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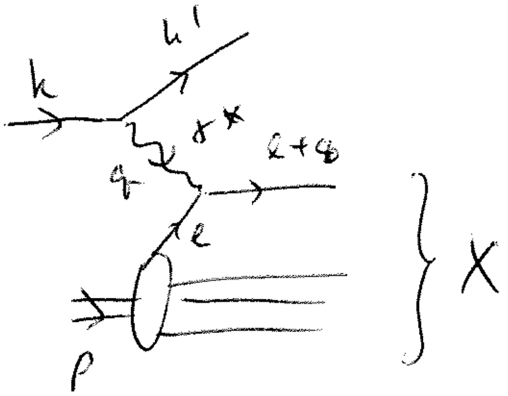
IPPP Lectures: "Small x and Saturation
& QCD Masterclass

Deep Inelastic Scattering



$$Q^2 \equiv -q^2 \geq 0.$$

$$W^2 \equiv (p+q)^2 \sim \text{proton} + \gamma^* \text{ CMS energy}$$

$$x \equiv \frac{Q^2}{2p\cdot q} = \frac{Q^2}{W^2 + Q^2 - M_p^2}$$

Bjorken x variable

use light-cone variables: $v^{\pm} = \frac{v^0 \pm v^3}{\sqrt{2}}$

$$\begin{array}{l} \quad\quad\; + \quad\quad\; - \quad\quad\; \perp \\ p^\mu = \left(p^+, \frac{M^2}{2p^+}, \vec{0}_\perp\right) \\ q^\mu = \left(-\frac{Q^2}{2q^-}, q^-, \vec{0}_\perp\right) \end{array} \Bigg] \leftarrow \text{frame choice.}$$

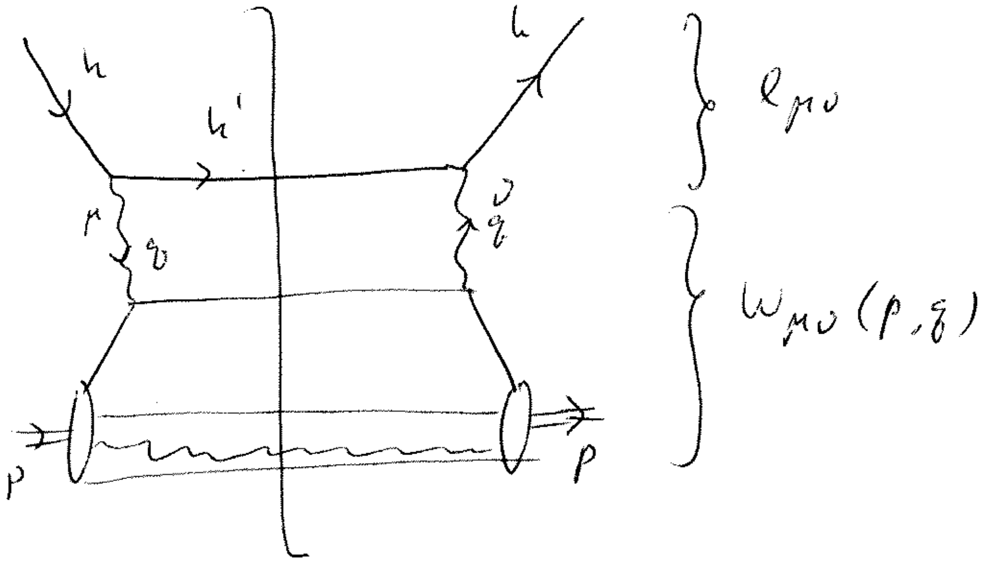
q^- is large, Q^2 is large

$$0 = (l+q)^2 = l^2 + \underbrace{2l\cdot q}_{2l^+q^- - \frac{Q^2}{q^-}l^-} + Q^2 \simeq 2l^+q^- - Q^2$$

$$\Rightarrow l^+ = \frac{Q^2}{2q^-} \Rightarrow \frac{l^+}{p^+} = \frac{Q^2}{2p^+q^-} \simeq \frac{Q^2}{2p\cdot q} = x$$

$$\Rightarrow x = \frac{l^+}{p^+}$$ ~ fraction of the LC proton's momentum carried by struck quark.

Square the handbag diagram:



$\Rightarrow$ can write the cross section as

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{\alpha_{EM}^2}{Q^2 E E'}\, \ell_{\mu\nu}\, W^{\mu\nu}}$$ (rest frame of the proton)

where the hadronic tensor is

$$\boxed{W_{\mu\nu} \equiv \frac{1}{4\pi M_p} \int d^4x\, e^{iq\cdot x} \langle P | j_\mu(x)\, j_\nu(0) | P \rangle}$$

$j_\mu(x)$ = EM current

$\sim$ all QCD interactions are in $W_{\mu\nu}$.

$$x = \frac{Q^2}{W^2 + Q^2 - M_p^2} \underset{\substack{\uparrow \\ W^2 \gg Q^2, M_p^2}}{\simeq} \frac{Q^2}{W^2} \Rightarrow$$

$\boxed{\text{High energy } W^2 \Updownarrow \text{ Low } x}$

Dipole Picture of DIS

$$W_{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{iq\cdot x} \langle P | j_\mu(x) \, j_\nu(0) | P \rangle$$

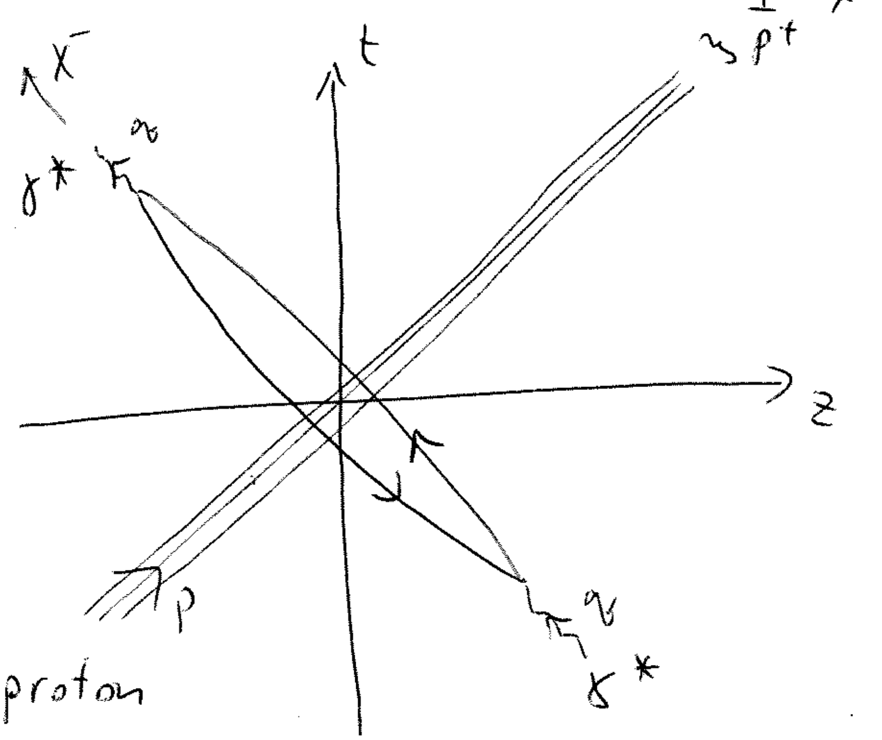
as $q^\mu = \left(-\frac{Q^2}{2q^-}, q^-, \vec{0}_\perp \right) \Rightarrow$

$$\Rightarrow W_{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{i\left(q^- x^+ - \frac{Q^2}{2q^-} x^-\right)} \langle P | j_\mu(x) j_\nu(0) | P \rangle$$

Let's visualize the process in space-time:

proton extent in x^- direction is:

$\sim r_p \frac{m}{p^+} \simeq \frac{1}{p^+}$.



Separation between the two EM currents:

$$x^- \approx \frac{2q^-}{Q^2} \Rightarrow \frac{2q^-}{Q^2} \gtrless \frac{1}{p^+} \Rightarrow 1 \gtrless \frac{Q^2}{2p^+q^-} = x$$

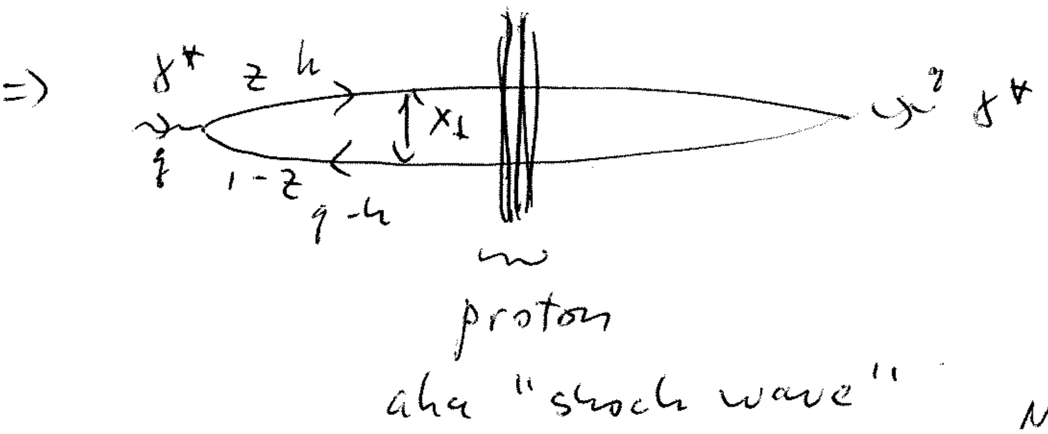
$\uparrow$ uncertainty principle

$$\Rightarrow \text{if } x \ll 1 \Rightarrow \frac{2q^-}{Q^2} \gg \frac{1}{p^+}$$

$\Rightarrow \gamma^*$ splits into a $q\bar{q}$ pair long before interacting with the proton, and, for forward amplitude, they $\underbrace{\text{they}}_{(q\bar{q})}$ merge back into γ^* long after the interaction.

