QCD

Masterclass 2024

Lecture 1

- Today
  - $\mathcal{L}$ (QCD) & its vertices
  - Color structure examples
  - Complete set of color rules

See also: Cvitanovic, Group Theory Birdtracks, Lie's and all exceptional groups, birdtracks.eu
- Previous lecture notes from Stefan Keppeler, 1707.07280
- P. Cvitanovic Group theory for Feynman diagrams in non-abelian gauge theories, PR D, vol 14 p1536-1553 (1976)
- (Dixon TASI Lecture notes hep-ph/9601359)
- Peigne Introduction to color in QCD: Initiation to the birdtrack pictorial technique 2302.07575

For implementations:
- Sjodahl, Color Math, 1211.2099, Mathematica implementation
- Vermaseren, FORM
- Sjodahl Colorfull (C++, including trace bases)

The QCD Lagrangian and its vertices

$$\mathcal{L} = \bar\psi\,(i\underbrace{\not{D}}_{\gamma^\mu\partial_\mu - i\gamma^\mu g\,\underbrace{A^a_\mu t^a}_{A_\mu}} - m)\,\psi \qquad - \frac{1}{2}\,\mathrm{Tr}\left[F^{\mu\nu}F_{\mu\nu}\right]$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

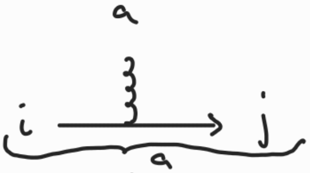
$$\mathcal{L} = \underbrace{\bar\psi\,(i\not{\partial} - m)\,\psi}_{\text{free fermion}} \quad + \underbrace{g\,A^a_\mu\,(\bar\psi\gamma^\mu t^a \psi)}_{\rightsquigarrow \text{ fermion gluon vertex}}$$

$$F_{\mu\nu} = \sum_a F^a_{\mu\nu}\, t^a$$

$$\begin{aligned} F_{\mu\nu} &= \frac{-1}{ig}\,[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A^a_\mu t^a, A^b_\nu t^b] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\, A^a_\mu A^b_\nu \underbrace{[t^a, t^b]}_{if^{abc}t^c} \end{aligned}$$

$$\begin{aligned} - \Bigg[& \underbrace{\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2}_{\text{free gluons}} \\ & \underbrace{+ g\, f^{abc}\,(\partial_\mu A^a_\nu)\, A^{\mu b} A^{\nu c}}_{\text{triple gluon vertex}} \\ & \underbrace{+ \frac{1}{4}\, g^2\, f^{eab} A^a_\mu A^b_\nu\, f^{ecd} A^{\mu c} A^{\nu d} \Bigg]}_{\text{four-gluon vertex}} \end{aligned}$$

- quark-gluon vertex $ig\gamma^\mu$



$\equiv t^a_{ij}$

def of graphical notation

(in conventions here)

Here t^a_{ij} generate the group $SU(3)$, i.e. they give infinitesimal rotations of complex 3-component vectors

special $\Rightarrow \det[\,]=1$ unitary 3×3 matrices

span traceless hermitian matrices orthogonal under $\mathrm{Tr}[t^a t^b]$

$t^1 = \frac{1}{2}\begin{bmatrix}\boxed{\begin{matrix}0&1\\1&0\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & 0\end{bmatrix}$ $=\sigma^1$

$t^2 = \frac{1}{2}\begin{bmatrix}\boxed{\begin{matrix}0&-i\\i&0\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & 0\end{bmatrix}$ $=\sigma^2$

$t^3 = \frac{1}{2}\begin{bmatrix}\boxed{\begin{matrix}1&0\\0&-1\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & 0\end{bmatrix}$ $=\sigma^3$

$t^4 = \frac{1}{2}\begin{bmatrix}\boxed{0}&0&\boxed{1}\\0&0&0\\\boxed{1}&0&\boxed{0}\end{bmatrix}$ $=\sigma^1$

$t^5 = \frac{1}{2}\begin{bmatrix}&&-i\\&&\\i&&\end{bmatrix}$

$t^6 = \frac{1}{2}\begin{bmatrix}&&\\&0&1\\&1&0\end{bmatrix}$

$t^7 = \frac{1}{2}\begin{bmatrix}&&\\&0&-i\\&i&0\end{bmatrix}$

$t^8 = \underbrace{\frac{1}{2\sqrt{3}}\begin{bmatrix}1&&\\&1&\\&&-2\end{bmatrix}}_{\text{Why?}}$ s.t.

$$\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$$

$\mathrm{Tr}[t^8 t^8]$

$= \frac{1}{2^2}\frac{1}{\sqrt{3}^2}\underbrace{(1^2+1^2+2^2)}_{6}$

$= \frac{1}{4}\frac{1}{3}6 = \frac{1}{2}$

Ex: $\mathrm{Tr}[t^1 t^1] = \frac{1}{2^2}\underbrace{\mathrm{Tr}[\overset{\mathbb{1}_{2\times2}}{(\sigma^1)^2}]}_{2} = \frac{1}{2}$

... but it's nicer if

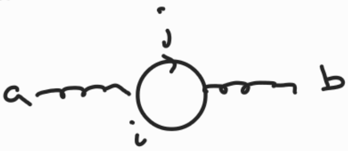
$\mathrm{Tr}[t^a t^a] = \delta^{ab}$ so we will later use this

Note that $\mathrm{Tr}[t^a t^b] = \underset{\uparrow}{T_R}\,\delta^{ab}$ $\Rightarrow$

any normalization

recall def of notation above ↓

$t^a_{ij}\, t^b_{ji} =$



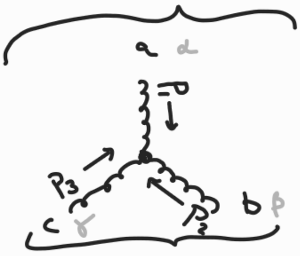
$= T_R \underbrace{\delta^{ab}}_{\shortparallel}$

$\Rightarrow$

$= T_R$

- triple-gluon vertex

antisym in 1 <-> 2 etc.



$$-ig\underbrace{\left(g^{\alpha\beta}(p_1-p_2)^\gamma + g^{\beta\gamma}(p_2-p_3)^\alpha + g^{\gamma\alpha}(p_3-p_1)^\beta\right)}_{\text{antisym in } 1 \leftrightarrow 2 \text{ etc.}}$$

$$\equiv i \underbrace{f^{abc}}_{SU(3)}$$

def of notation

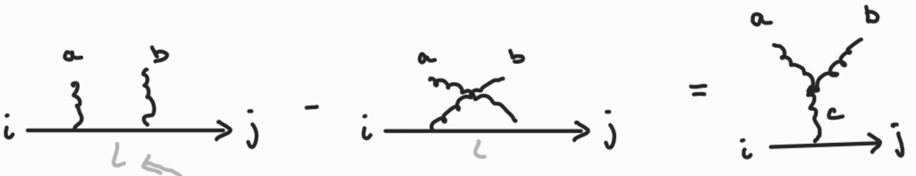
SU(3) structure constants

def of structure constants

$$[t^a, t^b] = if^{abc}\, t^c$$

$$(t^a t^b - t^b t^a)_{ij} = i f^{abc}\, t^c_{ij}$$

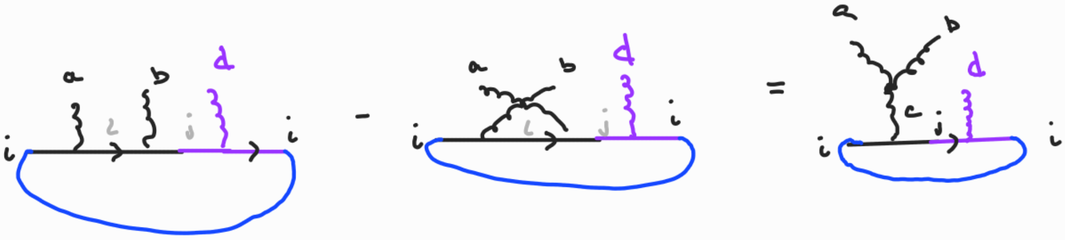
$$t^a_{il}\, t^b_{lj} - t^b_{il}\, t^a_{lj} = i f^{abc}\, t^c_{ij}$$



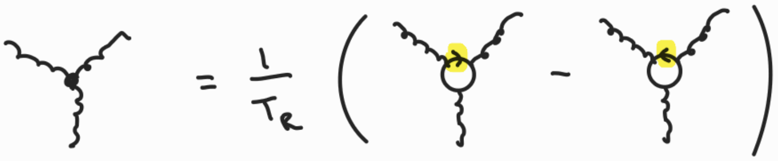
Label irrelevant omit

Multiply with t^d and take the trace

$$t^a_{il}\, t^b_{lj}\, t^d_{ji} - t^b_{il}\, t^a_{lj}\, t^d_{ji} = i f^{abc}\, \underbrace{t^c_{ij}\, t^d_{ji}}_{T_R\, \delta^{cd}}$$

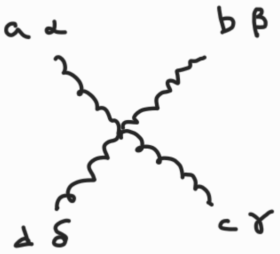


- Letting position denote index (abd here) we have



[Note: Different conventions for arrow directions in different sources]

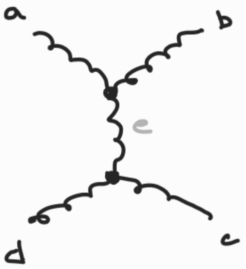
- four-gluon vertex



= [vertex must be fully sym. when changing color & kinematics]

a&b

$$= \underbrace{if^{abe}\ if^{ecd}} \times i g_s^2 \times \underbrace{\left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right)}_{\text{α (~a) is once contracted with γ(~c) and once δ (d) As color is antisym in a↔b kinematics must be in α↔β}}$$



a&c

$$+ \underbrace{if^{ace}\ if^{edb}} \times i\, g_s^2 \times \underbrace{\left(\delta^{\alpha\delta} \delta^{\beta\gamma} - \delta^{\alpha\beta} \delta^{\gamma\delta}\right)}_{\text{antisym in } \alpha \leftrightarrow \gamma}$$

a&d

$$+ \underbrace{if^{ade}\ if^{ebc}} \times i g_s^2 \times \underbrace{\left(g^{\delta\beta} g^{\alpha\gamma} - g^{\delta\alpha} g^{\gamma\beta}\right)}_{\text{antisym in } \alpha \leftrightarrow \delta}$$

Note: no "new" color structure

Dealing with color

- We always sum over outgoing color
- We always average over incoming color
- We always sum over color of all internal states

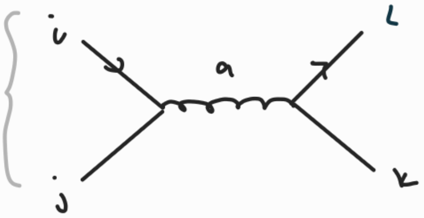
⇒

We always sum over all color

The spin analogy is to never measure spin z, only total spin

Ex Consider the color structure for $q\bar{q} \to q\bar{q}$ (different flavor)

not full feynman diagram, only color



$$= t^a_{ij} \, t^a_{kl}$$

def of notation

We want the color-summed/averaged version of this

$$\frac{1}{3}\sum_{i=1}^{3}\frac{1}{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left|\underbrace{\sum_{g=1}^{8} t^g_{ij} \, t^g_{kl}}_{A}\right|^2$$

can pick rep. of generators and sum over $3^4 \cdot 8 = 648$ terms

... but I don't want to

Algebraic way

$$\frac{1}{3^2}\sum_{ijkl=1}^{3}\left(\sum_{g=1}^{8} t^g_{ij}\, t^g_{kl}\right)^*\left(\sum_{h=1}^{8} t^h_{ij}\, t^h_{kl}\right)$$

generators are hermitian

$$= \frac{1}{3^2}\sum_{ijkl}\sum_{gh} t^g_{ji}\, t^g_{lk}\, t^h_{ij}\, t^h_{kl}$$

