BFKL Solution

of $\vec{x}_{1'0'}$: the resulting cross section does not depend on the directions of $\vec{x}_{10}$ either, since there is no preferred direction left in the transverse space. Defining

$$n(x_{10}, x_{1'0'}, Y) = \int d^2b \int_0^{2\pi} \frac{d\phi_{1'0'}}{2\pi} n(\vec{x}_{10}, \vec{x}_{1'0'}, \vec{b}_\perp, Y), \tag{4.89}$$

we see that this new quantity satisfies

$$\frac{\partial}{\partial Y} n(x_{10}, x_{1'0'}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \times [n(x_{12}, x_{1'0'}, Y) + n(x_{20}, x_{1'0'}, Y) - n(x_{10}, x_{1'0'}, Y)] \tag{4.90}$$

with initial condition (cf. Eq. (3.25))

$$n(x_{10}, x_{1'0'}, Y = 0) = \frac{4\pi\alpha_s^2 C_F}{N_c} x_<^2 \left(\ln \frac{x_>}{x_<} + 1 \right), \tag{4.91}$$

where $x_{>(<)} = \max(\min)\{|\vec{x}_{10}|, |\vec{x}_{1'0'}|\}$.

The solution of Eq. (4.90) can be found by noticing that in the angular-averaged case the eigenfunctions of the integral kernel are simple powers of the dipole size,

$$\left(x_{01}^2\right)^{1/2+i\nu} \tag{4.92}$$

with eigenvalues

$$\frac{\alpha_s N_c}{\pi} \chi(0, \nu), \tag{4.93}$$

where (cf. Eqs. (3.81), (3.74))

$$\chi(0, \nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right). \tag{4.94}$$

To prove this we need to evaluate the following integral:

$$\int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \left[\left(x_{12}^2\right)^{1/2+i\nu} + \left(x_{20}^2\right)^{1/2+i\nu} - \left(x_{10}^2\right)^{1/2+i\nu} \right]. \tag{4.95}$$

This can be done by noticing that the integral (4.95) is equivalent to that in Eq. (3.64) with $n = 0$. Alternatively, one can use the trick presented in appendix section A.3; in order to make each term in Eq. (4.95) finite we insert a UV regulator ρ. After that, with the help of Eqs. (A.18), (A.21), (A.24), and (A.29) one can rewrite Eq. (4.95) as

$$2\pi \left[2^{1+2i\nu} \frac{\Gamma\left(\frac{1}{2} + i\nu\right)}{\Gamma\left(\frac{1}{2} - i\nu\right)} x_{10}^2 \int_0^\infty dk k^{-2i\nu} \left(\ln \frac{2}{k\rho} + \psi(1) \right) J_0(k x_{10}) - x_{10}^{1+2i\nu} \ln \frac{x_{10}^2}{\rho^2} \right]. \tag{4.96}$$

Integrating over k in Eq. (4.96) using Eq. (A.18) yields

$$2\pi x_{10}^{1+2i\nu} \chi(0, \nu), \tag{4.97}$$

as desired.

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BFKL solution

We see that, as for to the BFKL equation (3.58), the eigenfunctions of Eq. (4.90) are powers (though of the transverse dipole size instead of the transverse momentum), with exactly the same eigenvalues, (4.93) as in that case.[6] We conclude that Eq. (4.90) is equivalent to the BFKL equation!

In fact, the substitution (Levin and Ryskin 1987)

$$n(x_{10}, x_{1'0'}, Y) = \int d^2k \left(1 - e^{i\vec{k}_\perp \cdot \vec{x}_{10}}\right) \frac{1}{k_\perp^2} f(\vec{k}_\perp, x_{1'0'}, Y) \tag{4.98}$$

turns Eq. (4.90) into the BFKL equation (3.58) for the function f (Kovchegov and Weigert 2007b). Verification of this statement is left as an exercise for the reader.

Using the eigenfunctions and the eigenvalues of the integral kernel in Eq. (4.90), we can write down the solution of Eq. (4.90) as

$$n(x_{10}, x_{1'0'}, Y) = \int\limits_{-\infty}^{\infty} d\nu\, C_\nu(x_{1'0'})\, x_{10}^{1+2i\nu}\, e^{\bar{\alpha}_s \chi(0,\nu) Y}, \tag{4.99}$$

where the coefficient $C_\nu(x_{1'0'})$ is fixed by the initial conditions (4.91) as follows:

$$C_\nu(x_{1'0'}) = \frac{16\alpha_s^2 C_F}{N_c} \frac{1}{(1+4\nu^2)^2} x_{1'0'}^{1-2i\nu}. \tag{4.100}$$

The general solution of Eq. (4.90) is then

$$n(x_{10}, x_{1'0'}, Y) = \frac{16\alpha_s^2 C_F}{N_c} x_{10} x_{1'0'} \int\limits_{-\infty}^{\infty} d\nu \left(\frac{x_{10}}{x_{1'0'}}\right)^{2i\nu} \frac{e^{\bar{\alpha}_s \chi(0,\nu) Y}}{(1+4\nu^2)^2}. \tag{4.101}$$