$\Big(W_{\mu\nu} = 2\,\mathrm{Im}\,(i\, T_{\mu\nu}),\ T_{\mu\nu} =$ same as $W_{\mu\nu}$ but with T-product $\Big)$

$\Rightarrow$



Gribov 1970

Bjorken & Kogut 1973

Frankfurt, Strikman 1988

Mueller 1990

Nikolaev & Zakharov 1991

total DIS cross section is

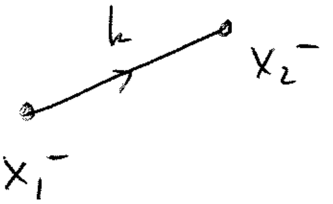
$$\sigma^{\gamma^* p} = \int \frac{d^2 x_\perp}{4\pi} \int_0^1 \frac{dz}{z(1-z)} \left| \Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z) \right|^2 \cdot \sigma^{q\bar{q}N}(x_\perp, s)$$

$z = \frac{k^-}{q^-}$ ~ light-cone momentum fraction $0 < z < 1$

$x_\perp$ = transverse size of the dipole.

$\psi^{\gamma^* \to q\bar{q}}(x_\perp, z)$ = virtual photon's LC wave function.

Take a scalar propagator: Fourier-transform into x^- space.



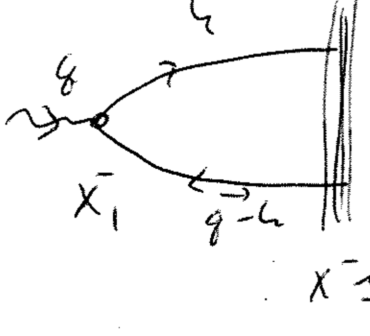
$$\int_{-\infty}^{\infty} \frac{dk^+}{2\pi}\, e^{-ik^+(x_2^- - x_1^-)} \frac{i}{k^2 - m^2 + i\varepsilon} =$$

$$= \theta(x_2^- - x_1^-)\, \frac{1}{2k^-}\, \theta(k^-)\, e^{-i\frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)}$$

$$- \theta(x_1^- - x_2^-) \cdot \frac{1}{2k^-}\, \theta(-k^-)\, e^{-i\frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)}$$

$$= e^{-i\frac{k_\perp^2 + m^2}{2k^-}(x_2^- - x_1^-)}\, \frac{1}{2k^-}\left[\theta(x_2^- - x_1^-)\, \theta(k^-) - \theta(x_1^- - x_2^-) \cdot \theta(-k^-)\right]$$

~ different x^- orderings come in with different contributions; particle is on mass shell.



know that $x_1^- < 0$ (restricted)!

$$\int_{-\infty}^{0} dx_1^-\, e^{i\left(\frac{k_\perp^2 + m^2}{2k^-} + \frac{(\vec{q}_\perp - \vec{k}_\perp)^2 + m^2}{2(q^- - k^-)}\right)x^-} \cdot e^{-i\left(\frac{-Q^2}{2q^-}\right)x^-} \cdot e^{\varepsilon x_1^-}$$

← external ← reg.

$$= \frac{-i}{\underbrace{\frac{k_\perp^2+m^2}{2k^-} + \frac{(\vec{q}_\perp-\vec{k}_\perp)^2+m^2}{2(q^--k^-)} + \frac{Q^2}{2q^-} - i\epsilon}_{\text{energy denominator}}}$$

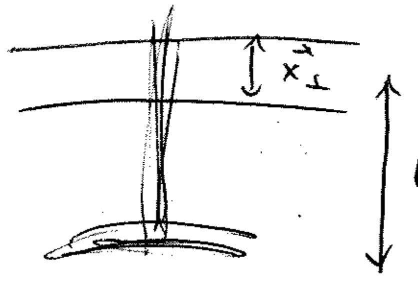
Particles are on mass shell, but the "+" momentum component is not conserved.

$\Rightarrow \psi^{\gamma^* \to q\bar{q}}(x_\perp, z)$ is well known (at LO & NLO).

$\sigma^{q\bar{q}N}$ = total cross section of a dipole scattering on the target.

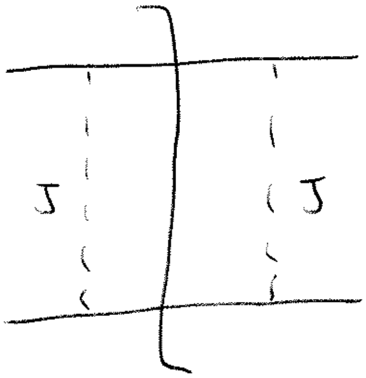
$N(\vec{x}_\perp, \vec{b}_\perp, s)$ = (Im part of) dipole forward scattering amplitude.

$$\boxed{\sigma^{q\bar{q}N} = 2\int d^2b_\perp\, N(\vec{x}_\perp, \vec{b}_\perp, s)}$$ Optical Theorem



$\vec{b}_\perp \sim$ impact parameter.

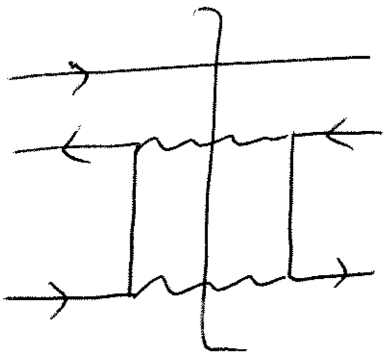
How does the dipole interact with the proton?



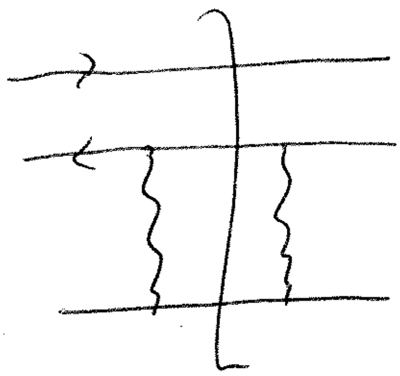
$$\Rightarrow \boxed{\sigma \sim s^{2(J-1)}} \sim \text{high-}s \text{ limit}$$

J = spin of exchanged particle

$\Rightarrow$



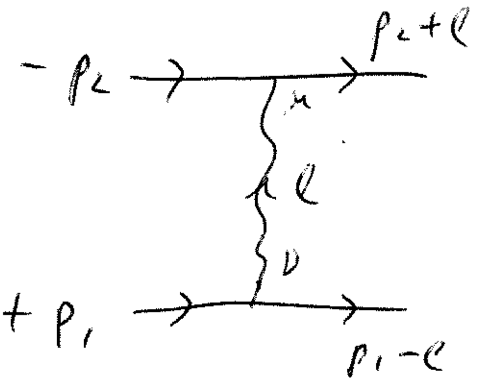
$$\sigma \sim s^{2\left(\frac{1}{2}-1\right)} \sim \frac{1}{s} \sim \text{decreases with } s$$



$$\sigma \sim s^{2(1-1)} = s^0 = 1$$

$\Rightarrow \sigma$ = const with energy

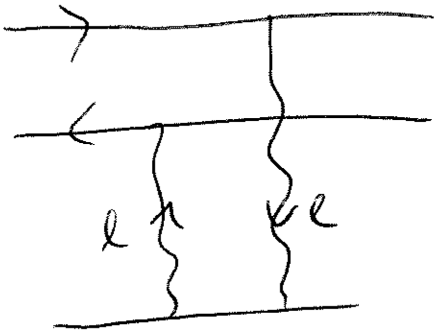
$\Rightarrow$ gluon exchange dominates



$$0 = (p_2+\ell)^2 \simeq 2p_2^-\ell^+ \Rightarrow \ell^+ \simeq 0$$

$$0 = (p_1-\ell)^2 \simeq -2p_1^+\ell^- \Rightarrow \ell^- \simeq 0$$

$$\Rightarrow \frac{-i\, g_{\mu\nu}}{\ell^2} \simeq \frac{i}{\underline{\ell}^2}\, g_{\mu\nu} \sim \text{Coulomb (Glauber) gluon}$$



$+ \ldots \quad \propto \int_{\Lambda} \frac{d^2 l_\perp}{(l_\perp^2)^2} \left(2 - e^{i \vec{l}_\perp \cdot \vec{x}_\perp} - e^{-i \vec{l}_\perp \cdot \vec{x}_\perp}\right)$

$$\propto x_\perp^2 \ln\left(\frac{1}{x_\perp \Lambda}\right)$$

$$\Rightarrow \sigma^{q\bar{q}N} = \frac{\pi \alpha_s^2 C_F}{N_c} x_\perp^2 \ln \frac{1}{x_\perp^2 \Lambda^2} = \frac{\alpha_s \pi^2}{N_c} x_\perp^2 \underbrace{x G_N}_{x \text{ times gluon PDF}}$$

more careful calculation with proton = quark model.

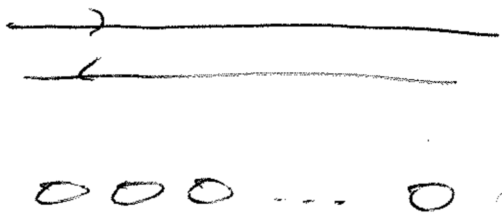
$$\frac{d\sigma^{q\bar{q}A}}{d^2 b} = T(\vec{b}_\perp)\, \sigma^{q\bar{q}N} = 2N(\vec{x}_\perp, \vec{b}_\perp, s)$$

← nucleus

$$T(\vec{b}) \equiv \int_{-\infty}^{\infty} dz\, \rho(\vec{b}_\perp, z) \sim \text{nuclear profile function}$$

ρ = # density of nucleons, approximately constant.

