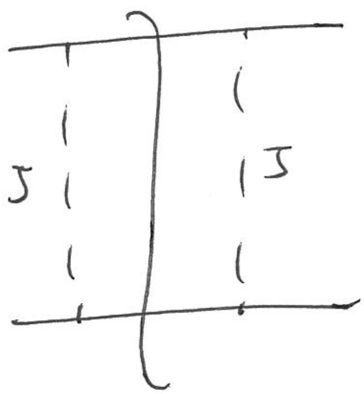
Last time | # High Parton Density QCD (cont'd)

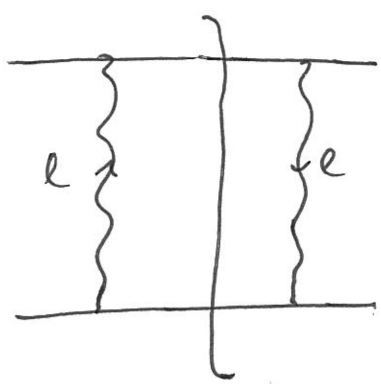
~ talked about high energy / small x limit & resummation of $\alpha_s \ln \frac{1}{x}$ instead of $\alpha_s \ln Q^2/Q_0^2$ done by DGLAP

~ started thinking about high energy scattering



$\Rightarrow \sigma \sim s^{2(J-1)}$ for exchange of particles with spin J

$\Rightarrow$ in QCD gluon exchanges dominate at high energy



$$\frac{d\sigma}{d^2 l_\perp} \underset{\text{high energy limit}}{=} \frac{2\alpha_s^2 C_F}{N_c} \frac{1}{(l_\perp^2)^2}$$

In general, $A \equiv \frac{M}{2s}$, $d\sigma = \langle |A|^2 \rangle \frac{d^2 l_\perp}{(2\pi)^2}$.

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The square of the amplitude in Eq. (3.17) leads to the following high energy cross section:

$$\sigma^0_{qq\to qq} = \frac{2\alpha_s^2 C_F}{N_c} \int \frac{d^2 l_\perp}{(l_\perp^2)^2}. \tag{3.18}$$

We see that, in agreement with the rule in Eq. (3.6), the cross section due to two t-channel gluon exchanges is independent of energy at high energy. This feature of QCD was first noticed by Low (1975) and Nussinov (1976). The two t-channel gluon exchange cross section is sometimes called the *Low–Nussinov pomeron*, since this result was the first successful attempt to describe hadronic cross sections in the framework of perturbative QCD: in pre-QCD language hadronic cross sections were described as being due to the t-channel exchange of a hypothetical particle with the quantum numbers of the vacuum called *the pomeron*, named after I. Y. Pomeranchuk (1958). The contribution of the pomeron to the scattering amplitude is

$$M \sim s^{\alpha(t)}, \tag{3.19}$$

where s and t are Mandelstam variables and $\alpha(t)$ is the "angular momentum" of the pomeron, usually referred to as the *pomeron trajectory*. The contribution of a single pomeron exchange to the total cross section is

$$\sigma_{tot} \sim s^{\alpha(0)-1}. \tag{3.20}$$

Here $\alpha(0)$ is the value of the pomeron trajectory at $t = 0$, which is the point where it intercepts the angular momentum axis in the (t, α)-plane. Therefore $\alpha(0)$ is referred to as the *pomeron intercept* and is sometimes denoted by α_P. As one can see from Eq. (3.20), the pomeron intercept always comes in the combination $\alpha(0) - 1$: according to a common notation, we will often refer to $\alpha(0) - 1 = \alpha_P - 1$ as itself the pomeron intercept. Frequently one uses a linear expansion of the pomeron trajectory near $t = 0$:

$$\alpha(t) \approx \alpha(0) + \alpha' t. \tag{3.21}$$

The parameter α' is called the *slope* of the pomeron trajectory. A tantalizing feature of strong interactions is that the linear approximation (3.21) actually describes the pomeron trajectory $\alpha(t)$ rather well at all values of t. This observation gave rise to the development of string theory, which started out as a candidate theory for strong interactions (see e.g. Green, Schwarz, and Witten (1987)).