For $x_{10} \approx x_{1'0'}$ we can use the diffusion approximation from Sec. 3.3.4: expanding $\chi(0, \nu)$ around $\nu = 0$ using Eq. (3.84) and integrating over ν we obtain

$$\begin{aligned} n(x_{10}, x_{1'0'}, Y) = {} & \frac{16\alpha_s^2 C_F}{N_c} x_{10} x_{1'0'} \sqrt{\frac{\pi}{14\zeta(3)\bar{\alpha}_s Y}} \\ & \times \exp\left[(\alpha_P - 1)Y - \frac{\ln^2(x_{10}/x_{1'0'})}{14\zeta(3)\bar{\alpha}_s Y}\right]. \end{aligned} \tag{4.102}$$

Readers who performed Exercise 3.5 will recognize Eq. (4.102) as the answer for the onium–onium scattering cross section obtained there using the standard Feynman diagram approach. Now we see that a calculation based on LCPT wave functions gives the same result. Note that the single-dipole distribution n_1 is only one component of the onium wave function. This wave function also contains multi-dipole distributions n_2, n_3, etc. Hence, as we will shortly see, the dipole approach, while in a certain limit equivalent to BFKL, in fact contains more information.

[6] We have verified this statement so far only in the case where the angular dependence has been integrated out: we will consider the general angular-dependent case in the next section.

Start from the BK eq'n: Traveling Wave Solution

$$\frac{\partial}{\partial Y} N(x_{10}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \Big[N(x_{21}, Y) + N(x_{20}, Y) - N(x_{10}, Y) - N(x_{21}, Y) N(x_{20}, Y) \Big]$$

Fourier transform:

$$N(x_\perp, Y) = x_\perp^2 \int \frac{d^2k}{2\pi} e^{i\vec{k}_\perp \cdot \vec{x}_\perp} \tilde{N}(k_\perp, Y), \quad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$$

$$\Rightarrow \frac{\partial \tilde{N}(k_\perp, Y)}{\partial Y} = \bar{\alpha}_s \underbrace{\chi\left(0, \frac{i}{2}\left(1 + \frac{\partial}{\partial \ln k_\perp}\right)\right)}_{\downarrow} \tilde{N}(k_\perp, Y) - \bar{\alpha}_s \tilde{N}^2(k_\perp, Y)$$

$$\chi\left(-\frac{\partial}{\partial \ln k_\perp^2}\right), \quad \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\rho = \ln k_\perp^2 / Q_{s0}^2$$

$$\frac{\partial \tilde{N}(\rho, Y)}{\partial Y} = \bar{\alpha}_s \chi\left(-\frac{\partial}{\partial \rho}\right) \tilde{N}(\rho, Y) - \bar{\alpha}_s \tilde{N}^2(\rho, Y)$$

Expand the kernel in Taylor series:

$$\chi(\gamma) = \chi(\gamma_{cr}) + (\gamma - \gamma_{cr}) \chi'(\gamma_{cr}) + \frac{1}{2}(\gamma - \gamma_{cr})^2 \chi''(\gamma_{cr}) + \ldots$$

where γ_{cr} is defined by $\chi(\gamma_{cr}) = \gamma_{cr} \chi'(\gamma_{cr})$

Change variables to:

$$t = \frac{1}{2}\bar{\alpha}_s \chi''(\gamma_{cr})\gamma_{cr}^2 Y$$

$$x = \gamma_{cr}\rho + \bar{\alpha}_s\left[\chi''(\gamma_{cr})\gamma_{cr}^2 - \chi(\gamma_{cr})\right]Y$$

$$u(t,x) \equiv \frac{2}{\chi''(\gamma_{cr})\gamma_{cr}^2}\tilde{N}(\rho,Y)$$

Munier & Peschanski, 2003

$\Rightarrow$ get $\boxed{\partial_t u(t,x) = \partial_x^2 u + u(1-u)}$

Fisher 1937

F-KPP equation

Kolmogorov, Petrovsky, Piskunov 1937

Traveling wave solution: $t \to \infty \Rightarrow$

$$u(t,x)\Big|_{t\to\infty} \propto f\left(x - 2t + \frac{3}{2}\ln t + \mathcal{O}(t)\right)$$

↑ function of one variable.

$$x - 2t + \frac{3}{2}\ln t = \gamma_{cr}\ln\frac{x_\perp^2}{Q_s^2(Y)} + \text{const.}$$

with $Q_s^2(Y) = Q_{s0}^2 \exp\left\{\bar{\alpha}_s\frac{\chi(\gamma_{cr})}{\gamma_{cr}}Y - \frac{3}{2\gamma_{cr}}\ln\bar{\alpha}_s Y\right\}$

$\Rightarrow u = e^{-x+2t}$ solves linear F-KPP $\Rightarrow$

$\Rightarrow \boxed{\tilde{N}(\rho,Y) \propto \left(\frac{Q_s^2(Y)}{x_T^2}\right)^{\gamma_{cr}}}$ ~ geometric scaling !