$T(\vec{b}) \propto A^{1/3}$



~ many ($\sim A^{1/3}$) nucleons at fixed $\vec{b}$.

$$\Rightarrow \quad N(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{1}{2} T(\vec{b}_\perp)\, \sigma^{q\bar{q}N}$$

$\Rightarrow$ for a 2-gluon exchange

$$N(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{\alpha_s \pi^2}{2N_c} T(\vec{b}_\perp)\, x_\perp^2\, x G_N(x, 1/x_\perp^2)$$

with $x G_N(x, Q^2) = \frac{\alpha_s C_F}{\pi} \ln \frac{Q^2}{\Lambda^2}$

for proton = quark.

Unitarity & Black Disk Limit

$$|\psi_f\rangle = \hat{S}\,|\psi_i\rangle = |\psi_i\rangle + \underbrace{(\hat{S} - \mathbb{1})}_{i\hat{T}}|\psi_i\rangle$$

total cross section:

$$\sigma_{tot} \propto \left|(\hat{S} - \mathbb{1})|\psi_i\rangle\right|^2 = 2 - S - S^*$$

$$S = \langle\psi_i|\,\hat{S}\,|\psi_i\rangle, \quad \underbrace{\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{1}}_{\text{unitarity}}$$

Elastic cross section:

$$\sigma_{el} \sim \left| \langle \psi_i | (\hat{S} - \mathbb{1}) | \psi_i \rangle \right|^2 = | 1 - S |^2$$

inelastic cross section:

$$\sigma_{inel} = \sigma_{tot} - \sigma_{el} \propto 1 - | S |^2 .$$

We write:

$$\boxed{\begin{aligned} \sigma_{tot} &= 2 \int d^2 b_\perp \left[1 - \mathrm{Re}\, S \right] \\ \sigma_{el} &= \int d^2 b_\perp \left| 1 - S(\vec{b}_\perp) \right|^2 \\ \sigma_{inel} &= \int d^2 b_\perp \left[1 - | S(\vec{b}_\perp) |^2 \right] \end{aligned}}$$

$$\mathbb{1} = \langle \psi_i | \hat{S}^+ \hat{S} | \psi_i \rangle = \sum_X \langle \psi_i | \hat{S}^+ | X \rangle \langle X | \hat{S} | \psi_i \rangle$$

$$\geq | S |^2 \Rightarrow \boxed{| S | \leq 1}$$ ~ unitarity

If $S = -1 \Rightarrow \sigma_{tot} = 4 \pi R^2 = \sigma_{el}$, $\sigma_{inel} = 0$

~ low-energy limit

$1 - |S|^2 \geq |1-S|^2 \Rightarrow -|S|^2 \geq -2\,\mathrm{Re}\,S + |S|^2 \Rightarrow \mathrm{Re}\,S \geq |S|^2 \geq 0.$

High energy: require that $\sigma_{inel} \geq \sigma_{el}$

$$\Rightarrow \mathrm{Re}\,S \geq 0 \quad \Rightarrow \quad \sigma_{tot} = 2\int d^2b_\perp \left[1 - \mathrm{Re}\,S\right] \leq 2\underbrace{\int d^2b}_{2\pi R^2}$$

$$\Rightarrow \boxed{\sigma_{tot} \leq 2\pi R^2}$$ ~ black disk limit

$$\sigma_{el} = \sigma_{inel} = \pi R^2$$ ~ in the black disk regime

(Note that σ_{el} is 50% of the total cross section.)

We had $\sigma^{q\bar{q}A} = 2\int d^2b_\perp\, N(\vec{x}_\perp, \vec{b}_\perp, s)$

$$\Rightarrow \boxed{N = 1 - \mathrm{Re}\,S}$$ ~ Im part of the forward T-matrix element

$$\Rightarrow \text{if } \mathrm{Re}\,S \geq 0 \Rightarrow \boxed{N \leq 1}$$ ~ unitarity constraint.

$$N^{2\text{-gluons}}(\vec{x}_\perp, \vec{b}_\perp, s) = \frac{\alpha_s \pi^2}{2N_c} T(\vec{b}_\perp)\, x_\perp^2 \times G_N\left(x, \frac{1}{x_\perp^2}\right)$$

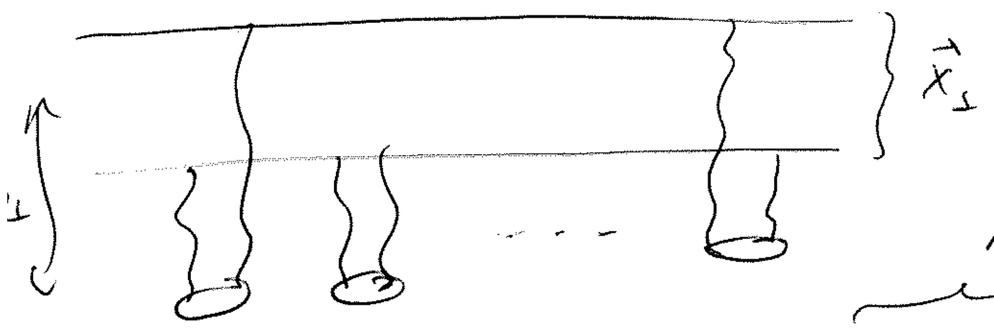
$$\Rightarrow N^{2\text{-gluons}} \propto A^{1/3} x_\perp^2$$ ~ grows with A and with $x_\perp$

$\Rightarrow$ N^2-gluons can violate unitarity

($\Rightarrow$ bad!).

Gribov - Glauber - Mueller Picture

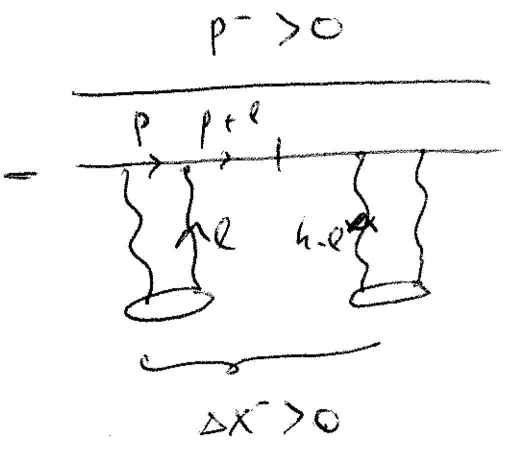
If the interaction with 1 nucleon becomes strong $\Rightarrow$ need to account for multiple interactions:



$$N(\vec{x}_\perp, \vec{b}_\perp, s) = 1 - \underbrace{e^{-\frac{1}{2}\overbrace{T(\vec{b})\,\sigma_{q\bar{q}N}}^{N_{2G} = N^2\text{-gluons}}}}_{\text{multiple exchanges simply exponentiate}}$$

$$\Rightarrow \boxed{N(\vec{x}_\perp, \vec{b}_\perp, s) = 1 - e^{-\frac{\alpha_s \pi^2}{2N_c} x_\perp^2 \, x G_N(x, \frac{1}{x_\perp^2})}}$$

GGM formula (Mueller, 1990)



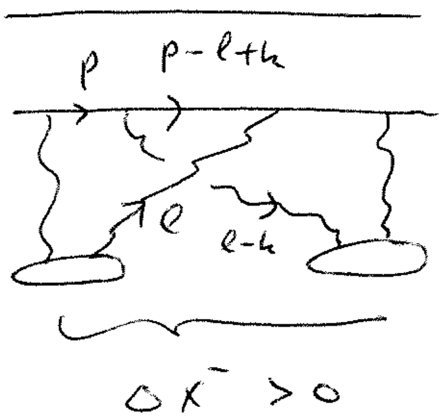
$$\Rightarrow \propto \int_{-\infty}^{\infty} \frac{d\ell^+}{2\pi} e^{-i\ell^+ \Delta x^-} \frac{i}{(p+\ell)^2 + i\varepsilon}$$

$$\approx \int_{-\infty}^{\infty} \frac{d\ell^+}{2\pi} e^{-i\ell^+ \Delta x^-} \frac{i}{2p^-\ell^+ + i\varepsilon} =$$

$$= \frac{1}{2p^-} \Rightarrow \text{pick the pole,}$$

putting $(p+\ell)^2 = 0 \Rightarrow$ intermediate quark goes on mass shell.

<u>Out of order:</u>

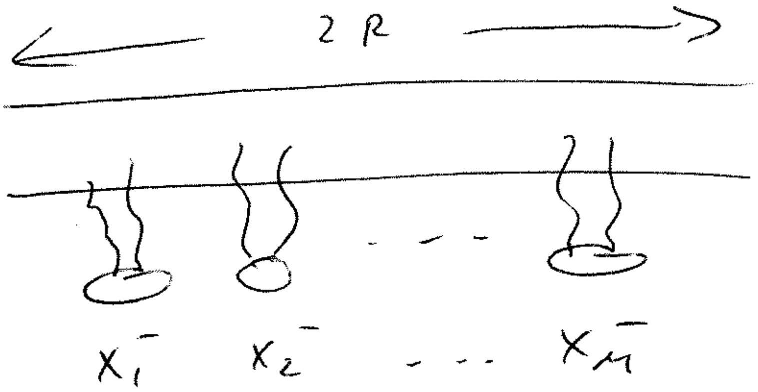


$$\propto \int_{-\infty}^{\infty} \frac{d\ell^+}{2\pi} e^{-i\ell^+ \Delta x^-} \frac{i}{(p-\ell+k)^2 + i\varepsilon}$$

$$= \int_{-\infty}^{\infty} \frac{d\ell^+}{2\pi} \underset{\substack{\Delta x^- > 0 \\ \Downarrow \\ \text{close in lower} \\ \text{half-plane}}}{e^{-i\ell^+ \Delta x^-}} \underbrace{\frac{i}{-2p^-\ell^+ + \ldots + i\varepsilon}}_{\substack{\text{pole in upper} \\ \text{half-plane}}} = 0$$