From Eq. (3.18) it is clear that the Low–Nussinov pomeron has intercept $\alpha(0) - 1 = 0$. In high energy proton–proton (pp) (and proton–antiproton, $p\bar{p}$) collisions, analysis of the experimental data showed that the total cross section grows approximately as follows (Donnachie and Landshoff 1992):

$$\sigma^{pp}_{tot} \sim s^{0.08}. \tag{3.22}$$

That is, using pre-QCD language, the pomeron intercept $\alpha_P - 1 = 0.08$. Since soft non-perturbative QCD physics is probably responsible for much of the total pp cross section observed at many modern-day accelerators, the pomeron with intercept $\alpha_P - 1 = 0.08$ is usually called the "soft pomeron".

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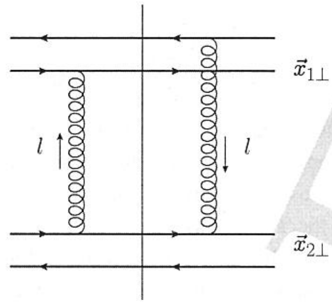


Fig. 3.4. A diagram contributing to the onium–onium high energy scattering cross section at leading order. The arrows next to the gluon lines indicate the direction of momentum flow and the vertical straight line denotes the final state cut.

We see that the prediction of Low and Nussinov that $\alpha_P - 1 = 0$, while it does not give the correct pomeron intercept, is not far from it, in the sense of giving a cross section that at least does not decrease with energy. (Of course there is no *a priori* reason to expect a perturbative calculation to describe the total pp scattering cross section, but it is good to have at least qualitative agreement between the two.) As we will see below, higher-order perturbative corrections to the cross section (3.18) generate a positive order-α_s contribution to the $\alpha_P - 1 = 0$ result. Note that the fact that experimental measurement of the total pp scattering cross section (3.22) gives a result that does not fall off with energy but instead rises slowly with s, when combined with the above rule for counting powers of s (see (3.6)), demonstrates that there must exist a spin-1 particle responsible for strong interactions – the gluon. This is exactly the argument for the existence of gluons mentioned in Sec. 1.1.

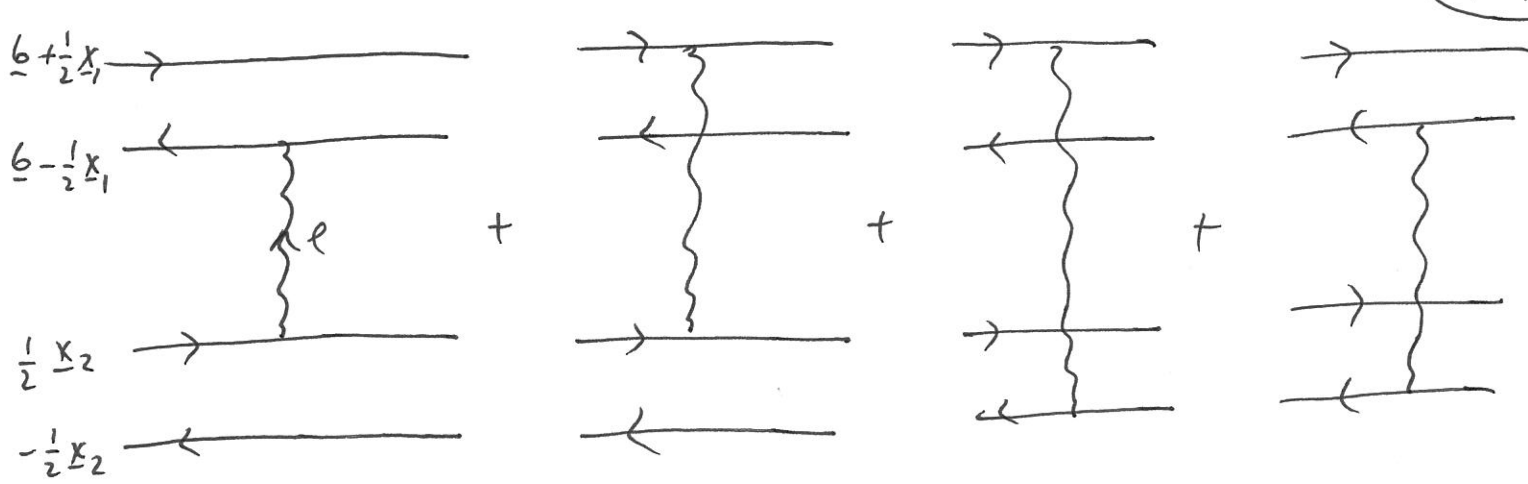
The $l_\perp$-integral in Eq. (3.18) has an infrared (IR) divergence. This is natural since we are calculating a cross section for the scattering of free color charges (quarks). To make the cross section IR-finite we need to remember that the scattering quarks are part of the onium wave functions. Suppose that the $q\bar{q}$ pairs have separations $\vec{x}_{1\perp}$ and $\vec{x}_{2\perp}$ in transverse coordinate space, though the impact parameter between the two onia has been integrated out. By summing diagrams with all possible gluon connections to quarks and antiquarks, one of which is shown in Fig. 3.4, one can then show that the total onium–onium scattering cross section is