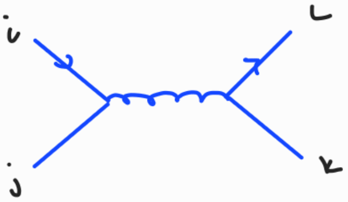
Do you see what this is?

$$= \frac{1}{3^2}\sum_{gh} \underbrace{Tr[t^g t^h]}_{T_R\, \delta^{gh}}\ \underbrace{Tr[t^g t^h]}_{T_R\, \delta^{gh}}$$

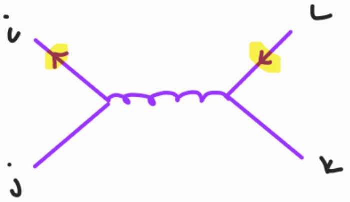
$$= \frac{1}{3^2} T_R^2 \underbrace{\sum_g \delta^{gg}}_{N_c^2-1=8} = T_R^2\, \frac{N_c^2-1}{N_c^2}$$

Bird track way

$$A = \sum_{g=1}^{8} t^g_{ij}\, t^g_{kl} =$$



$$A^* = \sum_{g=1}^{8} t^g_{ji}\, t^g_{lk} =$$

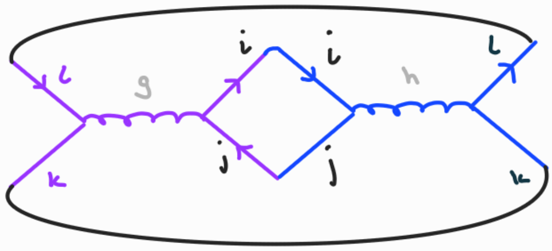


generators are hermitian

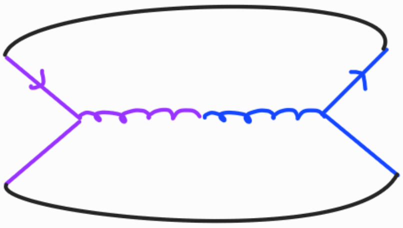
We always sum over all contracted indices ⇒ no need for dummy indices

$$\sum_{ijklgh} t^g_{lk}\, t^g_{ji}\, t^h_{ij}\, t^h_{kl}$$

$$\frac{1}{N_c^2}\sum_{color} A^* A = \frac{1}{N_c^2}$$



$$= \frac{T_R}{N_c^2}$$



$$= \frac{T_R}{N_c^2}$$

$$= \frac{T_R^2}{N_c^2}$$

$$\underbrace{}_{\sum_a \delta^{aa} = N_g}$$

$$= N_c^2 - 1$$

$$= \frac{T_R^2}{N_c^2}\left(N_c^2 - 1\right)$$

NOTE: The scalar product is a real number, so it doesn't matter which color structure we conjugate

Graphical color rules

Generally to contract any color structure it is enough to know a few rules:

- Dimension:

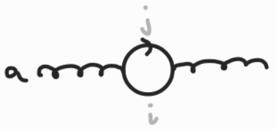
$$= \sum_i \delta_{ii} = N_c$$

$$= \sum_a \delta^{aa} = N_g = N_c^2 - 1$$

- Trace:

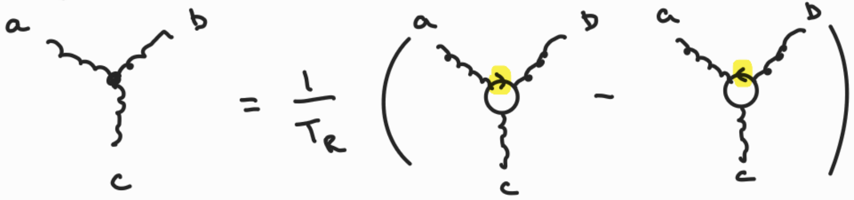
$$= \sum_i t^a_{ii} = \mathrm{Tr}\,[t^a] = 0$$

SU(N) generators are traceless

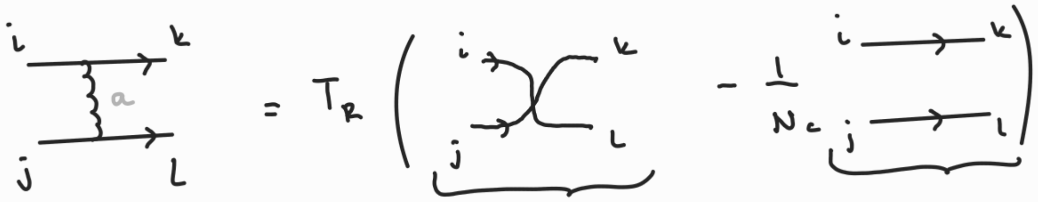


$$= \sum_{ij} t^a_{ij}\, t^b_{ji} = \mathrm{Tr}\,[t^a t^b] = T_R\, \delta^{ab}$$

- Algebra



- Fierz :

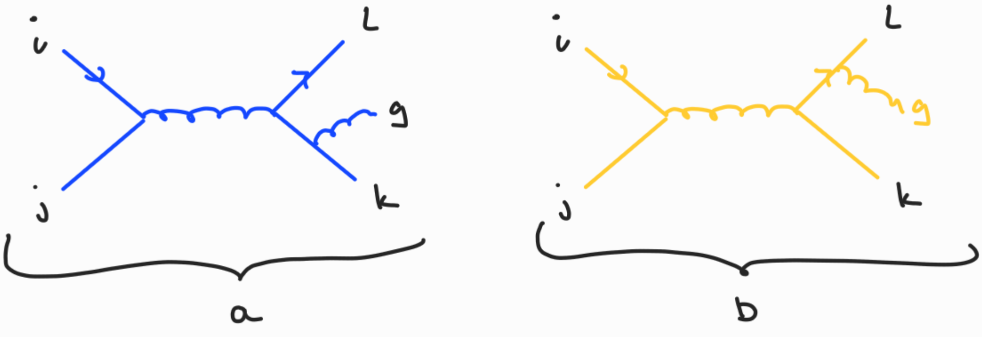


Note: You may have seen Fierz drawn in a twisted way

$$t^a_{ik}\, t^a_{jl} = T_R \left(\delta_{il}\, \delta_{jk} - \frac{1}{N_c}\, \delta_{ik}\, \delta_{jl} \right)$$

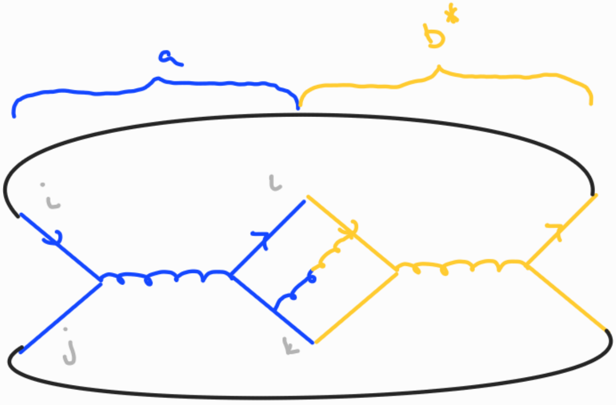
(We will motivate the fierz identity later as a completeness relation, but one can also prove it using matrices.)

Ex Calculate the interference between the diagrams

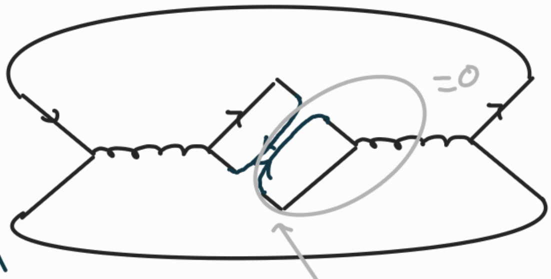


swap arrows & mirror

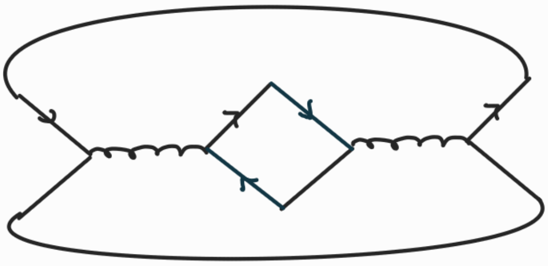
$\sum_{\text{color}} A A^* =$



Fierz $= T_R \Bigg($



$- \frac{1}{N_c}$



$\Bigg)$

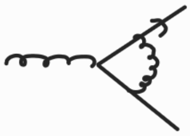
color lines can only go one way due to quark arrows

$$= \frac{-T_R^3}{N_c} \left(N_c^2 - 1\right)$$

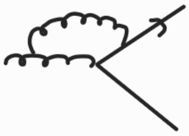
L1: Exercises

1) Calculate the color factors associated with vertex corrections:

(a)



(b)



2) Prove that the quadratic Casimir operator* $\sum_a (t^a)^2$ is indeed a Casimir operator, proportional to the identity

$= C_F$

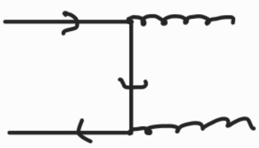
Calculate C_F.

for the adjoint (gluon) representation we instead have

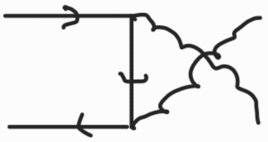
$= C_A$

Find the value of C_A

3) Calculate the interference term between



and



* A Casimir operator is an operator that commutes with all the elements in the algebra (and therefore also with all group elements).

Lecture 2

Today: systematic treatment of color

- The color space is a vectorspace of total color siglets
- ... ⇒ we can use our full intuition for vector spaces
- ... in particular bases, trace bases
- Gluon exchange & gluon emission
- Kidonakis, Oderda, Sterman, Evolution of color exchange in QCD hard scattering, hep-ph/9803241
- M. Sjodahl, Color structure for soft gluon Resummation: A general recipe, 0906.1121

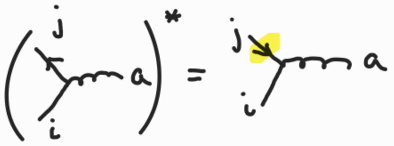
The color singlet vector space

- The only color in $\mathcal{L}(QCD)$ sit in t^a_{ij}, if^{abc}
- We are interested in color summed/averaged quantities

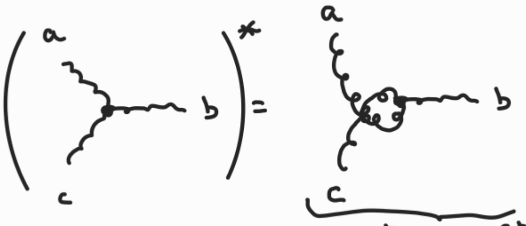
  ⇝ only need fully contracted color structure
- Counting incoming plus outgoing color, the total amplitude is a color singlet
- When calculating $\sum_{\text{color}} A^* A$, we note that

$$(t^a_{ij})^* = t^a_{ji} \Rightarrow$$

hermitian generators



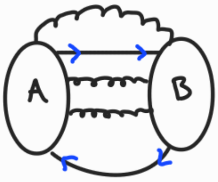
$$(if^{abc})^* = -if^{abc} \Rightarrow$$



$if^{acb} = -if^{abc}$

⇒ The conjugated amplitude can be obtained by swapping arrow directions and mirroring (⇒ swap sign of f^{abc}).

- Therefore the color summed scaler product between color structures A and B can always be represented by A contracted with B



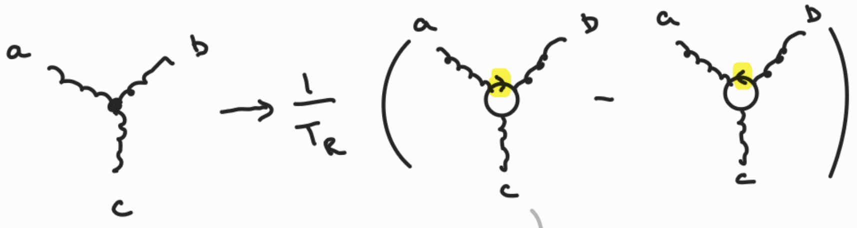
where in B arrows are swapped and (gluon) vertices are mirrored.