For M scatterings: $\quad N_{LO} = \frac{1}{2} T(\vec{b}_\perp) \sigma^{q\bar{q}N} = \int dx^- \rho(\vec{b}_\perp, x^-)$ (12'')

$= \cancel{(N_{LO})^M}$



$$= \int_{-R}^{R} dx_1^- \int_{x_1^-}^{R} dx_2^- \ldots \int_{x_{M-1}^-}^{R} dx_M^- \; \rho(\vec{b}_\perp, x_1^-) \rho(\vec{b}_\perp, x_2^-) \ldots$$

$$\ldots \rho(\vec{b}_\perp, x_M^-) \cdot \left(\frac{-1}{2} \sigma^{q\bar{q}N}\right)^M = \frac{1}{M!} \left[\int_{-R}^{R} dx^- \rho(\vec{b}_\perp, x^-)\right]^M \cdot$$

$$\cdot \left(\frac{1}{2}(-) \sigma^{q\bar{q}N}\right)^M = \frac{1}{M!} \left[-\frac{1}{2} \sigma^{q\bar{q}N} T(\vec{b}_\perp)\right]^M =$$

$$= \frac{1}{M!} \left[-N_{LO}\right]^M \Rightarrow \sum_{M=0}^{\infty} \frac{1}{M!} \left[-N_{LO}\right]^M = e^{-N_{LO}} = S'$$

$$\Rightarrow S' = 1 - N \Rightarrow N = 1 - e^{-N_{LO}} .$$

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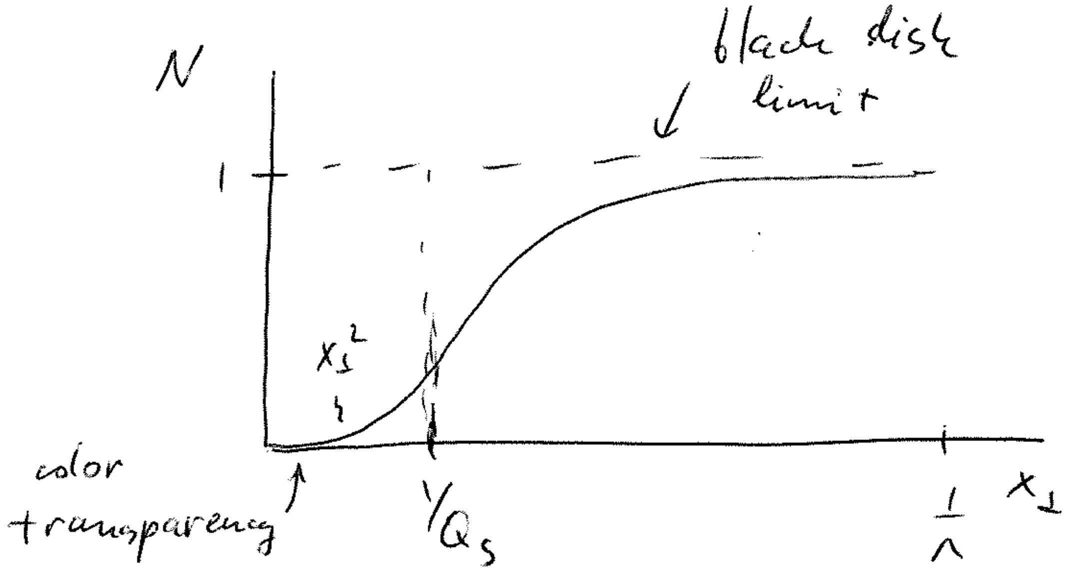
resummation parameter: $\alpha_s^2 A^{1/3}$.

$$N(x_\perp, \vec{b}_\perp, s) = 1 - e^{-\frac{\alpha_s^2 C_F \pi}{2 N_c} T(\vec{b}_\perp)\, x_\perp^2 \ln\frac{1}{x_\perp^2 \Lambda^2}}$$



Note that now $N \leq 1 \Rightarrow$ unitarity is not violated, even for large A and $x_\perp$ (as long as $x_\perp \ll \frac{1}{\Lambda}$).

$x_\perp \sim$ small $\Rightarrow N \sim x_\perp^2$ (color transparency)

$x_\perp \sim$ large $\Rightarrow N \simeq 1$ (black disk limit)

$\Rightarrow$ transition between the two regimes happens at $x_\perp \simeq \frac{1}{Q_s}$ with $Q_s^2 = \frac{4\pi \alpha_s^2 C_F}{N_c} T(\vec{b}_\perp)$ ← saturation scale

such that $N(\vec{x}_\perp, \vec{b}_\perp) = 1 - e^{-\frac{1}{4} x_\perp^2 Q_s^2 \ln\left(\frac{1}{x_\perp \Lambda}\right)}$

$Q_s^2 \propto T(\vec{b}_\perp) \propto A^{1/3} \Rightarrow \boxed{Q_s^2 \propto A^{1/3}}$

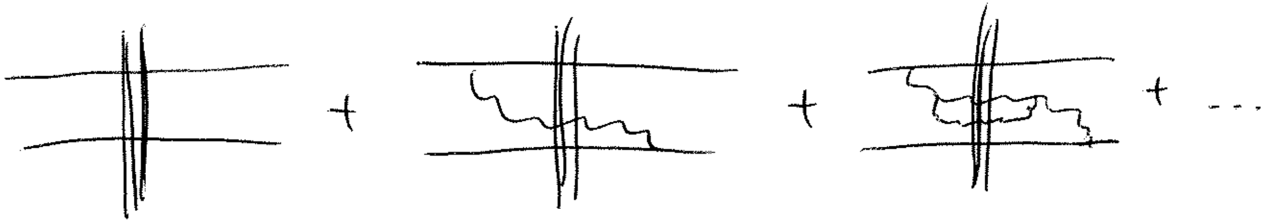
$\Rightarrow$ for large nuclei, $Q_s^2 \gg \Lambda^2 \Rightarrow$

transition to saturation regime is perturbative. $\Rightarrow$ large nuclei are perturbative at high energies!

Unitarity is not violated due to this saturation regime.

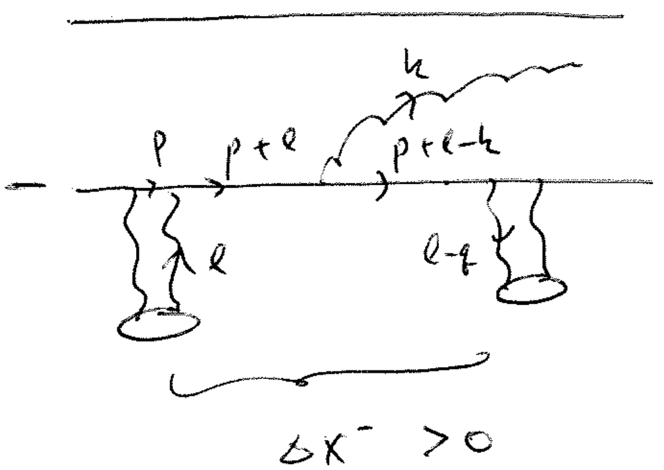
Small-x Evolution

The above calculation does not include any energy dependence in the resulting cross section. Energy dependence comes in through quantum corrections, which bring in powers of $\alpha_s \ln \frac{1}{x}$. Those are given by the long-lived s-channel gluons:



There are no emissions from inside the shock wave: those are energy suppressed.

$p^- > 0$



$$\propto \int_{-\infty}^{\infty} \frac{dl^+}{2\pi}\, e^{-il^+\Delta x^-} \frac{1}{(p+l)^2 + i\varepsilon} \cdot \frac{1}{(p+l-k)^2 + i\varepsilon} \approx$$

$$\approx \int_{-\infty}^{\infty} \frac{dl^+}{2\pi}\, e^{-il^+\Delta x^-} \frac{1}{2p^- l^+ + i\varepsilon} \; \frac{1}{2p^-(l^+ - k^+) + \ldots + i\varepsilon}$$

$$\approx -i\,\frac{1}{2p^-} \cdot \left[\frac{1}{-2p^- k^+} + \frac{1}{2p^- k^+} e^{-ik^+\Delta x^-} \right] =$$

$$= -i\,\frac{1}{2p^-} \cdot \frac{1}{2p^- k^+} \underbrace{\left[e^{-ik^+\Delta x^-} - 1 \right]}$$ as $\Delta x^- \approx \frac{1}{p^+}$

