, k_2\rangle = \langle k_1', k_2' | T^+ T | k_1, k_2\rangle.$$

as $\langle p_1, \dots, p_n | \; T | k_1, k_2\rangle = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i p_i)\, M_{2\to n}$

$$\Rightarrow \quad \text{lhs} = \left[-iM(k_1, k_2 \to k_1', k_2') + iM^*(k_1', k_2' \to k_1, k_2)\right](2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2').$$

$$\text{rhs} = \sum_n \langle k_1', k_2' | T^+ \underbrace{|n\rangle\langle n|}_{\text{complete set of states}} T | k_1, k_2\rangle =$$

$$= \sum_n \int \prod_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3\, 2E_{q_i}} \langle k_1', k_2' | T^+ | q_1 \dots q_n\rangle\langle q_1 \dots q_n | T | k_1, k_2\rangle.$$

$$= \sum_n \int \prod_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3\, 2E_{q_i}} \; M(k_1, k_2 \to q_i)\, M^*(k_1', k_2' \to q_i) \cdot (2\pi)^4 \cdot$$

$$\cdot\, \delta^{(4)}(k_1 + k_2 - \sum_i q_i)\,(2\pi)^4\, \delta(k_1 + k_2 - k_1' - k_2').$$

Equating lhs = rhs & dropping δ-function yields:

$$-i\left[M(k_1,k_2\to k_1,k_2)-M^*(k_1,k_2\to k_1,k_2)\right]=\sum_n\int\prod_{i=1}^{n}\frac{d^3q_i}{(2\pi)^3\,2\mathcal{E}_{q_i}}\cdot$$

$$\cdot\left|M(k_1,k_2\to\{q_i\})\right|^2\,(2\pi)^4\,\delta^{(4)}\Big(k_1+k_2-\sum_i q_i\Big)$$

where we replaced $k_1', k_2' \to k_1, k_2$.

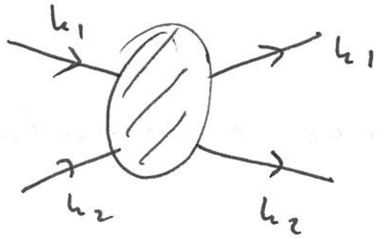
Right-hand side now looks just like $2\to n$ cross section summed over all n $\Rightarrow$ it is

$\sigma_{tot}\cdot 2\mathcal{E}_{k_1}\,2\mathcal{E}_{k_2}\,|\vec v_1-\vec v_2|$. We write

$$\boxed{\sigma_{tot}=\frac{1}{2\mathcal{E}_{k_1}\,2\mathcal{E}_{k_2}\,|\vec v_1-\vec v_2|}\cdot 2\,\mathrm{Im}\,M(k_1,k_2\to k_1,k_2)}$$

Optical Theorem (cf. E&M, QM)

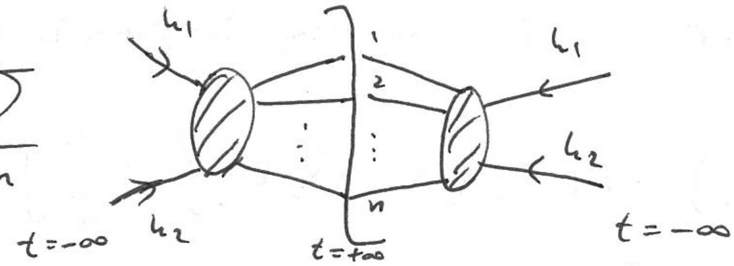
$M(k_1,k_2\to k_1,k_2)$ ~ forward scattering amplitude (final state = initial state)



Optical Th'm is very useful, true for ∀ # of external legs.

The cross section is the amplitude squared. Let us represent it diagrammatically as:

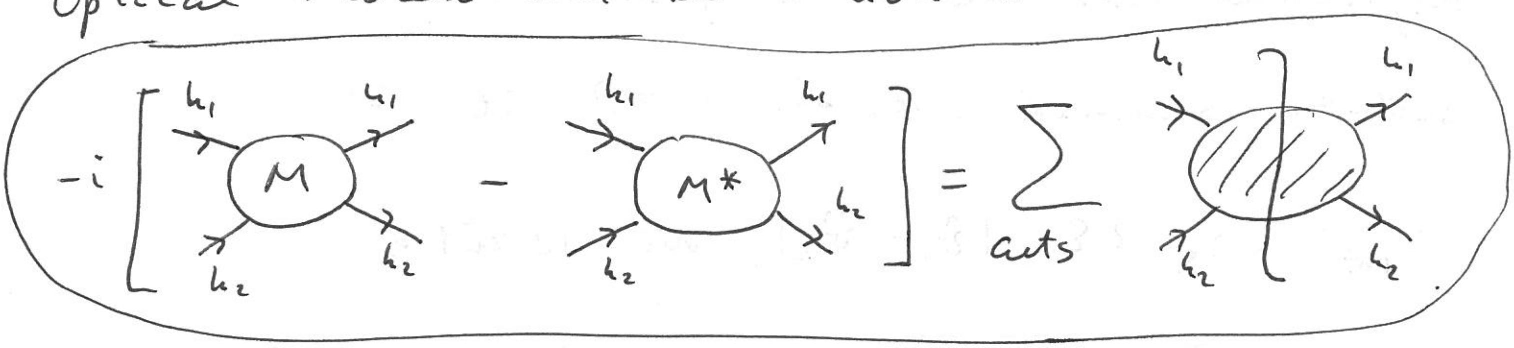
$\sigma_{total} = \sum_n$



amplitude $M_{2\to n}$ | complex-conjugate amplitude $M^*_{2\to n}$

"Cut" denotes the final state.