- This is indeed a (real) scaler product
  - $A \cdot B = B \cdot A$ (real as polynomial in N_c)
  - Distributive for vector addition
    $A \cdot (B + C) = A \cdot B + A \cdot C$ } obvious
  - Scalar multiplication
    $(aA) \cdot (bB) = ab\, A \cdot B$ } obvious
  - positive semidefinite
    $A \cdot A \geq 0.$
- Note: As we have a scalar product and a vector space we can use all our intuition from linear algebra.

Color contraction

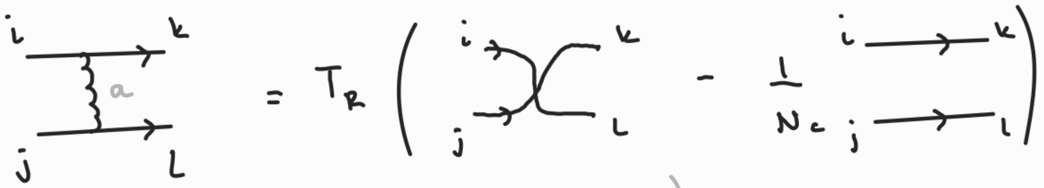
All QCD color structure can be contracted by

1) Applying the replacement



(⇝ no 3g vertices)

2) Internal gluons (all gluons in a scalar product) can be removed using the Fierz identity



(⇝ no internal gluons)

3) After this there are only contracted quark-lines remaining, and they simply give  $= N_c$.

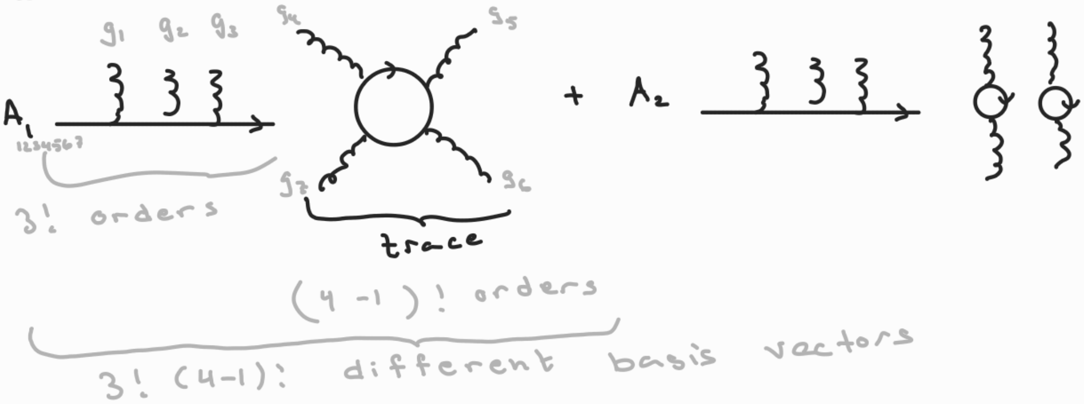
This proves that we can contract any color structure.

Note 1: Only considering color (no kinematic factors) the scalar product is real (no factors of i, the $if^{abc} \rightarrow$ traces), i.e. we have a real vector space.

Note 2: In particular we can construct a basis for the color vector space.

Note 3: This contraction strategy also instructs us <u>how</u> to construct a spanning set of color structures: Remove 3g vertices with 1) and internal gluons with 2). This results in a sum of products of open and closed quark-lines

Note that the order of gluons matters



3! orders

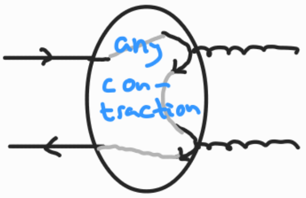
trace

(4 -1)! orders

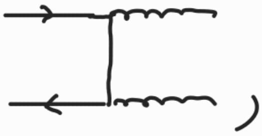
3! (4-1)! different basis vectors

+ all other combinations & orders

Ex A basis for $q\bar{q} \to gg$ is given by



$V_1 =$

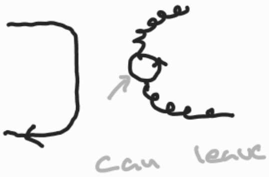


,

$V_2 =$

,

$V_3 =$

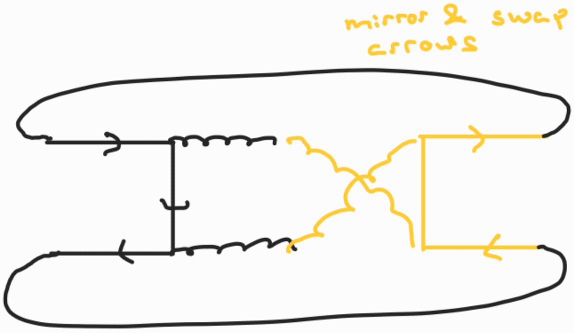


can leave out

not there at tree level

This basis is not orthogonal, from the exercise yesterday:

$v_1 \cdot v_2 =$



$=$

Fierz

$= T_R \Big($

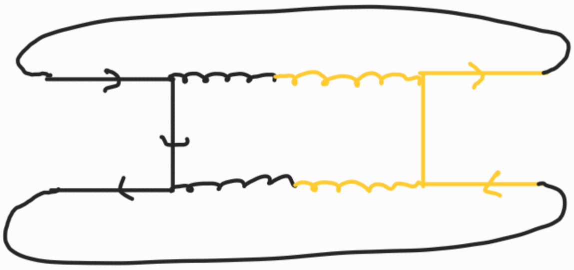
$- \frac{1}{N_c}$

$\Big) = - \frac{T_R}{N_c}$

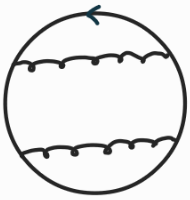
$$= - \frac{T_R^2}{N_c} (N_c^2 - 1) \sim N_c$$

... but the leading N_c is on the diagonal,

$v_1 \cdot v_1 =$



$=$



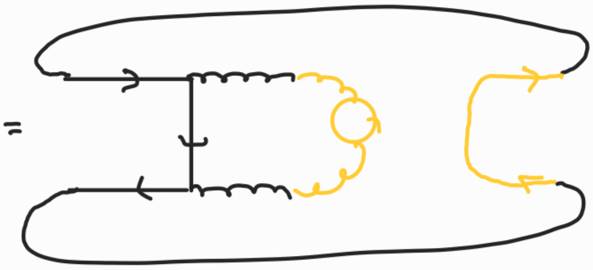
$= C_f^2$

$$= C_f^2 N_c = T_R^2 \frac{(N_c^2 - 1)^2}{N_c^2} N_c$$

$$= T_R^2 \frac{(N_c^2 - 1)^2}{N_c} \sim N_c^3$$

Similarly

$V_3 \cdot V_1 =$



$$= T_R C_f N_c = T_R^2 (N_c^2-1) \sim N_c^2$$

With the above we can write down the scalar product matrix

$$\begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & v_1\cdot v_3 \\ v_2\cdot v_1 & v_2\cdot v_2 & v_2\cdot v_3 \\ v_3\cdot v_1 & v_3\cdot v_2 & v_3\cdot v_3 \end{pmatrix} =$$

$v_2\cdot v_2 = v_1\cdot v_1$

$$\begin{pmatrix} T_R^2 \frac{(N_c^2-1)^2}{N_c} & -T_R^2 \frac{(N_c^2-1)}{N_c} & T_R^2 (N_c^2-1) \\ -T_R^2 \frac{(N_c^2-1)}{N_c} & T_R^2 \frac{(N_c^2-1)^2}{N_c} & T_R^2 (N_c^2-1) \\ T_R^2 (N_c^2-1) & T_R^2 (N_c^2-1) & T_R^2 N_c (N_c^2-1) \end{pmatrix}$$

$v_2\cdot v_3 = v_1\cdot v_3$

given by symmetry

$v_3\cdot v_3$ trivial

tree-level

$$\sim \begin{pmatrix} N_c^3 & N_c & N_c^2 \\ N_c & N_c^3 & N_c^2 \\ N_c^2 & N_c^2 & N_c^3 \end{pmatrix}$$

Note that

- The leading color sits on the diagonal $\propto N_c^{N_g + N_{q\bar{q}}}$

  \# $q\bar{q}$-pairs, in + out, here N_c^{2+1}

- For tree-level color structure (as long as we have 0-1 $q\bar{q}$-pairs) the off-diagonal parts are suppressed by two powers of N_c (or more in general)

- Beyond leading order or for >1 $q\bar{q}$-pair the off-diagonal terms can be suppressed by only one power of N_c

- Moreover, the "bases" constructed in this way are overcomplete for > N_c gluons + $q\bar{q}$-pairs (for example 4 gluons)

  spanning sets

- \# basis vectors scales ~ $(N_g + N_{q\bar{q}})!$,

  for only gluons $\frac{N_g!}{e}$ is a superb approximation.

- For only gluons charge conjugation reduces the number of independent terms

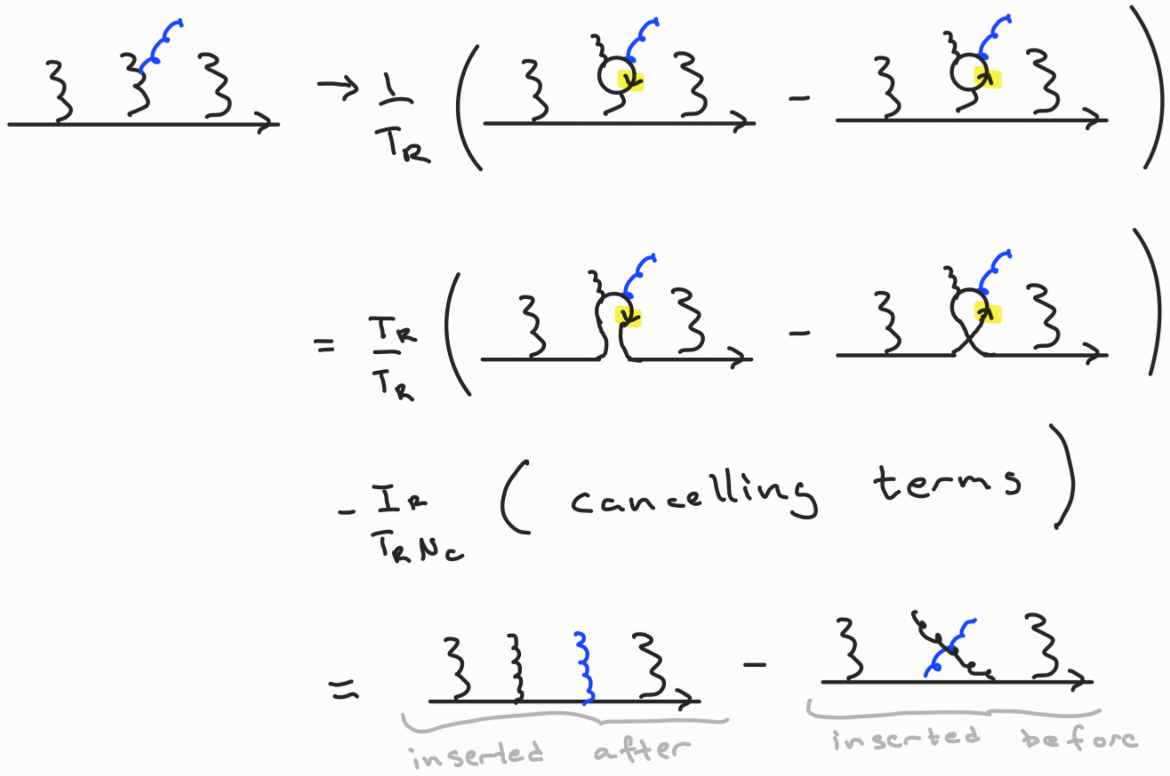
Why trace bases?