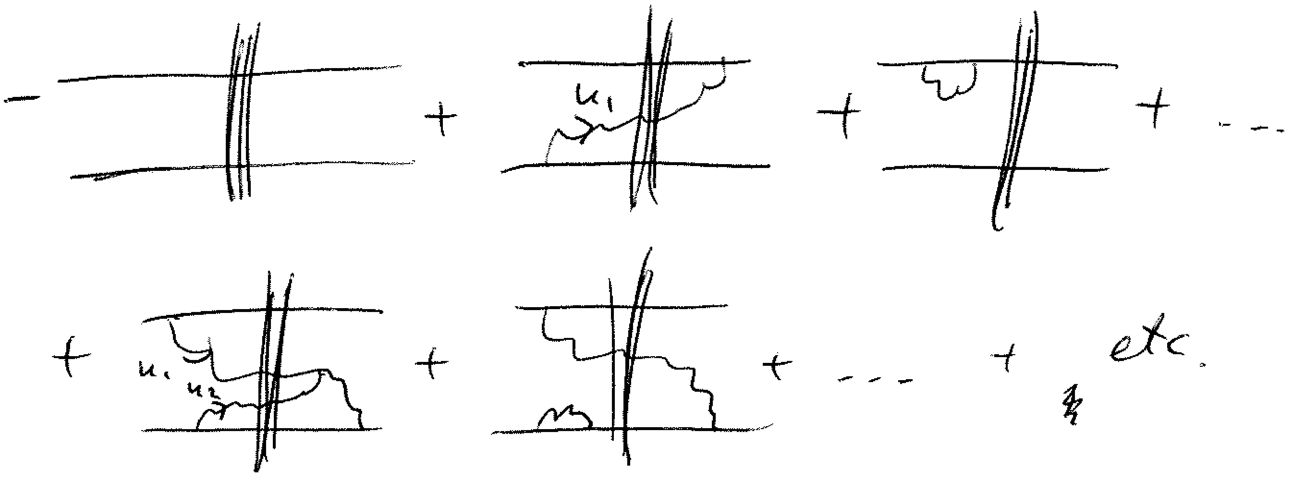
$$\approx -ik^+\Delta x^- \approx -i\,\underset{\sim}{\frac{k^+}{p^+}} \ll 1.$$

in fact, since you have $\propto A^{1/3}$ nucleons, x

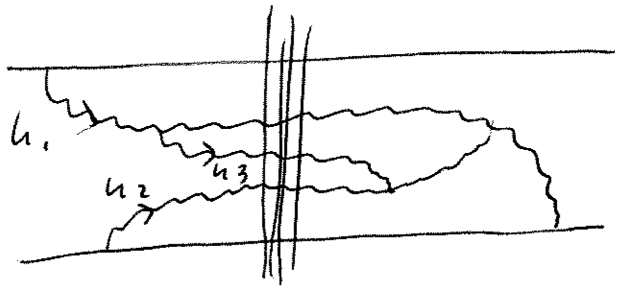
one gets up to $\frac{k^+}{p^+} A^{1/3} = x A^{1/3} \ll 1 \Rightarrow x \ll A^{-1/3}$

c.f. DIS estimate: $\frac{2q^-}{Q^2} \gg \frac{1}{p^+} A^{1/3} \Rightarrow 1 \gg x A^{1/3}$ ← now for nucleus ↑ same

$\Rightarrow$ We need to sum up an ∞ cascade of long-lived gluons



To give us powers of $\alpha_s \ln \frac{1}{x}$, the gluons "$-$" momenta have to be ordered,



$$k_1^- \gg k_2^- \gg k_3^- \gg \dots$$

Their transverse momenta are comparable,

$$k_{1\perp} \sim k_{2\perp} \sim \dots \sim k_{N\perp} \sim \dots$$

$\Rightarrow$ life-times are ordered:

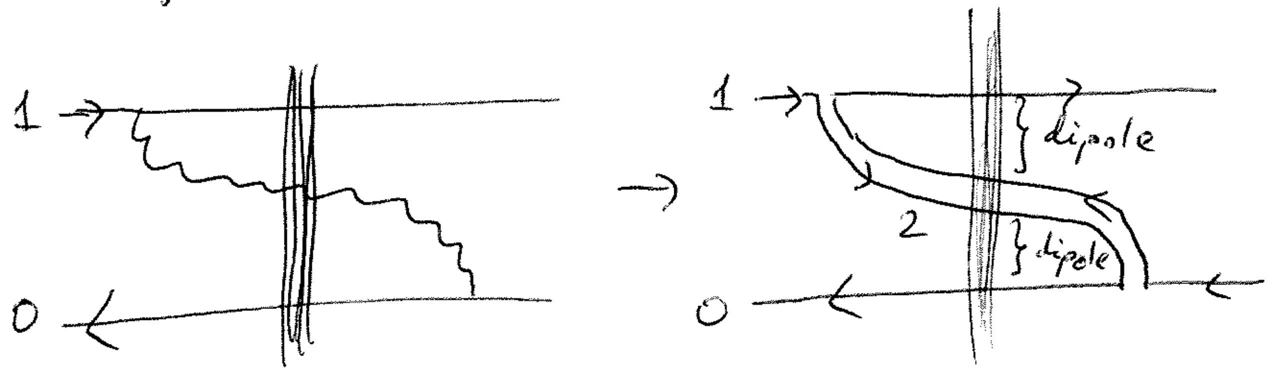
$$\boxed{\frac{2k_1^-}{k_{1\perp}^2} \gg \frac{2k_2^-}{k_{2\perp}^2} \gg \frac{2k_3^-}{k_{3\perp}^2} \gg \dots \quad .}$$

$\Rightarrow$ still, hard to resum a gluon cascade due to color factors 't Hooft 1977

Large-N_c limit: $\to$

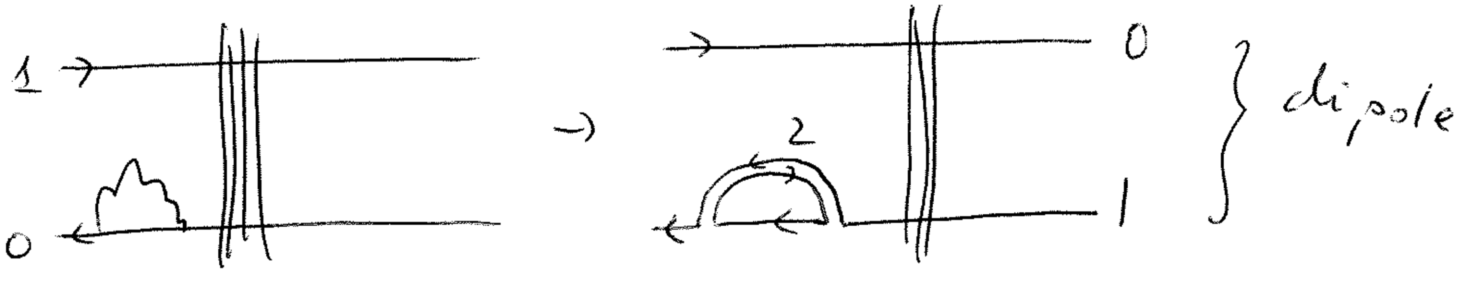
$$N_c \otimes \bar{N}_c = 1 \oplus (N_c^2 - 1) \approx N_c^2 - 1 .$$

Replace the gluon by a $q\bar{q}$ pair. Only planar diagrams contribute:



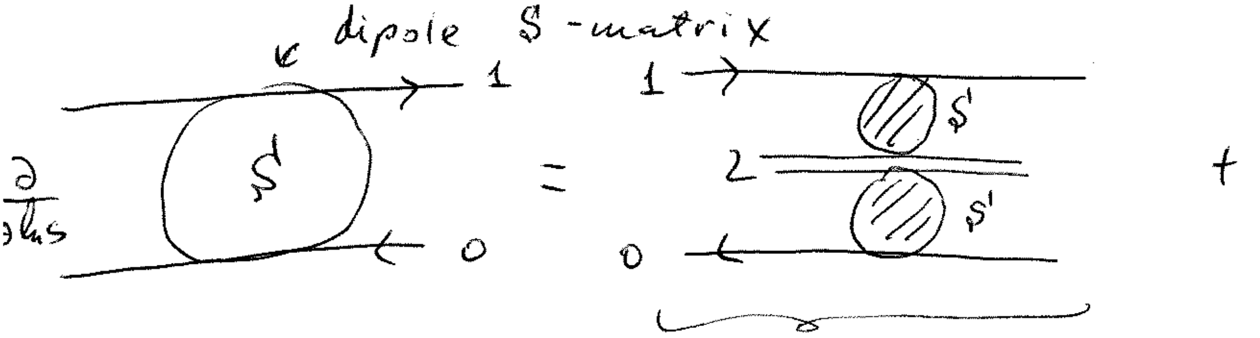
Mueller's dipole model   Mueller 1993

In one step of evolution a dipole may split into two dipoles: $01 \to 02 + 21$ (above). There are also virtual corrections:

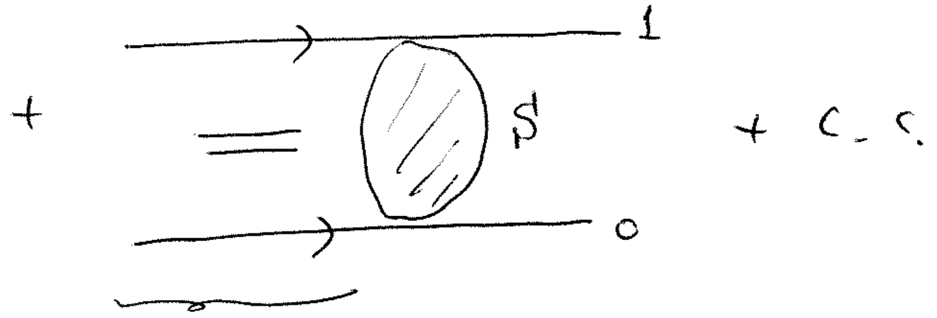


~ still have only 1 dipole going through the shock wave.

We are ready to write down a diagrammatic representation of resummation:



all diagrams with the gluon crossing the shock wave



virtual diagrams

↳ see pp. 18′ & 18″

Performing a diagrammatic calculation, we get ($Y = \ln s/_{\perp^2} = \ln\frac{1}{x}$)

$\vec{x}_{10} = \underline{x}_1 - \underline{x}_0$

$x_{10} = |\vec{x}_{10}|$, etc.

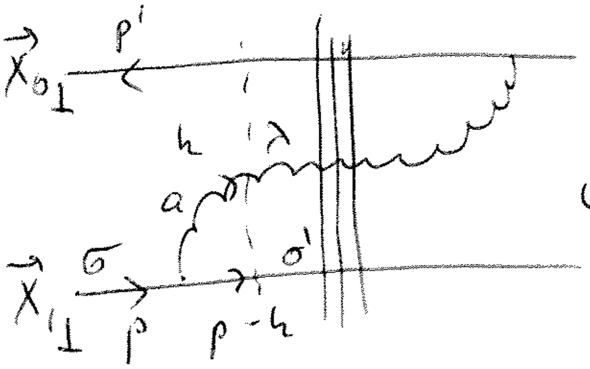
$$\boxed{\frac{\partial}{\partial Y} S_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \left[S_{21}(Y) S_{20}(Y) - S_{10}(Y) \right]}$$

$S_{10} = 1 - N_{10} \Rightarrow$

$$\boxed{\frac{\partial}{\partial Y} N_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \left[\underline{N_{21}(Y) + N_{20}(Y)} \underbrace{- N_{10}(Y)}_{\text{BFKL}} - N_{21}(Y) N_{20}(Y) \right]}$$

Balitsky '96; YK '99

Let's calculate a real gluon emission, e.g.