$$\sigma_{tot}^{onium+onium} = \int d^2x_{1\perp} d^2x_{2\perp} \int\limits_0^1 dz_1 dz_2 \, |\Psi(\vec{x}_{1\perp}, z_1)|^2 \, |\Psi(\vec{x}_{2\perp}, z_2)|^2 \hat{\sigma}_{tot}^{onium+onium} \tag{3.23}$$

with

$$\hat{\sigma}_{tot}^{onium+onium} = \frac{2\alpha_s^2 C_F}{N_c} \int \frac{d^2 l_\perp}{(l_\perp^2)^2} \left(2 - e^{-i\vec{l}_\perp \cdot \vec{x}_{1\perp}} - e^{i\vec{l}_\perp \cdot \vec{x}_{1\perp}}\right)\left(2 - e^{-i\vec{l}_\perp \cdot \vec{x}_{2\perp}} - e^{i\vec{l}_\perp \cdot \vec{x}_{2\perp}}\right), \tag{3.24}$$

at the lowest order in α_s. Here $\Psi(\vec{x}_\perp, z)$ is the onium light cone wave function with quark light cone momentum fraction z. The exact form of the wave function is not important



$$A = (-g^2) \int \frac{d^2 l_\perp}{(2\pi)^2} \, \frac{1}{l_\perp^2} \Big[- e^{i\mathbf{l}\cdot(\mathbf{b} - \frac{1}{2}\mathbf{x}_1 - \frac{1}{2}\mathbf{x}_2)} + e^{i\mathbf{l}\cdot(\mathbf{b} + \frac{1}{2}\mathbf{x}_1 - \frac{1}{2}\mathbf{x}_2)}$$

$$- e^{i\mathbf{l}\cdot(\mathbf{b} + \frac{1}{2}\mathbf{x}_1 - (-\frac{1}{2}\mathbf{x}_2))} + e^{i\mathbf{l}\cdot(\mathbf{b} - \frac{1}{2}\mathbf{x}_1 - (-\frac{1}{2}\mathbf{x}_2))} \Big] =$$

$$= g^2 \int \frac{d^2 l_\perp}{(2\pi)^2} \, \frac{1}{l_\perp^2} \, e^{i\mathbf{l}\cdot\mathbf{b}} \Big[e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} \Big] \Big[e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} \Big]$$

$$\Rightarrow \int |A|^2 d^2 b = g^4 \int d^2 b \int \frac{d^2 l}{(2\pi)^2} \, \frac{1}{l_\perp^2} \, e^{i\mathbf{l}\cdot\mathbf{b}} \Big[e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} \Big]$$

$$\cdot \Big[e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} \Big] \int \frac{d^2 l'}{(2\pi)^2} \, \frac{1}{l_\perp'^2} \, e^{-i\mathbf{l}'\cdot\mathbf{b}} \Big[e^{-i\mathbf{l}'\cdot\frac{1}{2}\mathbf{x}_1} - e^{i\mathbf{l}'\cdot\frac{1}{2}\mathbf{x}_1} \Big]$$