- Conceptually simple
- Effect of gluon emission and gluon exchange easily described

Gluon emission

Takes us from n to n+1 partons, i.e. a larger color space.

Consider



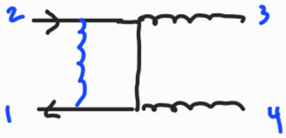
⇝ effect on basis vectors easy

Get at most 2 new basis vectors from each starting basis vector and each emitter.

Gluon exchange

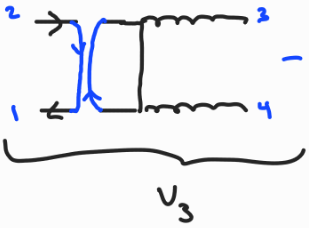
Consider

This would be $T_1 \cdot T_2$ in Andrew's lecture

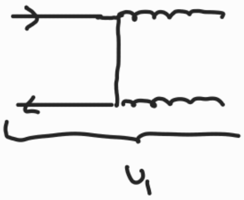


Assume $T_R = 1$

=



$- \frac{1}{N_c}$



so $\quad V_1 \longrightarrow -\frac{1}{N_c} V_1 + V_3$

Similarly $\quad V_2 \longrightarrow -\frac{1}{N_c} V_2 + V_3$

$$V_3 \rightarrow C_F V_3$$

We can collect this in a matrix describing the effect of gluon exchange

exchange between 1&2

$$\Gamma^{12} = \begin{pmatrix} -1/N_c & & 1 \\ & -1/N_c & 1 \\ & & C_F \end{pmatrix}$$

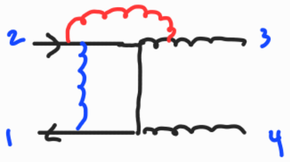
Generally we map from one space to the same space.

Generally one gets at most 4 new basis vectors (for each exchange and starting basis vector).

L2: Exercises

1a) In the basis above, calculate the matrix describing exchange between 2 and 3.

b) Calculate the color part of the two-loop exchange



2) Consider $qq \to qq$.

a) Construct the "trace" basis for $q_1 q_2 \to q_3 q_4$. (Trace basis is not a good name here.)

b) Calculate the scaler product matrix (color matrix)

Calculate the matrix describing gluon exchange between

i) q_1 & q_2

ii) q_1 & q_1

Comment: Generally the exchange between i and i gives a diagonal leading N_c contribution, C_F or C_A.

3) Construct the trace basis for $gg \to gg$

Lecture 3

- Color flow

  Maltoni, Paul, Stelzer, Willenbrook
  hep-ph/0209271

- Chirality flow
  - Lifson, Reuschle & Sjodahl ← massless QED & QCD
    2003.05877
  - Alnefjord, Lifson, Reuschle & Sjodahl ← full SM

    2011.10075

Color flow

Instead of using adjoint rep, translate everything to fundamental rep.,

!?

but this is not the same ?!

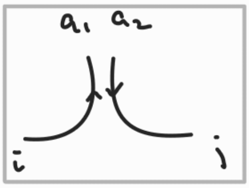
Can this work ? Yes, effectively.

How ? Shift ⌇ out of vertices

⇜ sits in propagators & external particles

- quark - gluon vertex

$$\to i\frac{g}{\sqrt{2}}\gamma^\alpha$$

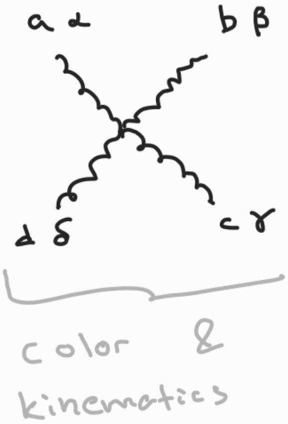


color (flow) structure

- triple - gluon vertex

$$= \longrightarrow i\frac{g}{\sqrt{2}}\left(g^{\alpha\beta}(p_1-p_2)^\gamma + g^{\beta\gamma}(p_2-p_3)^\alpha + g^{\gamma\alpha}(p_3-p_1)^\beta\right)$$

- 4 - gluon vertex



=

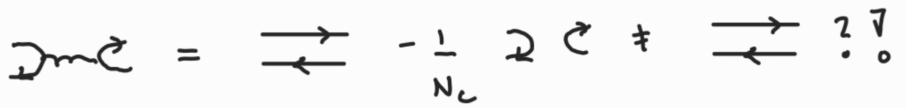
[You can prove this but it takes a bit of effort]

$$= \frac{ig^2}{2}\sum_{\substack{\text{six non-}\\\text{cyclic}\\\text{perms}}}\left[\left(2g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\beta}g_{\delta\gamma} - g_{\alpha\delta}g_{\beta\gamma}\right)\right.$$

$\times$

$$\left.\right]$$

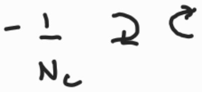
L3:2

... but 

So how can this work ?

easy to prove, try it

Remark: As soon as a gluon is exchanged between a gluon and a (quark or gluon) the $\frac{1}{N_c}$-terms cancel

⇒ we only need the 

part for propagators <u>between quarks</u> or for external gluons attached to quarks, but then we <u>do</u> need them.

⇝ insert again in propagator.

Note
- Color flow bases are very convenient for taking the leading N_c limit
- Color flow bases are very good for assigning explicit colors r/g/b for ⇝ good for samplings over color
- Products of δ^i_j are fast to evaluate ⇝ good for sampling color structure
- ... but the bases are even more overcomplete

De Angelis, Forshaw
Plätzer 2007.09648

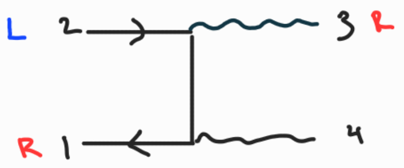
Forshaw, Holguin &
Plätzer, 2112.13124

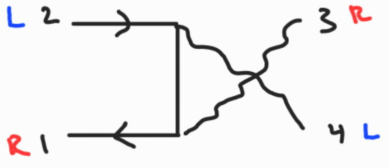
Chirality flow

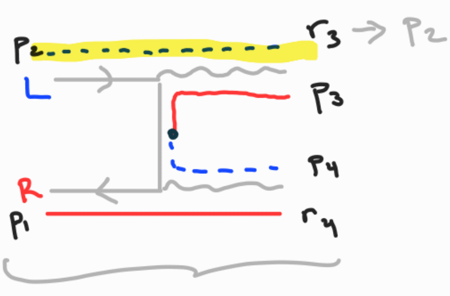
- see slides

Ex Consider $\overbrace{\underset{R}{e^+_{1+}} \; \underset{L}{e^-_{2-}}}^{}$ as incoming $\rightarrow \overset{R}{\gamma^-_3} \; \overset{L}{\gamma^+_4}$ as outgoing

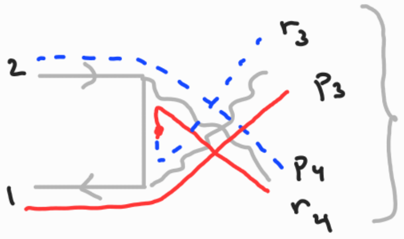
We have 2 diagrams





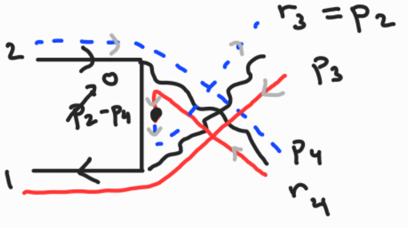


$[p_2 p_2] = 0$



can not make vanish due to contraction with p_3/p_4 (rather than r_3, r_4)

... but we can make the first vanish by letting $r_3 = p_2$, leaving



prop vert
(i) (i²)

$$= \underbrace{\frac{-2ie^2}{[3 \underset{=2}{r_3}] \cancel{\langle r_4 4\rangle}}}_{\text{from pol. vecs}} \frac{1}{(p_2-p_4)^2} [2 4]\cancel{\langle r_4 4\rangle}\Big(-\underbrace{[4 \overset{=2}{r_3}]}_{[24]}\langle 3 1\rangle\Big)$$

r_4 drops out as it must since unphysical

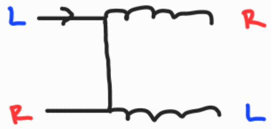
$$= \frac{-2ie^2 [24]^{\cancel{2}} \langle 31\rangle}{[32]\Big(\underbrace{-\cancel{[24]}\langle 42\rangle}_{S_{24}}\Big)} = \frac{2ie^2 [24]\langle 31\rangle}{[32]\langle 42\rangle}$$

$$S_{24} = (p_2+p_4)^2 = 2p_2\cdot p_4 = [24]\langle 42\rangle$$

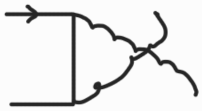
$$= -(-2p_2\cdot p_4) = -(p_2-p_4)^2$$

Ex: Consider $q_L \bar q_R \to g_{1^-_R}\, g_{2^+_L} \rightsquigarrow 3$ Feynman diagrams

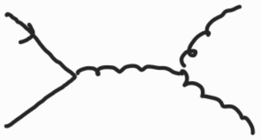
(1)



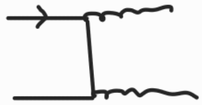
(2)



(3)



consider the color structure

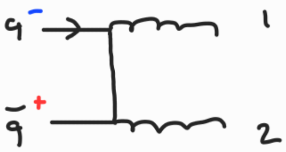


(Let $T_R = 1$)

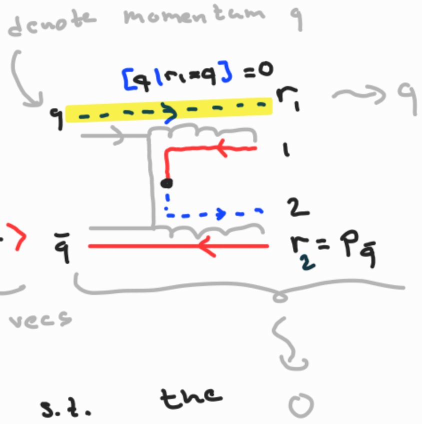
$\rightsquigarrow$ (1) & (3) contribute

Let's calculate (1)

c.f. slide 13-14



$$\rightsquigarrow \overset{\text{prop vert}}{\overset{(i)\,(i^2)}{}} \frac{-i g_s^2}{s_{1,q}} \underbrace{\frac{1}{[1 r_1]} \frac{1}{\langle r_2 2\rangle}}_{\text{from pol. vecs}}$$

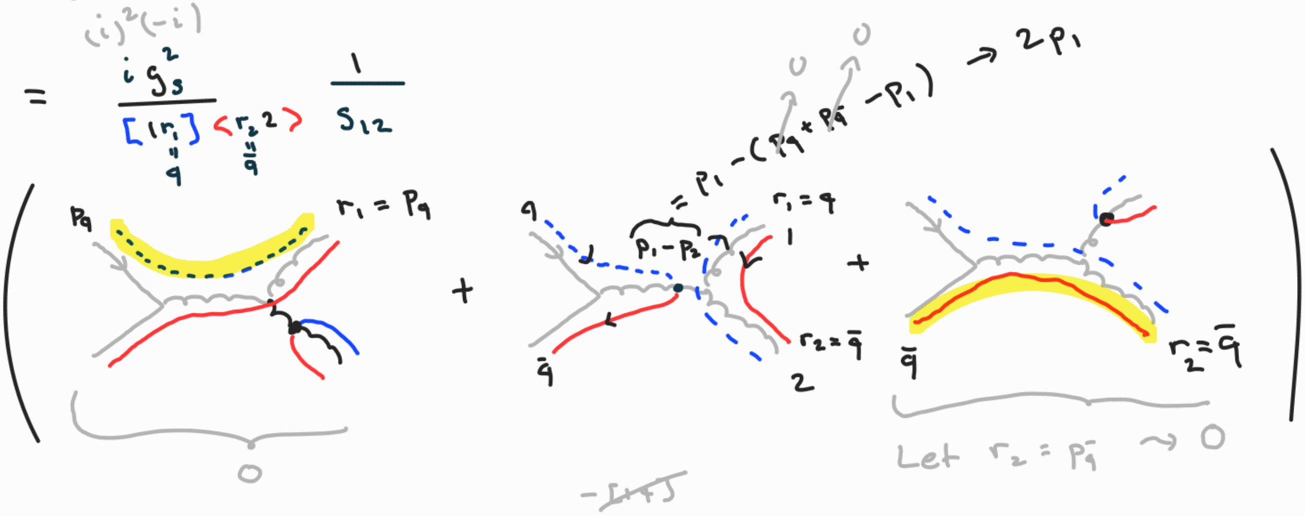


... but we can pick $r_1 = q$ s.t. the diagram vanishes.