LC wave function

use LCPT: $A^- = 0$ gauge

$$\psi = \underbrace{-g\, \bar{u}_{\sigma'}(p-k)\, \not{\varepsilon}_\lambda^{*}\, u_\sigma(p) \cdot t^a}_{\text{vertex}} \cdot \theta(k$$

$$\cdot \underbrace{\frac{1}{p^+ + p'^+ - k^+ - (p-k)^+ - p'^+ + i\varepsilon}}_{\text{energy denominator}} \cdot \underbrace{\frac{1}{2p^-}}_{\substack{2\times\text{incoming} \\ +\text{ momentum}}}$$

Strong ordering of "−" momenta: $\boxed{k^- \ll p^-}$

Energy denominator: $$\frac{1}{\frac{p_\perp^2}{2p^-} - \frac{k_\perp^2}{2k^-} - \frac{(\vec{p}_\perp - \vec{k}_\perp)^2}{2(p^- - k^-)}} \approx -\frac{2k}{k_\perp^2}$$

$k^- \ll p^-$

$k_\perp \sim p_\perp$

$$\Rightarrow \psi^{q\to qG} = g\, \bar{u}_{\sigma'}(p-k)\, \not{\varepsilon}_\lambda^{*}\, u_\sigma(p)\, t^a\, \theta(k^-)\, \frac{2k^-}{k_\perp^2} \cdot \frac{1}{2p^-}.$$

For simplicity, pick $p^\mu = (0^+, p^-, \vec{0}_\perp)$, $p^- \sim$ very large

$$\Rightarrow k^-, k_\perp \ll p^- \Rightarrow \bar{u}_{\sigma'}(p - \not{k})\, \gamma^\mu\, u_\sigma(p) \approx 2p^-\, \delta^{\mu -}\, \delta_{\sigma\sigma'}$$

eikonal approx   drop   Gordon identity

$$\Rightarrow \psi^{q\to qG} = g\, t^a\, \theta(k^-)\, \cancel{2p^-}\, \delta_{\sigma\sigma'} \left(\varepsilon_\lambda^{*}\right)^+ \frac{2k^-}{k_\perp^2}\, \frac{1}{\cancel{2p^-}} = \Big| \; \varepsilon_\lambda^+ = \frac{\vec{\varepsilon}_\lambda}{k}$$

$$= 2g\, t^a\, \delta_{\sigma\sigma'}\, \theta(k^-)\, \frac{\vec{\varepsilon}_\lambda^{\,*} \cdot \vec{k}_\perp}{k_\perp^2}.$$

$A^- = 0$ gauge $\Rightarrow \varepsilon_\lambda^- = 0$, $\quad k\cdot\varepsilon_\lambda = 0 \Rightarrow \underbrace{k^+\varepsilon_\lambda^-}_{0} + k^-\varepsilon_\lambda^+ - \vec{k}_\perp\cdot\vec{\varepsilon}_\lambda = 0$

$$\Rightarrow \varepsilon_\lambda^+ = \frac{\vec{\varepsilon}_\lambda\cdot\vec{k}_\perp}{k^-} \Bigg].$$

Def. $z \equiv k^-/p^-$

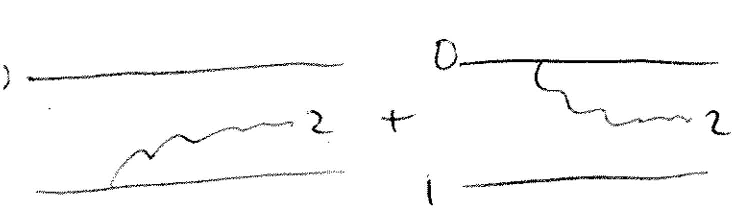
$$\psi^{q\to qG}(\vec{k}_\perp, z) = 2g\, t^a\, \delta_{\sigma\sigma'}\, \theta(z)\, \frac{\vec{\varepsilon}_\lambda^{\,*}\cdot\vec{k}_\perp}{k_\perp^2}$$

Fourier-transform: $\psi^{q\to qG}(\vec{x}_\perp, z) = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot\vec{x}_\perp}\, \psi^{q\to qG}(\vec{k}_\perp, z)$

$$\Rightarrow \boxed{\psi^{q\to qG}(\vec{x}_\perp, z) = i\frac{g}{\pi}\, t^a\, \delta_{\sigma\sigma'}\, \theta(z)\, \frac{\vec{x}_\perp}{x_\perp^2}\cdot\vec{\varepsilon}_\lambda^{\,*}}$$

(use $\int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot\vec{x}_\perp}\, \frac{\vec{k}_\perp}{k_\perp^2} = \frac{i}{2\pi}\, \frac{\vec{x}_\perp}{x_\perp^2}$).

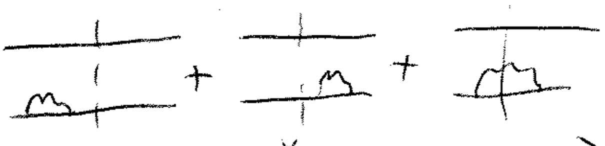
Def. $\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$



$$= i\frac{g}{\pi}\, t^a\, \vec{\varepsilon}_\lambda^{\,*}\cdot\left(\frac{\vec{x}_{21}}{x_{21}^2} - \frac{\vec{x}_{20}}{x_{20}^2}\right)$$

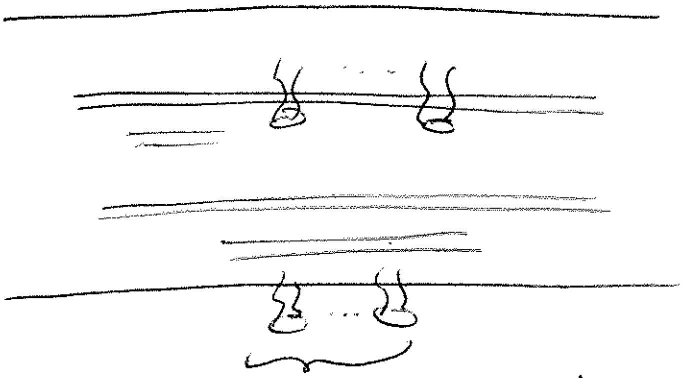
$\Rightarrow$ Square, sum over quantum #'s: $\frac{g^2}{\pi^2}\, C_F \left|\frac{\vec{x}_{21}}{x_{21}^2} - \frac{\vec{x}_{20}}{x_{20}^2}\right|$

$$= \frac{4\alpha_s C_F}{\pi}\, \frac{x_{10}^2}{x_{21}^2\, x_{20}^2} \otimes \int_{z_{min}}^{\sim z_1} \frac{dz_2}{z_2} \int \frac{d^2x_2}{4\pi} = \frac{\alpha_s C_F}{\pi^2} \int \frac{dz_2}{z_2} \int d^2x_2\, \frac{x_{10}^2}{x_{21}^2 x_{20}^2}$$

Virtual graphs:  $= 0$

$$V + V^* + R = 0 \Rightarrow \boxed{V + V^* = -R}$$

Initial condition $N_{10}(Y=Y_0) = \underbrace{1-e^{-\frac{1}{4}x_\perp^2 Q_{s0}^2 \ln\frac{1}{x_\perp \Lambda}}}_{\text{GGM f-la.}}$



each dipole interacts with the target via multiple rescatterings $\sim \left(\alpha_s^2 A^{1/3}\right)^{\cdots}$ powers

Evolution : resums powers of $\alpha_s N_c Y \sim \alpha_s N_c \ln\frac{1}{x}$.

initial conditions resum powers of $\alpha_s^2 A^{1/3}$.

$\sim$ Beyond large-N_c : JIMWLK functional differential / integral equation.

(small correction to N compared to BK, $< 0.1\%$)

BK solution $\sim$ see slides

$Q_s(Y) \propto e^{\Delta Y}$ $\sim$ saturation scale <u>grows</u> with energy (with $x \to 0$).

$\Rightarrow$ $\boxed{Q_s^2(x) \propto A^{1/3}\left(\frac{1}{x}\right)^{\Delta}}$ $\Rightarrow$ the higher the energy (the smaller the x) and/or the larger the nucleus, the more perturbative interactions get.