$$\cdot \Big[e^{-i\mathbf{l}'\cdot\frac{1}{2}\mathbf{x}_2} - e^{i\mathbf{l}'\cdot\frac{1}{2}\mathbf{x}_2} \Big] = g^4 \int \frac{d^2 l_\perp}{(2\pi)^2} \, \frac{1}{(l_\perp^2)^2} \Big| e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_1} \Big|^2$$

$$\cdot \Big| e^{i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} - e^{-i\mathbf{l}\cdot\frac{1}{2}\mathbf{x}_2} \Big|^2 = g^4 \int \frac{d^2 l_\perp}{(2\pi)^2} \, \frac{1}{(l_\perp^2)^2} \underbrace{\Big[2 - e^{i\mathbf{l}\cdot\mathbf{x}_1} - e^{-i\mathbf{l}\cdot\mathbf{x}_1} \Big]}_{\text{impact factor}} \cdot$$

$$\cdot \underbrace{\Big[2 - e^{i\mathbf{l}\cdot\mathbf{x}_2} - e^{-i\mathbf{l}\cdot\mathbf{x}_2} \Big]}_{\text{impact factor}} \Rightarrow \text{leads to Eq. (3.24)}.$$

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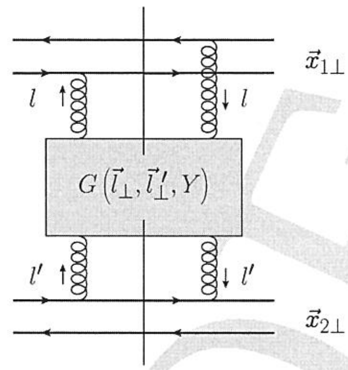

Fig. 3.5. A general representation of the onium–onium scattering cross section at high energy. The rectangle denotes all leading-$\ln s$ corrections to the two-gluon exchange cross section from Fig. 3.5.

for the moment. The summation and averaging over all appropriate quantum numbers is implicit in the $|\Psi|^2$ factors in Eq. (3.23).

The $l_\perp$-integral in Eq. (3.24) is now finite; if we average over the directions of $\vec{x}_{1\perp}$ and $\vec{x}_{2\perp}$ then it can be easily carried out, giving

$$\langle \hat{\sigma}_{tot}^{onium+onium} \rangle = \frac{4\pi \alpha_s^2 C_F}{N_c} x_<^2 \left(\ln \frac{x_>}{x_<} + 1 \right), \tag{3.25}$$

where $x_{>(<)} = \max\,(\min)\{|\vec{x}_{1\perp}|, |\vec{x}_{2\perp}|\}$ and $\langle \cdots \rangle$ denotes angular averaging.

We will now look for corrections to this lowest-order result.

3.3 The Balitsky–Fadin–Kuraev–Lipatov evolution equation

As discussed in Sec. 3.1, in high energy scattering (or at small Bjorken x) one would like to sum the longitudinal logarithms, i.e., the powers of $\alpha_s \ln s$ (or $\alpha_s \ln 1/x$). We will denote the sum of all such corrections to the Born-level onium–onium scattering cross section found above in Sec. 3.2 by the shaded rectangle in Fig. 3.5.

Generalizing the cross section in Eq. (3.24) we write

$$\hat{\sigma}_{tot}^{onium+onium} = \frac{2\alpha_s^2 C_F}{N_c} \int \frac{d^2 l_\perp d^2 l'_\perp}{l_\perp^2 l_\perp'^2} \left(2 - e^{-i\vec{l}_\perp \cdot \vec{x}_{1\perp}} - e^{i\vec{l}_\perp \cdot \vec{x}_{1\perp}}\right)$$
$$\times \left(2 - e^{-i\vec{l}'_\perp \cdot \vec{x}_{2\perp}} - e^{i\vec{l}'_\perp \cdot \vec{x}_{2\perp}}\right) G\left(\vec{l}_\perp, \vec{l}'_\perp, Y\right), \tag{3.26}$$

where l and l' are the momenta of the gluon lines on each side of the shaded rectangle, as illustrated in Fig. 3.5. We also define the *rapidity* variable $Y = \ln(s|\vec{x}_{1\perp}||\vec{x}_{2,\perp}|)$; it is important that $Y \sim \ln s$, though the exact cutoff under the logarithm of the energy is not important in the leading-logarithmic approximation that we would like to apply here. The shaded rectangle in Fig. 3.5 brings in a factor $G(\vec{l}_\perp, \vec{l}'_\perp, Y)$. The lowest-order expression