We still need to calculate (3):

gluon vert p25

$$= \overset{\text{vert prop}}{\overset{(i)^2(-i)}{}} \frac{i g_s^2}{[1 \underset{q}{\underset{\shortparallel}{r_1}}] \langle \underset{\bar q}{\underset{\shortparallel}{r_2}} 2\rangle} \frac{1}{s_{12}}$$



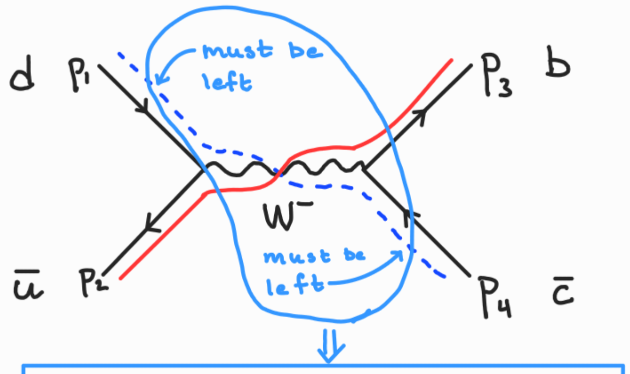
$$= \frac{i g_s^2}{[1 \cancel{q}]\langle \bar q 2\rangle} \frac{1}{\langle 12\rangle [21]}\; 2 \overset{\cancel{-[1q]}}{\overbrace{[q1]}} \langle 1 \bar q\rangle [2q] \langle 1 \bar q\rangle$$

$$q = p_1 + \overset{0}{\cancel{p_2}} - \overset{0}{\cancel{p_{\bar q}}}$$
$$= [\cancel{21}]\langle 1 \bar q\rangle$$

$$= -i g_s^2 \underbrace{\frac{\langle \bar q 1\rangle^{2} [2q]}{\langle \bar q 2\rangle \langle 12\rangle [\cancel{21}]}}_{\text{done if we want}} \underbrace{\frac{\langle q \bar q\rangle \langle q1\rangle}{\langle q\bar q\rangle \langle q1\rangle}}_{1} = \frac{-i g_s^2 \overset{-}{\langle \bar q 1\rangle}^3 \overset{+}{\langle q1\rangle}}{\underset{+}{\langle q1\rangle} \underset{+}{\langle 12\rangle} \underset{-}{\langle 2 \bar q\rangle} \underset{+}{\langle \bar q q\rangle}}$$

Example: W-exchange

Consider $W^\pm$ exchange, say $d\bar{u} \to b\bar{c}$



the Dirac spinor

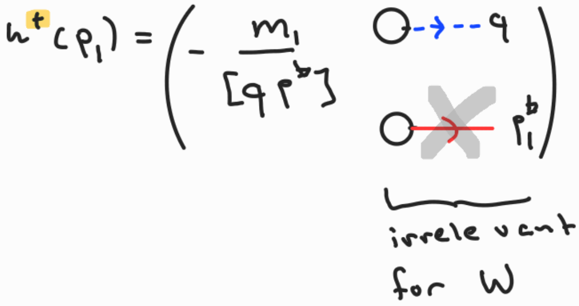
p_2 u_1 → left spinor only

→ inflowing fermion is left.

Only one possible chirality flow

Assume that particle 1 has positive spin along $s_1^\mu = \frac{1}{m_1}\left(p_1^\mu - \frac{m_1^2}{p_1\cdot q} q^\mu\right)$

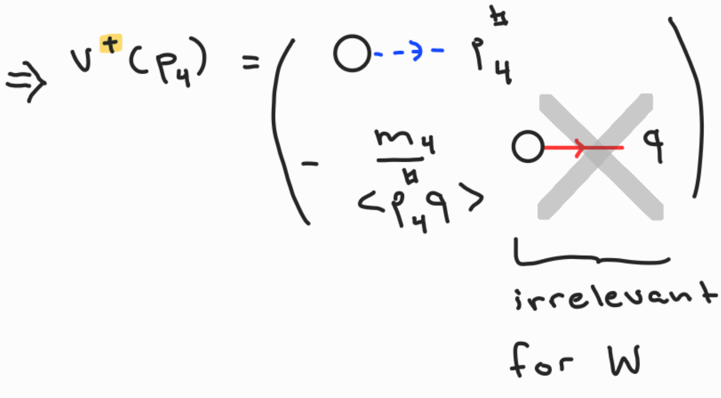
p.19



How is the spin of particle 4 directed

Particle 4 can not have q for its left spinor as $[qq]=0$

⇝ must have q for the right spinor



⇒ particle 4 has positive spin along $s_4^\mu = \frac{1}{m_4}\left(p_4^\mu - \frac{m_4^2}{p_4\cdot q} q^\mu\right)$

L3: Exercises

1) Prove that

$$\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\,\tau_{\mu}^{\dot{\gamma}\eta} = \delta_{\alpha}^{\eta}\,\delta^{\dot{\gamma}}_{\dot{\beta}}$$

You may use the SU(N) fierz identity for SU(2).

2) Prove that the amplitude for

$$\underbrace{\underset{R}{e^{+}_{1+}}\;\underset{L}{e^{-}_{2-}}}_{\text{as incoming}} \to \overbrace{\overset{R}{\gamma^{-}_{3}}\;\overset{R}{\gamma^{-}_{4}}}^{\text{as outgoing}}$$

is zero.

(We know that this must vanish, as only 1 left)

3) Calculate the amplitude for massless

$$e^{+}_{1+}\,e^{-}_{2-} \longrightarrow \underset{L}{q_{3+}}\,\underset{R}{\bar{q}_{4-}}\,\underset{R}{g_{5}}$$

Pick some reference vector that simplifies the calculation.

Lecture 4

- Orthogonal multiplet bases
  (Keppeler & Sjodahl 1207.0609, Sjodahl & Thorén 1809.05002)
- Symmetrizers & anti-symmetrizers
  (See Stefan Keppeler's lecture notes 1707.07280)

Orthogonal bases

As we have seen the trace bases are not orthogonal, and severely overcomplete for many partons.

How can we construct orthogonal bases?

Gram-Schmidt works of course but creates a mess

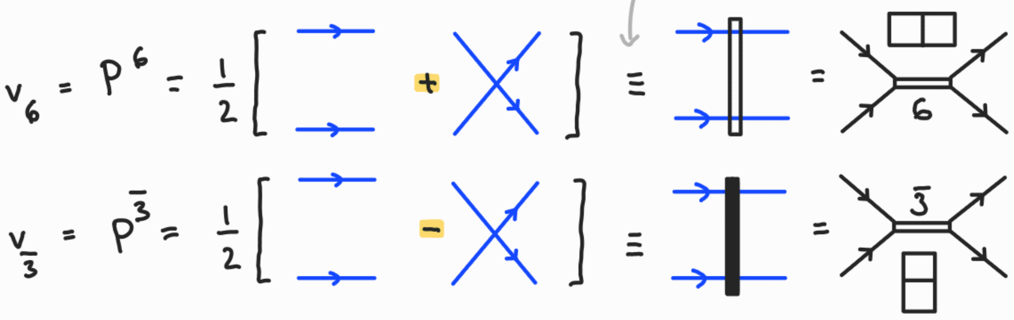
Consider the basis from L2: exercise 2

with scalar product matrix $\begin{pmatrix} N_c^2 & N_c \\ N_c & N_c^2 \end{pmatrix}$

How can we make this orthogonal?

Symmetrize!

def of notation



includes $\frac{1}{2} = \frac{1}{2!}$

This is our first multiplet basis

Where did the dimensions come from?

For the symmetric state, we can pick

obviously sym   sym. by construction

$|rr\rangle, |gg\rangle, |bb\rangle, \frac{1}{\sqrt{2}}(|rg\rangle+|gr\rangle), \frac{1}{\sqrt{2}}(|rb\rangle+|br\rangle), \frac{1}{\sqrt{2}}(|gb\rangle+|bg\rangle)$

i.e. 6 states that are symmetric under $1 \leftrightarrow 2$.

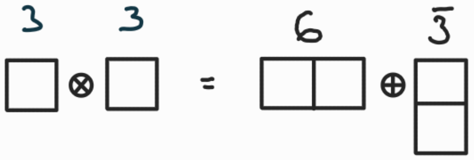
For the antisym. we can pick

$\frac{1}{\sqrt{2}}(|rg\rangle-|gr\rangle), \frac{1}{\sqrt{2}}(|rb\rangle-|br\rangle), \frac{1}{\sqrt{2}}(|gb\rangle-|bg\rangle)$

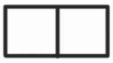
[Clearly the same color does not work]

i.e. 3 antisym. states. One can prove that this is the $\bar{3}$ rep.

We write this as

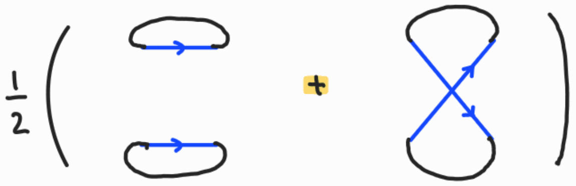


Note that $3 \times 3 = 6 + 3$

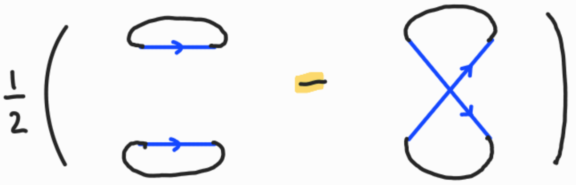
where  means that we should symmetrize and  antisymmetrize in the quarks.

The dimension is also given by the trace of the projector

The projector will do nothing on a sym. state and put an antisym state ⇝ 0



$= \frac{1}{2}(N_c^2+N_c) = 6$

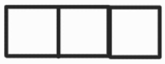


$= \frac{1}{2}(N_c^2-N_c) = 3$

Traceing is like summing over sym states

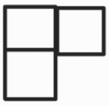
Now consider 3 quarks, then we have states which are

sym.

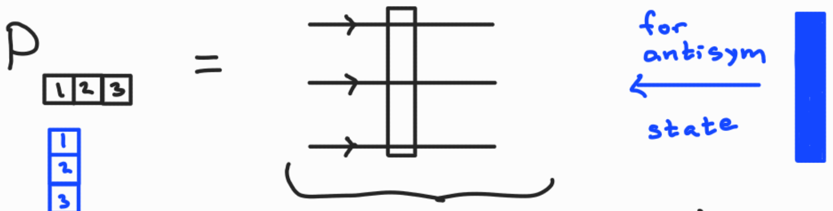


antisym.