Take the BFKL solution:

$$N(x_\perp, Y) = \int_{a-i\infty}^{a+i\infty} \frac{d\gamma}{2\pi i}\, C_\gamma\, e^{\frac{\alpha_s N_c}{\pi}\chi(\gamma) Y} \cdot \left(x_\perp^2 Q_{s0}^2\right)^\gamma$$

$\Rightarrow$ completeness of BFKL eigenfunctions requires $a = \frac{1}{2}$

$\Rightarrow \gamma = \frac{1}{2} + i\nu$, ν = real

$$N(x_\perp, Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\, C_\nu\, e^{\frac{\alpha_s N_c}{\pi}\chi(\nu) Y} \left(x_\perp Q_{s0}\right)^{1+2i\nu}.$$

saddle point condition $\frac{d}{d\gamma}\left[\frac{\alpha_s N_c}{\pi}\chi(\gamma) Y + \gamma \ln\left(x_\perp^2 Q_{s0}^2\right)\right] = 0.$

$$\Rightarrow \frac{\alpha_s N_c}{\pi}\chi'(\gamma_{cr}) Y + \ln\left(x_\perp^2 Q_{s0}^2\right) = 0. \qquad (*)$$

Along saturation boundary $N\left(x_\perp = \frac{1}{Q_s(Y)}, Y\right) = \text{const}$

$$\Rightarrow \frac{\alpha_s N_c}{\pi}\chi(\gamma_{cr}) Y + \gamma_{cr} \ln\left(\underbrace{x_\perp^2}_{=\frac{1}{Q_s(Y)}} Q_{s0}^2\right) = 0 \qquad (**)$$

From $(*)$ and $(**)$, we get

$$\boxed{\chi'(\gamma_{cr}) = \frac{\chi(\gamma_{cr})}{\gamma_{cr}}} \Rightarrow \boxed{\gamma_{cr} \simeq 0.628}$$

$$\Rightarrow \boxed{Q_s^2(Y) = Q_{s0}^2\, e^{\frac{\alpha_s N_c}{\pi}\chi'(\gamma_{cr}) Y}} \Rightarrow Q_s(Y) \approx Q_{s0}\, e^{2.44\frac{\alpha_s N_c}{\pi} Y}$$

$$Y = \ln\frac{1}{x} \Rightarrow Q_s(x) \simeq Q_{s0}\left(\frac{1}{x}\right)^{2.44\frac{\alpha_s N_c}{\pi}}.$$

(Gribov, Levin, Ryskin '83; Iancu, Itakura, McLerran '02; Mueller & Triantafyllopoulos '02)

$$N(x_\perp, Y) \propto e^{\frac{\alpha_s N_c}{\pi}\chi(\gamma_{cr})Y} (x_\perp^2 Q_{s0}^2)^{\gamma_{cr}}$$

with $Q_s^2(Y) = Q_{s0}^2\, e^{\frac{\alpha_s N_c}{\pi}\chi'(\gamma_{cr})Y} = Q_{s0}^2\, e^{\frac{\alpha_s N_c}{\pi}\frac{\chi(\gamma_{cr})}{\gamma_{cr}}Y}$

$$\Rightarrow e^{\frac{\alpha_s N_c}{\pi}\chi(\gamma_{cr})Y} = \left(\frac{Q_s^2(Y)}{Q_{s0}^2}\right)^{\gamma_{cr}}$$

$$\Rightarrow N(x_\perp, Y) \propto \left(\frac{Q_s^2(Y)}{\cancel{Q_{s0}^2}}\right)^{\gamma_{cr}} \cdot (x_\perp^2 \cancel{Q_{s0}^2})^{\gamma_{cr}}$$

$$\Rightarrow \boxed{N(x_\perp, Y) \propto \underbrace{(Q_s^2(Y)\, x_\perp^2)}_{\text{one parameter}}{}^{\gamma_{cr}}}$$

IIM '02

MT '02.

Geometric scaling: instead of being a function of 2 parameters $x_\perp$ & Y, the dipole amplitude is a function of just one parameter, $x_\perp^2 Q_s^2(Y)$

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BFKL Solution

of $\vec{x}_{1'0'}$: the resulting cross section does not depend on the directions of $\vec{x}_{10}$ either, since there is no preferred direction left in the transverse space. Defining

$$n(x_{10}, x_{1'0'}, Y) = \int d^2b \int_0^{2\pi} \frac{d\phi_{1'0'}}{2\pi} n(\vec{x}_{10}, \vec{x}_{1'0'}, \vec{b}_\perp, Y), \tag{4.89}$$

we see that this new quantity satisfies

$$\frac{\partial}{\partial Y} n(x_{10}, x_{1'0'}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \times [n(x_{12}, x_{1'0'}, Y) + n(x_{20}, x_{1'0'}, Y) - n(x_{10}, x_{1'0'}, Y)] \tag{4.90}$$

with initial condition (cf. Eq. (3.25))

$$n(x_{10}, x_{1'0'}, Y = 0) = \frac{4\pi\alpha_s^2 C_F}{N_c} x_<^2 \left(\ln \frac{x_>}{x_<} + 1\right), \tag{4.91}$$

where $x_{>(<)} = \max(\min)\{|\vec{x}_{10}|, |\vec{x}_{1'0'}|\}$.

The solution of Eq. (4.90) can be found by noticing that in the angular-averaged case the eigenfunctions of the integral kernel are simple powers of the dipole size,

$$\left(x_{01}^2\right)^{1/2+i\nu} \tag{4.92}$$

with eigenvalues

$$\frac{\alpha_s N_c}{\pi} \chi(0, \nu), \tag{4.93}$$

where (cf. Eqs. (3.81), (3.74))

$$\chi(0, \nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right). \tag{4.94}$$

To prove this we need to evaluate the following integral:

$$\int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \left[\left(x_{12}^2\right)^{1/2+i\nu} + \left(x_{20}^2\right)^{1/2+i\nu} - \left(x_{10}^2\right)^{1/2+i\nu}\right]. \tag{4.95}$$

This can be done by noticing that the integral (4.95) is equivalent to that in Eq. (3.64) with $n = 0$. Alternatively, one can use the trick presented in appendix section A.3; in order to make each term in Eq. (4.95) finite we insert a UV regulator ρ. After that, with the help of Eqs. (A.18), (A.21), (A.24), and (A.29) one can rewrite Eq. (4.95) as

$$2\pi\left[2^{1+2i\nu} \frac{\Gamma\left(\frac{1}{2} + i\nu\right)}{\Gamma\left(\frac{1}{2} - i\nu\right)} x_{10}^2 \int_0^\infty dk k^{-2i\nu} \left(\ln \frac{2}{k\rho} + \psi(1)\right) J_0(k x_{10}) - x_{10}^{1+2i\nu} \ln \frac{x_{10}^2}{\rho^2}\right]. \tag{4.96}$$

Integrating over k in Eq. (4.96) using Eq. (A.18) yields

$$2\pi x_{10}^{1+2i\nu} \chi(0, \nu), \tag{4.97}$$

as desired.

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We see that, as for to the BFKL equation (3.58), the eigenfunctions of Eq. (4.90) are powers (though of the transverse dipole size instead of the transverse momentum), with exactly the same eigenvalues, (4.93) as in that case.[6] We conclude that Eq. (4.90) is equivalent to the BFKL equation!

In fact, the substitution (Levin and Ryskin 1987)

$$n(x_{10}, x_{1'0'}, Y) = \int d^2k \left(1 - e^{i\vec{k}_\perp \cdot \vec{x}_{10}}\right) \frac{1}{k_\perp^2} f(\vec{k}_\perp, x_{1'0'}, Y) \tag{4.98}$$

turns Eq. (4.90) into the BFKL equation (3.58) for the function f (Kovchegov and Weigert 2007b). Verification of this statement is left as an exercise for the reader.

Using the eigenfunctions and the eigenvalues of the integral kernel in Eq. (4.90), we can write down the solution of Eq. (4.90) as

$$n(x_{10}, x_{1'0'}, Y) = \int\limits_{-\infty}^{\infty} d\nu C_\nu(x_{1'0'}) x_{10}^{1+2i\nu} e^{\bar{\alpha}_s \chi(0,\nu) Y}, \tag{4.99}$$

where the coefficient $C_\nu(x_{1'0'})$ is fixed by the initial conditions (4.91) as follows:

$$C_\nu(x_{1'0'}) = \frac{16\alpha_s^2 C_F}{N_c} \frac{1}{(1+4\nu^2)^2} x_{1'0'}^{1-2i\nu}. \tag{4.100}$$

The general solution of Eq. (4.90) is then

$$n(x_{10}, x_{1'0'}, Y) = \frac{16\alpha_s^2 C_F}{N_c} x_{10} x_{1'0'} \int\limits_{-\infty}^{\infty} d\nu \left(\frac{x_{10}}{x_{1'0'}}\right)^{2i\nu} \frac{e^{\bar{\alpha}_s \chi(0,\nu) Y}}{(1+4\nu^2)^2}. \tag{4.101}$$

For $x_{10} \approx x_{1'0'}$ we can use the diffusion approximation from Sec. 3.3.4: expanding $\chi(0, \nu)$ around $\nu = 0$ using Eq. (3.84) and integrating over ν we obtain

$$\begin{aligned} n(x_{10}, x_{1'0'}, Y) = {} & \frac{16\alpha_s^2 C_F}{N_c} x_{10} x_{1'0'} \sqrt{\frac{\pi}{14\zeta(3)\bar{\alpha}_s Y}} \\ & \times \exp\left[(\alpha_P - 1)Y - \frac{\ln^2(x_{10}/x_{1'0'})}{14\zeta(3)\bar{\alpha}_s Y}\right]. \end{aligned} \tag{4.102}$$

Readers who performed Exercise 3.5 will recognize Eq. (4.102) as the answer for the onium–onium scattering cross section obtained there using the standard Feynman diagram approach. Now we see that a calculation based on LCPT wave functions gives the same result. Note that the single-dipole distribution n_1 is only one component of the onium wave function. This wave function also contains multi-dipole distributions n_2, n_3, etc. Hence, as we will shortly see, the dipole approach, while in a certain limit equivalent to BFKL, in fact contains more information.

[6] We have verified this statement so far only in the case where the angular dependence has been integrated out: we will consider the general angular-dependent case in the next section.