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For example, the tree-level diagrams in Fig. B.1 yield

$$\mathrm{Im}_s\, A\big(s', t; \mathrm{Fig.\, B1a}\big) = \pi \lambda^2 \delta\left(m^2 - s'\right), \tag{B.16a}$$

$$\mathrm{Im}_u\, A\big(u', t; \mathrm{Fig.\, B1b}\big) = \pi \lambda^2 \delta\left(m^2 - u'\right). \tag{B.16b}$$

Substituting each of these imaginary parts into the right-hand side of Eq. (B.15) yields the appropriate amplitude after straightforward integration over the delta functions.

Note that a dispersion relation in the form Eq. (B.15) cannot be used in QCD since we know that QCD amplitudes grow in proportion to the energy s at large s (see e.g. Eq. (3.17)), making the integrals in Eq. (B.15) divergent. Therefore, we have to alter Eq. (B.15) by subtracting, for example, the amplitude $A(s = 0, t)$ obtained by putting $s = 0$ in Eq. (B.15). Doing this, we obtain the *subtracted dispersion relation*

$$A(s, t) = A(s = 0, t) + \frac{1}{\pi}\left\{ s \int_{s_{min}}^{+\infty} ds' \frac{\mathrm{Im}_s\, A\big(s', t\big)}{s'(s' - s)} \right.$$
$$\left. + [u - u(s = 0)] \int_{u_{min}}^{+\infty} du' \frac{\mathrm{Im}_u\, A\big(u', t\big)}{[u' - u(s = 0)](u' - u)} \right\}. \tag{B.17}$$

Finally, subtracting $s\partial_s A(s = 0, t)$ from Eq. (B.17) (with $A(s, t)$ again given by Eq. (B.15)) we obtain the *double-subtracted dispersion relation*

$$A(s, t) = A(s = 0, t) + s\partial_s A(s = 0, t) + \frac{1}{\pi}\left\{ s^2 \int_{s_{min}}^{+\infty} ds' \frac{\mathrm{Im}_s\, A\big(s', t\big)}{s'^2(s' - s)} \right.$$
$$\left. + [u - u(s = 0)]^2 \int_{u_{min}}^{+\infty} du' \frac{\mathrm{Im}_u\, A\big(u', t\big)}{[u' - u(s = 0)]^2(u' - u)} \right\}. \tag{B.18}$$

This is exactly the dispersion relation used in Eq. (3.43). Note that in perturbative QCD $A(s = 0, t) = 0$.

B.2 Unitarity and the Froissart–Martin bound

The unitarity constraint (B.3) can be written in terms of scattering amplitudes as (see e.g. Peskin and Schroeder (1995))

$$M(k_1, k_2 \to k_1, k_2) - M^*(k_1, k_2 \to k_1, k_2)$$
$$= i \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} |M(k_1, k_2 \to q_1, \ldots, q_n)|^2 (2\pi)^4 \delta^4 \left(k_1 + k_2 - \sum_{j=1}^{n} q_j \right), \tag{B.19}$$

where $M(k_1, k_2 \to q_1, \ldots, q_n)$ is the $2 \to n$ scattering amplitude for the scattering of two particles with momenta k_1, k_2 into n particles with momenta $q_1, \ldots, q_n$, and $M(k_1, k_2 \to k_1, k_2)$ is the forward scattering amplitude; E_{q_i} is the energy of a particle with momentum q_i.