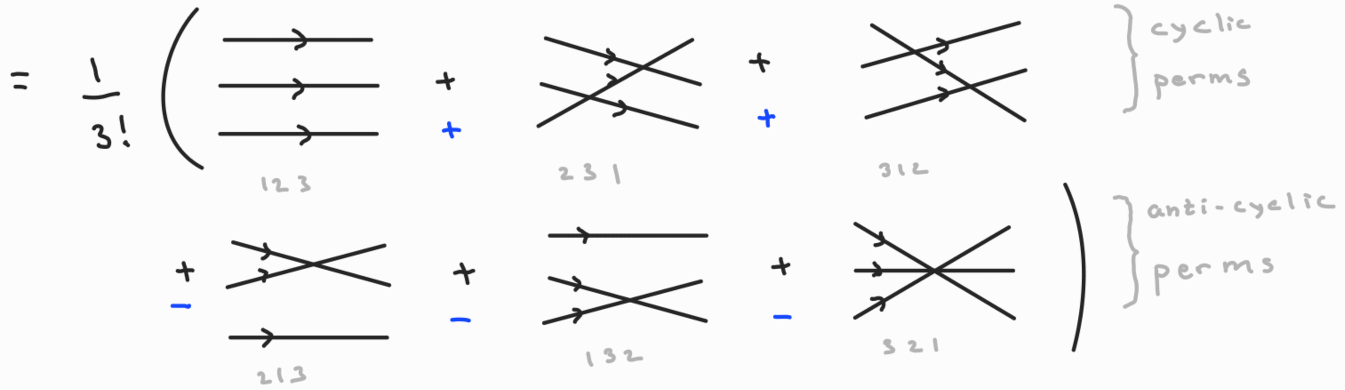
mixed symmetry:



- Symmetric state $\boxed{1}\boxed{2}\boxed{3}$



≡ fully sym state



obviously sym under any exchange of quarks

States: • $|rrr\rangle, |ggg\rangle, |bbb\rangle$ 3 options

- $\frac{1}{\sqrt{3}}\left(\underbrace{|rrg\rangle + |rgr\rangle + |grr\rangle}_{\text{or other set of colors}}\right)$ $\overset{\text{pick one color}}{3} \cdot \overset{\text{pick other color}}{2}$ options

- $\frac{1}{\sqrt{6}}\left(\underbrace{|rgb\rangle + |brg\rangle + |gbr\rangle}_{\text{cyclic}} + \underbrace{|grb\rangle + |bgr\rangle + |rbg\rangle}_{\text{anticyclic}}\right)$ } only one fully sym state when all colors are different

⇝ 10 options decuplet

The projector $P_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$ will project onto any of these states, whereas states which are not sym will vanish when acted on by $P_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$

- Anti-symmetric states

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ antisym in all indices

⇝ all indices different

⇝ only one state a singlet

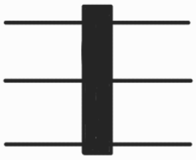
color i, r, g or b

$$\frac{1}{\sqrt{6}}\varepsilon^{ijk} c_i c_j c_k =$$

$$\frac{1}{\sqrt{6}}|rgb\rangle + |brg\rangle + |gbr\rangle - |grb\rangle - |bgr\rangle - |rbg\rangle$$

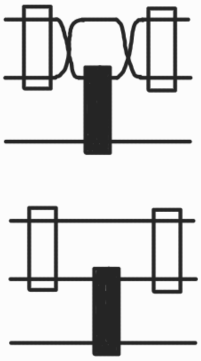
Projector

$$P_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} = Y_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} =$$



← includes $\frac{1}{3!}$

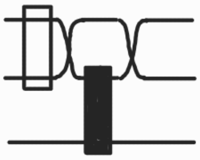
Alternatively one can use Hermitian Young operators



- Mixed states

4 compensates $\left(\frac{1}{2}\right)^2$

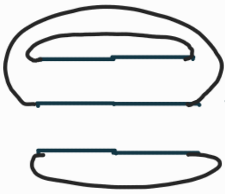
$$P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = Y_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = \frac{4}{\boxed{3}}$$



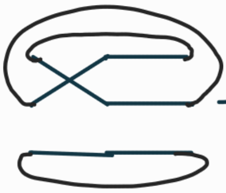
Hook length see below

$$\mathrm{Tr}\left(Y_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}\right) = \frac{4}{3}$$

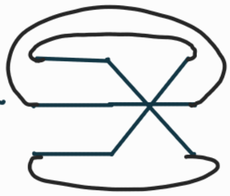
$$= \frac{4}{3}\Bigg($$



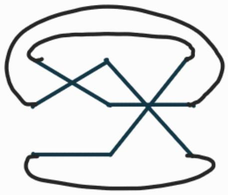
$+$



$-$



$-$



$$\Bigg)$$

$$= \frac{4}{3}\left(N_c^3 + N_c^2 - N_c^2 - N_c\right) = \frac{1}{3}\left(\underbrace{27-3}_{24}\right) = 8$$

This an octet

- Finally we have one more octet $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

The dimension of the rep is alternatively given by

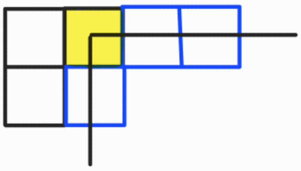
start in upper left corner

$$\frac{\prod_{\text{boxes}}\left(N_c+\begin{pmatrix}\text{steps}\\ \text{right}\end{pmatrix}-\begin{pmatrix}\text{steps}\\ \text{down}\end{pmatrix}\right)}{\underbrace{\prod_{\text{boxes}}\left(1+\begin{pmatrix}\text{boxes}\\ \text{below}\end{pmatrix}+\begin{pmatrix}\text{boxes}\\ \text{right}\end{pmatrix}\right)}_{\text{hook length}}}$$

$=$
↑
for

$$\begin{array}{|c|c|c|c|}\hline N_c & N_c+1 & N_c+2 & N_c+3\\ \hline N_c-1 & N_c \\ \cline{1-2}\end{array}$$

$$\frac{\begin{array}{|c|c|c|c|}\hline 3 & \cancel{4} & \cancel{5} & 6\\ \hline \cancel{2} & 3 \\ \cline{1-2}\end{array}}{\begin{array}{|c|c|c|c|}\hline \cancel{5} & \cancel{4} & 2 & 1\\ \hline \cancel{2} & 1 \\ \cline{1-2}\end{array}} = \frac{3\cdot 6\cdot 3}{2} = 27$$



hook length = # boxes through hook

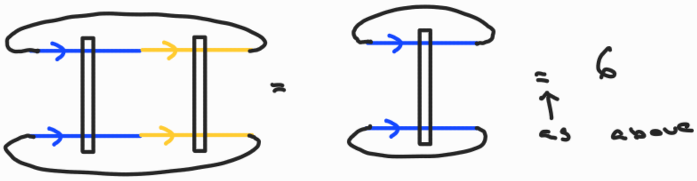
For the octet rep

$$\frac{\begin{array}{|c|c|}\hline 3 & \cdot 4\\ \hline 2 \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 3 & 1\\ \hline 1 \\ \cline{1-1}\end{array}} = 8$$

Above we can use the projectors as basis vectors in an orthogonal multiplet basis. This basis will be normalized s.t. $v_i \cdot v_i = d_i$, for example

$v_6 = P^6$, then

$v_6 \cdot v_6 =$



$=$ 6
↑
as above

$q\bar{q} \to q\bar{q}$

So far we have considered $qq \to qq$.

Note that an incoming quark comes with the same color factor as an outgoing anti-quark. Therefore we can comb the partons as we want

⇝ above basis works also for

$q\bar{q} \to q\bar{q}$.

Alternatively we can use $3 \otimes \bar{3} = 1 \oplus 8$ in birdtracks

spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$\underbrace{2 \otimes 2 = 1 \oplus 3}$

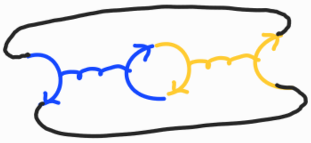
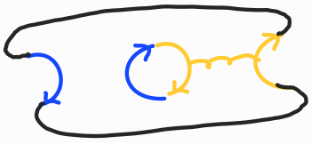
if counting dimension

$v_1 = P^1 = \frac{1}{N_c}$ 

, $v_8 = P^8 = \frac{1}{T_R}$ 

inserted to normalize as projector

Now,

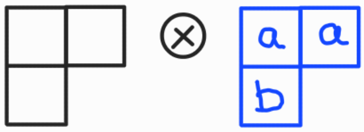
- $v_1^2 = \frac{1}{N_c^2} \bigcirc^2 = 1$
- $v_8^2 = \frac{1}{T_R^2}$  $= \frac{T_R^2(N_c^2-1)}{T_R^2} = 8$
- $v_1 \cdot v_8 = \frac{1}{N_c} \frac{1}{T_R}$  $= 0$

$\Rightarrow$ Scalar product matrix $\begin{pmatrix} 1 & \\ & N_c^2-1 \end{pmatrix}$

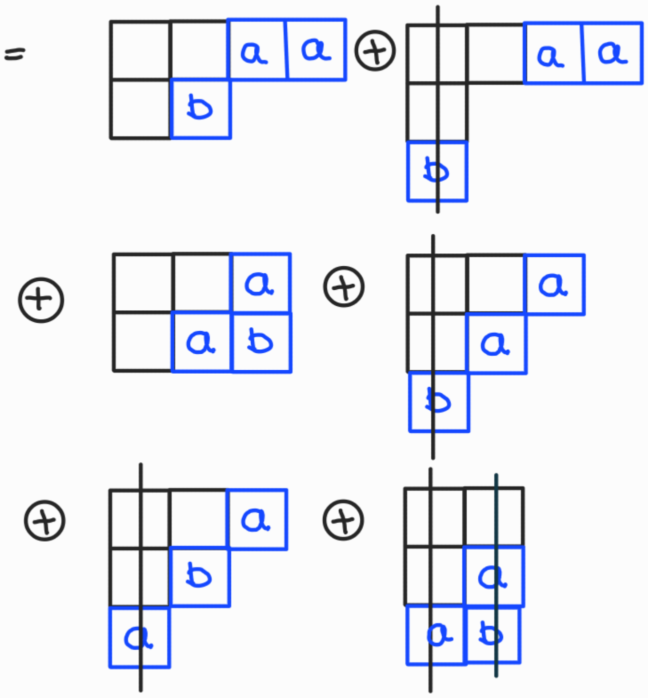
We can of course also make the basis orthonormal with $v_{rep} \to \frac{1}{\sqrt{d_{rep}}} v_{rep}$

gg → gg

To find out what basis vectors to expect, we use Young-tablean multiplication



(a) Put some label a in first row, some b in next etc.

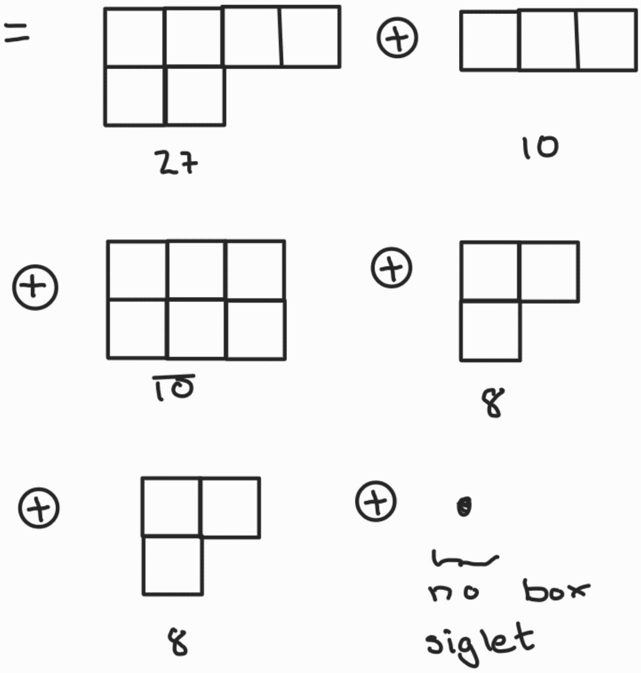


(b) Add boxes with the letters to the first diagram, first $\boxed{a}$, then $\boxed{b}$ etc. s.t.