Optical theorem can be drawn as



This is the essence of Cutkosky rules: Im part of a diagram is given by the sum over all the cuts.

Cutkosky rules: to get $2i\,\mathrm{Im}\,M$:

(i) Cut in all possible ways.

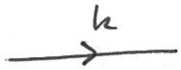
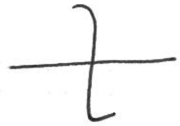
(ii) In each cut, replace cut propagators' denominators:

$$\frac{1}{p^2-m^2+i\epsilon} \to -2\pi i\,\delta(p^2-m^2)$$

(iii) Sum the contributions of all cuts.

& drop the overall "i" (from iM)

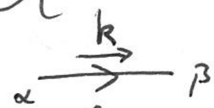
$$2i\,\mathrm{Im}\,\frac{1}{p^2-m^2+i\varepsilon} = -2\pi i\,\delta(p^2-m^2) \Rightarrow$$ a cut scalar propagator

gives  $\rightarrow$ 

$$iM = \frac{i}{k^2-m^2+i\varepsilon} \qquad i\cdot(-2\pi i)\,\delta(k^2-m^2) = 2\pi\,\delta(k^2-m^2)$$

puts particle on mass shell, as required

~~$iM = \frac{1}{k^2-m^2+i\varepsilon}$~~

Fermions:  $\rightarrow$ 

$$\frac{i(\gamma\cdot k+m)_{\beta\alpha}}{k^2-m^2+i\varepsilon} \qquad i(\not{k}+m)_{\beta\alpha}(-2\pi i)\,\delta(k^2-m^2) =$$

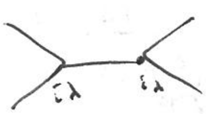
$$= \underbrace{(\not{k}+m)_{\beta\alpha}}_{\sum_r u_r(k)_\beta\,\bar{u}_r(k)_\alpha} 2\pi\,\delta(k^2-m^2)$$

Gauge bosons:  $\rightarrow$ 

$$\frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \qquad -ig_{\mu\nu}(-2\pi i)\,\delta(k^2) =$$

$$= -\underbrace{g_{\mu\nu}}(2\pi)\,\delta(k^2)$$

usual replacement $\rightarrow \sum_{\lambda=\pm} \varepsilon_\mu^{(\lambda)*}(k)\,\varepsilon_\nu^{(\lambda)}(k)$.



$$M = \frac{-\lambda^2}{s-m^2+i\varepsilon} \Rightarrow -i(M-M^*) = -i\cdot 2i\,\mathrm{Im}\,M =$$

$$= 2\,\mathrm{Im}\,M = 2(-\lambda^2)(-\pi)\,\delta(s-m^2)$$

$$\Downarrow$$

$$2\,\mathrm{Im}\,M = 2\lambda^2\pi\,\delta(s-m^2) \qquad = \lambda^2(2\pi)\,\delta(s-m^2).$$

Cutkosky: $iM = \frac{-i\lambda^2}{s-m^2+i\varepsilon} \rightarrow -\lambda^2(-2\pi i)\delta(s-m^2) = 2\pi i\lambda^2\delta(s-m^2) = 2i\,\mathrm{Im}\,M$

Regularization & Renormalization

Field - Strength Renormalization: the Electron Self - Energy.

When deriving LSZ formula we claimed that "dressing" particle propagator leads to (for scalars):

$$\int d^4x\, e^{ip\cdot x} \langle \Psi_0 | T\, \varphi_H(x)\, \varphi_H(0) | \Psi_0 \rangle = Z \frac{i}{p^2 - m_{phys}^2 + i\varepsilon} +$$

+ multiparticle contributions.

It appears that one can write

$$\varphi_H(x) \simeq \sqrt{Z}\, \varphi_{free}(x)$$

with some free field $\varphi_{free}(x)$ with mass m_{phys}.

Def. Z is called <u>field-strength renormalization.</u>

For Dirac fields $\psi(x)$ one has:

$$\int d^4x\, e^{ip\cdot x} \langle \Psi_0 | T\, \psi(x)\, \bar{\psi}(0) | \Psi_0 \rangle = Z_2 \frac{i(\not{p} + m_{phys})}{p^2 - m_{phys}^2 + i\varepsilon} + \text{multi-particle contrib}$$

Z_2 ~ new notation.

Let us see how Z_2 & m_{phys} arise.