Let us consider the case of high energy scattering, where k_1^+ and k_2^- are very large and so are $q_1^+ \approx k_1^+$ and $q_2^- \approx k_2^-$. Separating the *elastic* $2 \to 2$ contribution from the *inelastic* contributions ($2 \to 3, 2 \to 4$, etc.) on the right-hand side of Eq. (B.19), and integrating over the delta-function in that contribution, yields

$$2\, \mathrm{Im}\, A(k_1, k_2 \to k_1, k_2) = \int \frac{d^2 q_\perp}{(2\pi)^2} |A(k_1, k_2 \to q_1, q_2)|^2 + \text{inelastic terms}, \tag{B.20}$$

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where q is the momentum transfer four-vector, defined by

$$q = q_1 - k_1 = k_2 - q_2, \tag{B.21}$$

and we also define a new rescaled scattering amplitude

$$A(k_1, k_2 \to q_1, q_2) \equiv \frac{M(k_1, k_2 \to q_1, q_2)}{2\sqrt{2E_{k_1} 2E_{k_2} 2E_{q_1} 2E_{q_2}}} \approx \frac{M(k_1, k_2 \to q_1, q_2)}{2k_1^+ k_2^-}. \tag{B.22}$$

Since both the incoming and outgoing particles are on mass shell the momentum transfer q has only two free components, which we choose to be transverse and over which we integrated in Eq. (B.20).

The optical theorem then states that the total scattering cross section is given by (again, see e.g. Peskin and Schroeder (1995))

$$\sigma_{tot} = 2\,\mathrm{Im}\, A(k_1, k_2 \to k_1, k_2) \tag{B.23}$$

so that Eq. (B.20) simply implies that

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel}, \tag{B.24}$$

where σ_{el} is the elastic $2 \to 2$ cross section and σ_{inel} is the total inelastic cross section.

As we have seen above, in general the elastic amplitude $A(k_1, k_2 \to q_1, q_2)$ can be written as a function of the Mandelstam variables s and t. However, for our purposes it is convenient to go to impact parameter ($\vec{b}_\perp$) space, using

$$A(k_1, k_2 \to q_1, q_2) = \int d^2b\, e^{-i\vec{q}_\perp \cdot \vec{b}_\perp} A(s, \vec{b}_\perp), \tag{B.25}$$

which, when applied in Eq. (B.20) yields

$$2\,\mathrm{Im}\, A(s, \vec{b}_\perp) = |A(s, \vec{b}_\perp)|^2 + \text{inelastic terms}. \tag{B.26}$$

In arriving at Eq. (B.26) we have used the fact that the forward amplitude corresponds to the case of zero momentum transfer, $t = 0$, or, equivalently, $q_\perp = 0$, such that

$$A(k_1, k_2 \to k_1, k_2) = \int d^2b A(s, \vec{b}_\perp). \tag{B.27}$$

Note that the total cross section in impact parameter space is

$$\sigma_{tot} = 2 \int d^2b\ \mathrm{Im}\, A(s, \vec{b}_\perp). \tag{B.28}$$

We also see immediately from Eq. (B.26) that the elastic cross section is given by

$$\sigma_{el} = \int d^2b\, |A(s, \vec{b}_\perp)|^2. \tag{B.29}$$

Relating the inelastic terms in Eq. (B.26) to the corresponding cross section yields

$$2\mathrm{Im}A(s, \vec{b}_\perp) = |A(s, \vec{b}_\perp)|^2 + \frac{d\sigma_{inel}}{d^2b}. \tag{B.30}$$

The simple nonnegativity condition

$$\frac{d\sigma_{inel}}{d^2b} \geq 0 \tag{B.31}$$

used in Eq. (B.30) yields

$$\mathrm{Im}\, A(s, \vec{b}_\perp) \leq 2. \tag{B.32}$$

$$2\,\mathrm{Im}\,A = (\mathrm{Re}\,A)^2 + (\mathrm{Im}\,A)^2 + \frac{d\sigma_{inel}}{d^2b}$$

$$2\,\mathrm{Im}\,A - (\mathrm{Im}\,A)^2 - (\mathrm{Re}\,A)^2 \geq 0$$

$$1 - (1 - \mathrm{Im}\,A)^2 \geq (\mathrm{Re}\,A)^2 \geq 0 \Rightarrow$$

$$|1 - \mathrm{Im}\,A| \leq 1$$

$$\Downarrow$$

$$0 \leq \mathrm{Im}\,A \leq 2$$