1) At each stage the Young tab is admissible

2) Boxes with the same letter may not stand above each other

3) At any given box position, define $n_{a/b}$ to be # a/b above or to left. Then we must have $n_a \geq n_b \geq n_c$ etc.



calculate dims by hook rule

(c) Tabs are counted as different reps only if the differ labels

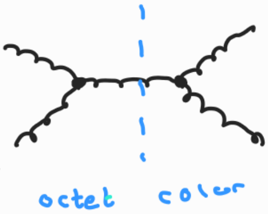
(d) Cancel columns with N_c boxes as they correspond to singlets.

Check: $8 \cdot 8 = 1 + 8 + 8 + 10 + 10 + 27$

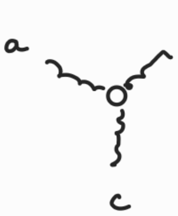
How can we find the projectors ?

- singlet $P^1 = \frac{1}{N_c^2-1}$

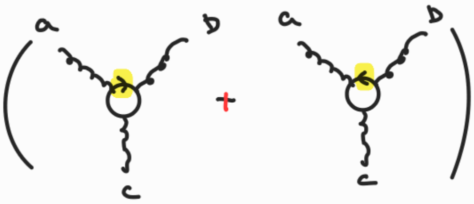
- octet $P^{8a} = \frac{1}{2N_c T_R}$



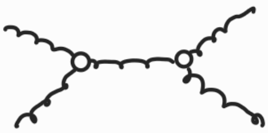
- Other octet, define sym. vertex



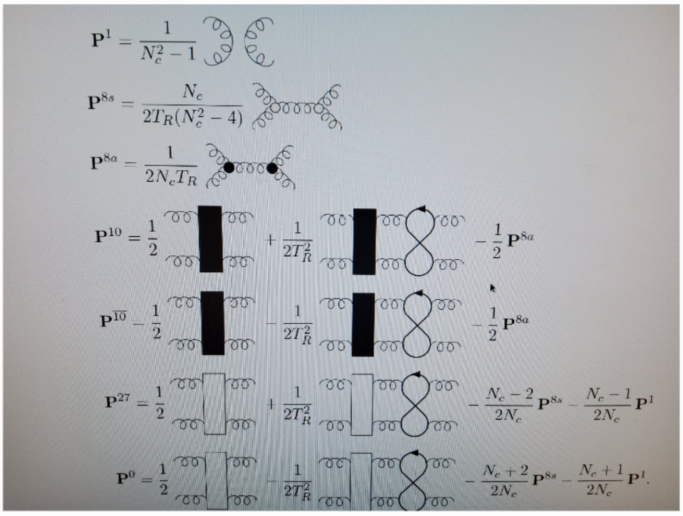
$\rightarrow \frac{1}{T_R}$



- $P^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$



- Higher reps



$\mathbf{P}^1 = \frac{1}{N_c^2-1}$

$\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$

$\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$

$\mathbf{P}^{10} = \frac{1}{2}$ $+ \frac{1}{2T_R^2}$ $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{\overline{10}} = \frac{1}{2}$ $- \frac{1}{2T_R^2}$ $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$ $+ \frac{1}{2T_R^2}$ $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^1$

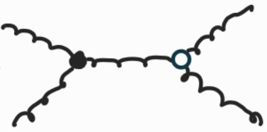
$\mathbf{P}^0 = \frac{1}{2}$ $- \frac{1}{2T_R^2}$ $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^1.$

- This 2g case first solved by Macfarlane, Sudbery & Weisz 1968 ($N_c=3$)
- Book by Cvitanovic
- General number of gluons Keppeler & Sjodahl 1207.0609

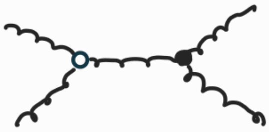
... but this is only 6 projectors for $N_c=3$, 7 for general N_c.

In an exercise from lecture 2 we had 9 vectors. What is missing?

$V^{8a\,8s} \propto$



$V^{8s\,8a} \propto$



Note: • For $N_c=3$, only 8 vector

- Actually $V^{8a\,8s}$, $V^{8s\,8a}$ can not appear from gluon vertices in the QCD Lagrangian. This comes from charge conjugation $q \leftrightarrow \bar{q}$
- V^{10} and $V^{\overline{10}}$ only appear together for the same rason

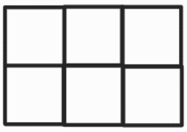
  $\rightsquigarrow$ 5 vectors for $N_c=3$

  $1,\ 8^s,\ 8^a,\ 10+\overline{10},\ 27$
- $V^{8a\,8s}$, $V^{8s\,8a}$ & $V^{10}-V^{\overline{10}}$ <u>can</u> appear in processes with quarks

L4: Exercises:

1) Prove that P^1 & P^8 from the $q\bar{q}$-example are indeed projectors

2) Confirm that the dimension of



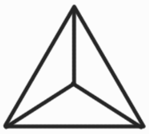
is 10.

3) For $gg \to gg$ verify the dimension of P^8 by taking the trace.

Verify that the normalization of $P^{8a}_{gg\to gg}$ makes $P^{8a}_{gg\to gg}$ a projector.

Lecture 5

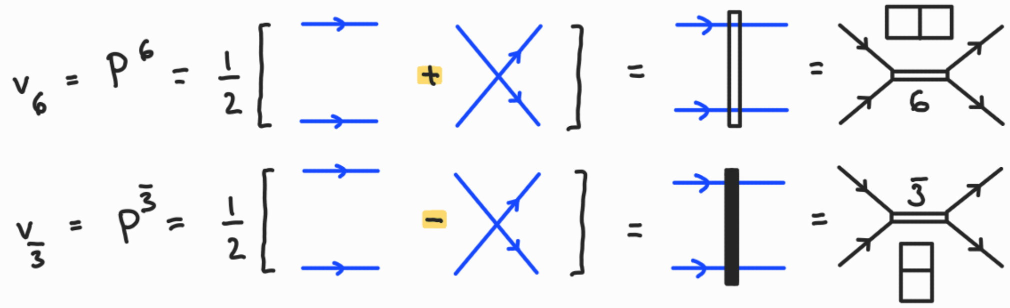
- Systematics of multiplet bases
  (Keppeler & Sjodahl 1207.0604, Sjodahl & Thorén 1809.05002)
- How to use a basis without constructing it / all you need is



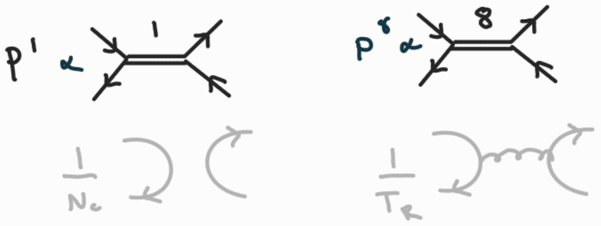
  (J. Alcock-Zeilinger, S. Keppeler, S. Plätzer & M. Sjodahl 2209.15013, 2312.16688)
- How to calculate with 6js
  (Book by Cvitanović)

So far we have encountered a few orthogonal multiplet bases

- $qq \to qq$

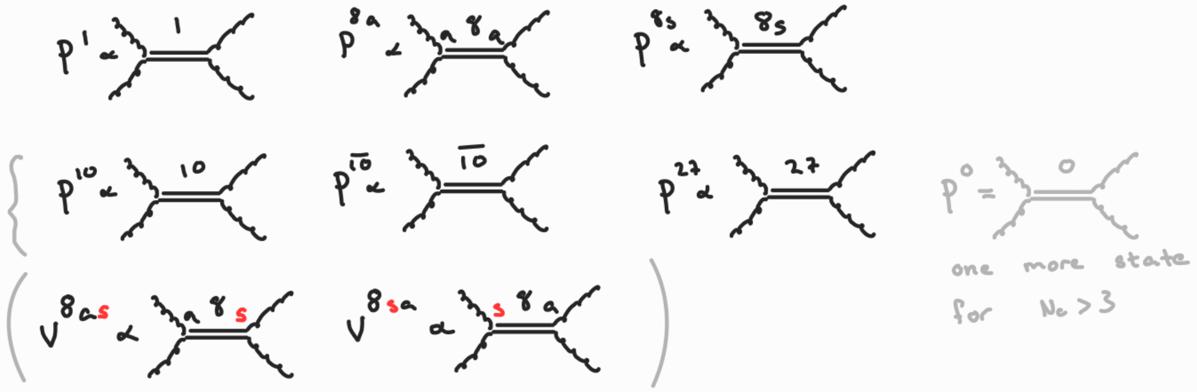


- $q\bar{q} \to q\bar{q}$

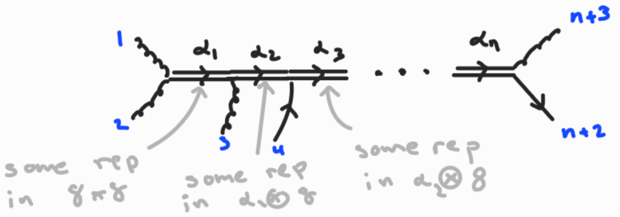


- $gg \to gg$

can use $P^{10} + P^{\overline{10}}$



- How does this work in general?



Note: fixed parton ordering ⇝ basis not unique

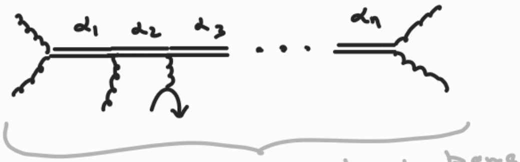
Orthogonal as contraction $d_i, \tilde{d}_i \neq d_i$ will vanish

Young tableau multiplication tells us what reps $d_1 \ldots d_n$ to expect.

Could also use in principle

Can we construct the bases?

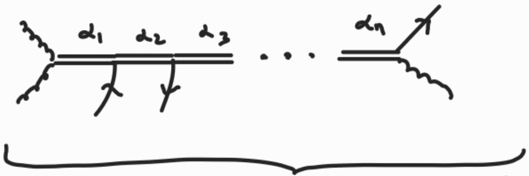
Yes, but non-trivial



only gluons on back bone

MS + S. Keppeler

arxiv: 1207.0609



quark or gluons anywhere

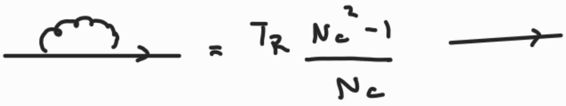
MS + J. Thorén

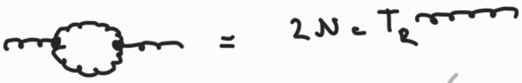
arXiv: 1809.05002

- For low reps, $3, \bar{3}, 8$ we found simplifying rules for various contractios:

  - Dimension

  - Two-vertex loops

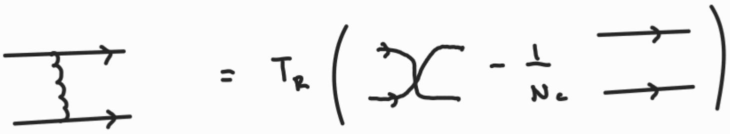




  - Three-vertex loops (vertex corrections)

etc.

  - Fierz



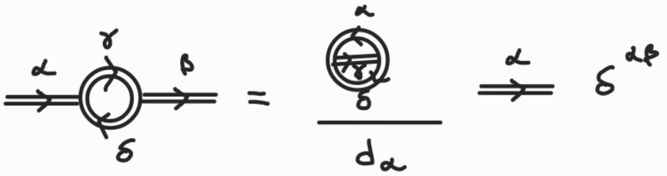
Can this be generalized?