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$\chi(n \neq 0, \nu = 0)$. We will therefore keep only the $n = 0$ term in Eq. (3.80) and write

$$G\left(\vec{l}_\perp, \vec{l}'_\perp, Y\right) \approx \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 l_\perp l'_\perp} \exp\left\{\bar{\alpha}_s \chi(0, \nu) Y + 2i\nu \ln \frac{l_\perp}{l'_\perp}\right\}; \tag{3.82}$$

here

$$\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}. \tag{3.83}$$

Expanding $\chi(n = 0, \nu)$ around the saddle point at $\nu = 0$ we get $4\ln 2 + 14\zeta(3)\left(\gamma - \frac{1}{2}\right)^2$

$$\chi(0, \nu) \approx 4\ln 2 - 14\zeta(3)\nu^2, \tag{3.84}$$

where $\zeta(z)$ is the Riemann zeta function. Using Eq. (3.84) in Eq. (3.82) we perform the ν-integration, obtaining (Balitsky and Lipatov 1978)

$$G\left(\vec{l}_\perp, \vec{l}'_\perp, Y\right) \approx \frac{1}{2\pi^2 l_\perp l'_\perp} \sqrt{\frac{\pi}{14\zeta(3)\bar{\alpha}_s Y}} \exp\left\{(\alpha_P - 1)Y - \frac{\ln^2(l_\perp/l'_\perp)}{14\zeta(3)\bar{\alpha}_s Y}\right\}, \tag{3.85}$$

where we have used, for the intercept of the perturbative BFKL pomeron,

$$\alpha_P - 1 = \frac{4\alpha_s N_c}{\pi} \ln 2. \tag{3.86}$$

The essential feature of Eq. (3.85) is that it shows that cross sections mediated by the BFKL ladder exchange grow as a power of the energy:

$$\sigma \sim e^{(\alpha_P - 1)Y} \sim s^{\alpha_P - 1}. \tag{3.87}$$

This behavior is reminiscent of pomeron exchange in pre-QCD language (see Eq. (3.20)). The BFKL ladder from Fig. 3.12 is therefore referred to as the "hard" (perturbative) pomeron or as the BFKL pomeron. We see that BFKL evolution modifies the energy-independent Low-Nussinov pomeron, which simply corresponds to a two-gluon exchange and has $\alpha_P - 1 = 0$, which makes the perturbative pomeron intercept $\alpha_P > 1$ as seen from Eq. (3.86). The numerical value of the BFKL intercept (3.86) is rather large: for $\alpha_s = 0.3$ one gets $\alpha_P - 1 \approx 0.79$, which is much larger than the "soft" pomeron intercept of 0.08 observed, say, for the total proton–proton scattering cross section (Donnachie and Landshoff 1992).

Double logarithmic approximation Let us consider the case $l_\perp \gg l'_\perp$. Now $\ln(l_\perp/l'_\perp)$ is large, and this may affect the location of the saddle point of the ν-integral in Eq. (3.80). The way the saddle point is shifted is shown in Fig. 3.14 for the $n = 0$ term in the series (3.80). As one can show analytically and as can be seen from Fig. 3.14, the effect of $(l_\perp/l'_\perp)^{2i\nu}$ in (3.80) is to shift the saddle point in the imaginary ν direction, moving it closer to the singularity of $\chi(0, \nu)$ at $\nu = i/2$. One can also show that the same is true for any integer n: the saddle point in the nth term in Eq. (3.80) is shifted toward the singularity of $\chi(n, \nu)$

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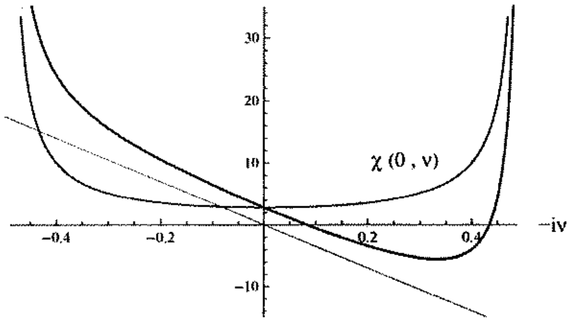

Fig. 3.14. The eigenvalue of the BFKL kernel $\chi(0, \nu)$ plotted as a function of $-i\nu$ (medium-bold line) for Re $\nu = 0$. The thin straight line is due to the linear term $2i\nu \ln(l_\perp/l'_\perp)$ in the exponent of Eq. (3.82). The boldest curve is a sum of the medium-bold line and the thin straight line: it represents the complete expression in the exponent of Eq. (3.82). A color version of this figure is available online at www.cambridge.org/9780521112574.

at $\nu = i(|n| + 1)/2$. However, near these saddle points the nth term in the series (3.80) scales as

$$\frac{1}{l_\perp^2}\left(\frac{l'^2_\perp}{l_\perp^2}\right)^{|n|}; \tag{3.88}$$

we see that terms with $|n| > 0$ are suppressed by powers of $l'^2_\perp/l_\perp^2 \ll 1$ compared with the $n = 0$ term (i.e., they are higher-twist corrections). Therefore the $n = 0$ term dominates again and, as before, we can work with Eq. (3.82).

Expanding the $n = 0$ eigenvalue of the BFKL kernel near $\nu = i/2$, we find that

$$\chi(0, \nu) \approx -\frac{i}{\nu - i/2}, \tag{3.89}$$

and the saddle point of the integral in Eq. (3.82) is then given by

$$\nu_{DLA} \approx \frac{i}{2} - i\sqrt{\frac{\bar{\alpha}_s Y}{\ln(l_\perp^2/l'^2_\perp)}}. \tag{3.90}$$

Distorting the ν-integration contour to run through ν_{DLA} and expanding the exponent of Eq. (3.82) up to terms of order $(\nu - \nu_{DLA})^2$, we integrate the result over ν, obtaining

$$G\left(\vec{l}_\perp, \vec{l}'_\perp, Y\right) \approx \frac{1}{2\pi^{3/2} l_\perp^2} \frac{(\bar{\alpha}_s Y)^{1/4}}{\ln^{3/4}(l_\perp^2/l'^2_\perp)} \exp\left\{2\sqrt{\bar{\alpha}_s Y \ln(l_\perp^2/l'^2_\perp)}\right\}. \tag{3.91}$$

Comparing the exponential in Eq. (3.91) with that in Eq. (2.143) or, since here we are assuming a fixed coupling constant, with Eq. (2.159), we see that the DLA limit is indeed the same when obtained from the DGLAP or the BFKL equations! Identifying Y in Eq. (3.91) with $\ln 1/x$ in Eq. (2.159) and the transverse logarithm $\ln(l_\perp^2/l'^2_\perp)$ in Eq. (3.91) with $\ln(Q^2/Q_0^2)$ in Eq. (2.159), we see complete agreement between the exponents in the two cases. The prefactor of Eq. (3.91) is different from what one would obtain in Eq. (2.159), Start from the BK eq'n: **Traveling Wave Solution**

$$\frac{\partial}{\partial Y} N(x_{10}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \Big[N(x_{21}, Y) + N(x_{20}, Y) - N(x_{10}, Y) - N(x_{21}, Y) N(x_{20}, Y) \Big]$$

Fourier transform:

$$N(x_\perp, Y) = x_\perp^2 \int \frac{d^2k}{2\pi} e^{i \vec{k}_\perp \cdot \vec{x}_\perp} \tilde{N}(k_\perp, Y), \quad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$$

$$\Rightarrow \frac{\partial \tilde{N}(k_\perp, Y)}{\partial Y} = \bar{\alpha}_s \underbrace{\chi\left(0, \frac{i}{2}\left(1 + \frac{\partial}{\partial \ln k_\perp}\right)\right)}_{\downarrow} \tilde{N}(k_\perp, Y) - \bar{\alpha}_s \tilde{N}^2(k_\perp, Y)$$

$$\chi\left(-\frac{\partial}{\partial \ln k_\perp^2}\right), \quad \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\rho = \ln k_\perp^2 / Q_{s0}^2$$

$$\frac{\partial \tilde{N}(\rho, Y)}{\partial Y} = \bar{\alpha}_s \chi\left(-\frac{\partial}{\partial \rho}\right) \tilde{N}(\rho, Y) - \bar{\alpha}_s \tilde{N}^2(\rho, Y)$$

Expand the kernel in Taylor series:

$$\chi(\gamma) = \chi(\gamma_{cr}) + (\gamma - \gamma_{cr}) \chi'(\gamma_{cr}) + \frac{1}{2}(\gamma - \gamma_{cr})^2 \chi''(\gamma_{cr}) + \ldots$$

where γ_{cr} is defined by $\chi(\gamma_{cr}) = \gamma_{cr} \chi'(\gamma_{cr})$

Change variables to:

$$t = \frac{1}{2}\bar{\alpha}_s \chi''(\gamma_{cr})\gamma_{cr}^2 Y$$

$$x = \gamma_{cr}\rho + \bar{\alpha}_s\left[\chi''(\gamma_{cr})\gamma_{cr}^2 - \chi(\gamma_{cr})\right] Y$$

$$u(t,x) \equiv \frac{2}{\chi''(\gamma_{cr})\gamma_{cr}^2}\tilde{N}(\rho,Y)$$

Munier & Peschanski, 2003

$\Rightarrow$ get $\boxed{\partial_t u(t,x) = \partial_x^2 u + u(1-u)}$

F-KPP equation

Fisher 1937.

Kolmogorov, Petrovsky, Piskunov 1937

Traveling wave solution: $t \to \infty \Rightarrow$

$$u(t,x)\Big|_{t\to\infty} \propto f\left(x - 2t + \frac{3}{2}\ln t + \mathcal{O}(t)\right)$$

function of one variable.

$$x - 2t + \frac{3}{2}\ln t = \gamma_{cr}\ln\frac{k_\perp^2}{Q_s^2(Y)} + \text{const.}$$

with $Q_s^2(Y) = Q_{s0}^2 \exp\left\{\bar{\alpha}_s\frac{\chi(\gamma_{cr})}{\gamma_{cr}}Y - \frac{3}{2\gamma_{cr}}\ln\bar{\alpha}_s Y\right\}$

$\Rightarrow u = e^{-x+2t}$ solves linear F-KPP $\Rightarrow$

$\Rightarrow$ $\boxed{\tilde{N}(\rho,Y) \propto \left(\frac{Q_s^2(Y)}{k_T^2}\right)^{\gamma_{cr}}}$ ~ geometric scaling !