Yes:

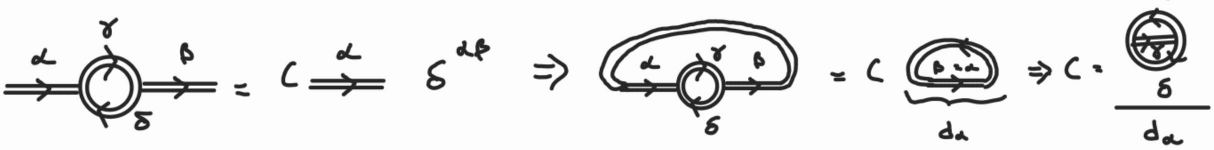
- Dimension 

$= d_\alpha$

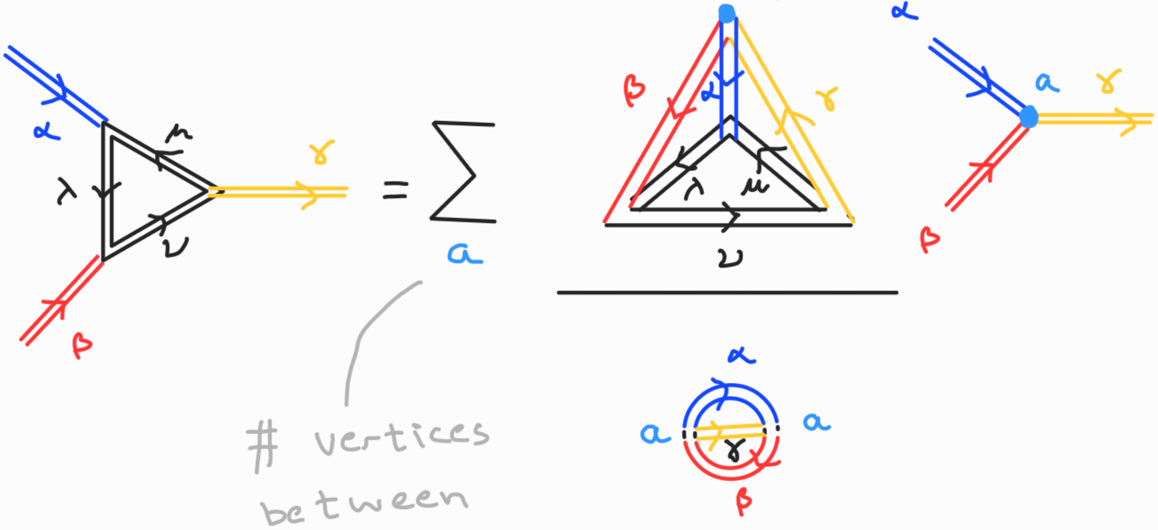
- Two-vertex loops



Proof:



- Three-vertex loops (vertex corrections)



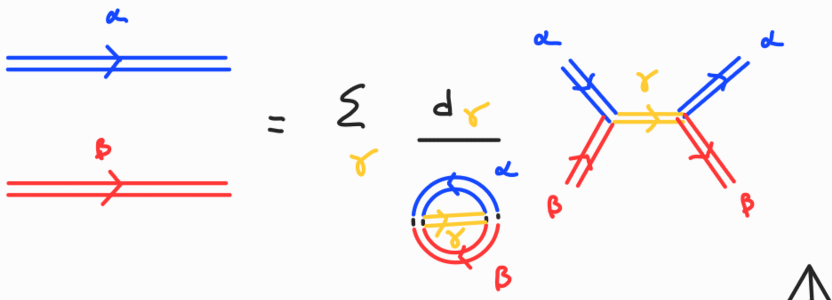
vertices between α β γ normally only 1

The symbol  is known as a

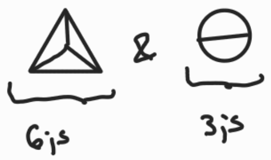
Wigner 6j symbol, 6j coefficient

or 6j for short.

- The completeness relation



It turns out that knowing



and using the above rules one can decompose any color structure in the corresponding multiplet basis.

Here

-  is just a number, and by changing the vertex normalization we can normalize it to 1.

-  is also just a number and it can be calculated once and for all using
  - explicit Clebsch-Gordan coefficients
  - Multiplet bases / (anti) symmetrizers
  - recursively, by constraining equations

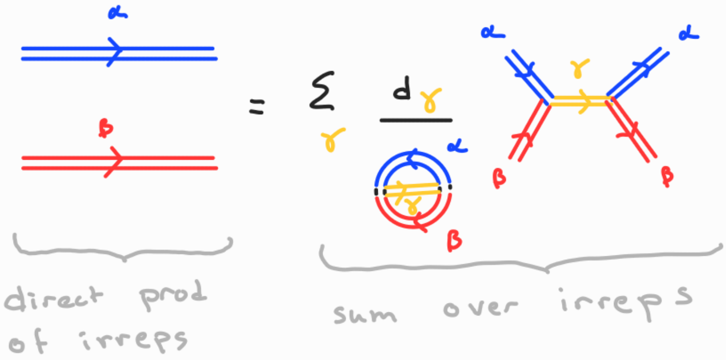
(J. Alcock-Zeilinger, S. Keppeler, S. Plätzer & M. Sjodahl 2209.15013, 2312.16688)

⇝ We actually never need to construct the basis! We can decompose in it anyway.

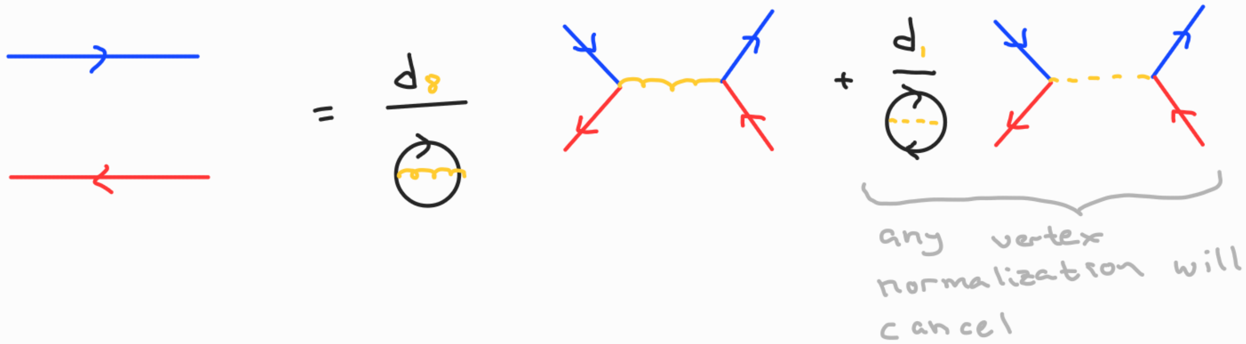
Back to Fierz:

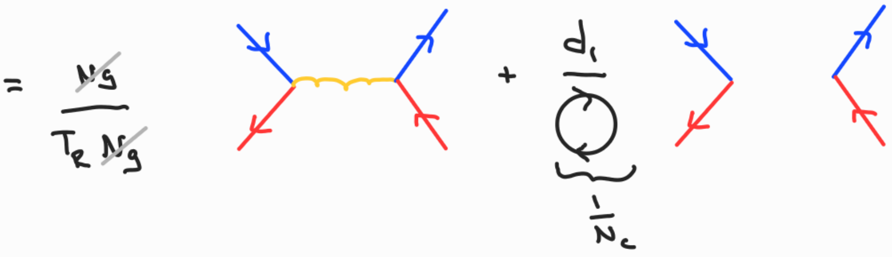
Prove that fierz identity is a special case of the completeness relation

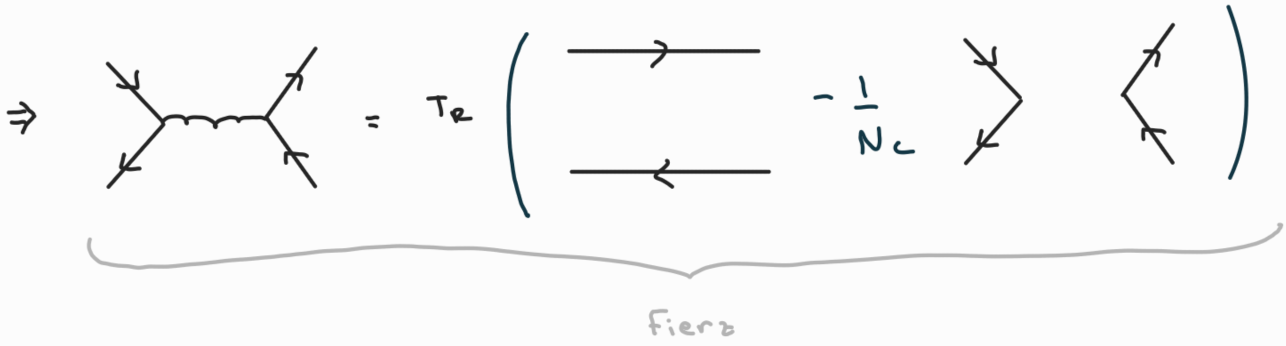
Apply



to $\alpha = \rightarrow$, $\beta = \leftarrow$

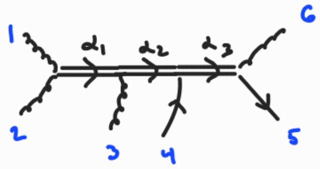




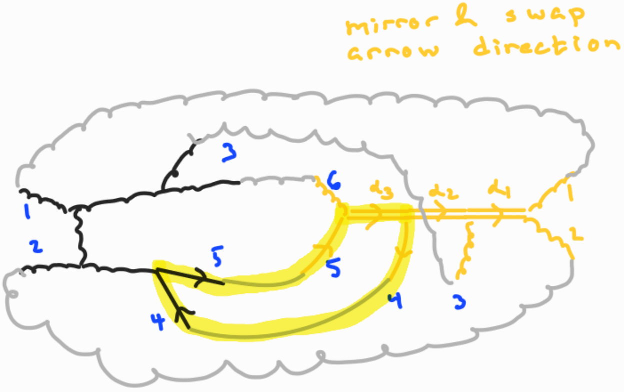


Ex: Decompose

in the basis



Note: one fixed order of partons, basis vectors characterized by reps $\alpha_1 \cdots \alpha_3$

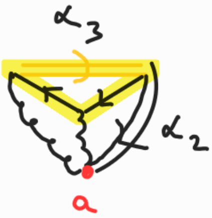


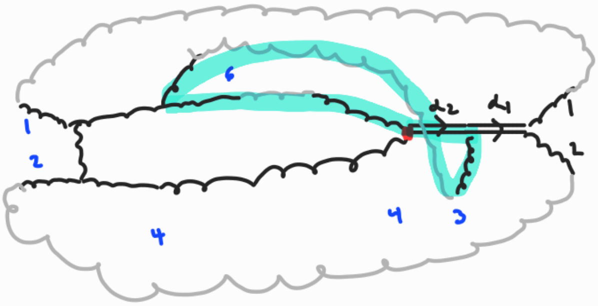
$=$ Let $3js = 1$,

$= 1$

$= \sum_a$

only 2 terms if $\alpha_1 = 8$

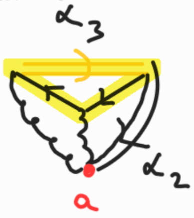




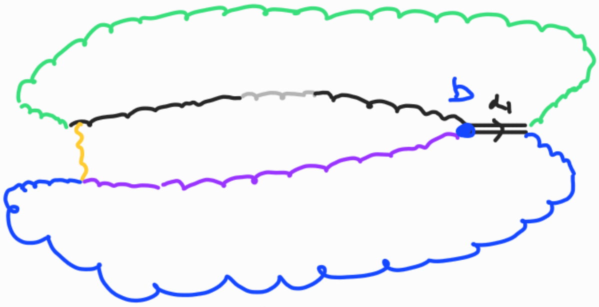
In general up to $N_c - 1$ instances of M in $8 \otimes M$

$= \sum_a \sum_b$

only 2 terms if $\alpha_1 = 8$



$-$

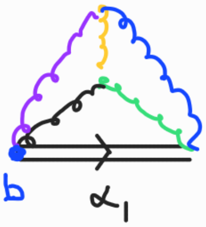


read lines in vertex in opposite order can make a difference, for example in if^{abc}

$= \underbrace{\sum_{ab}}$

mostly only 1 term in sum, up to 4 if one parton is a gluon

$-$



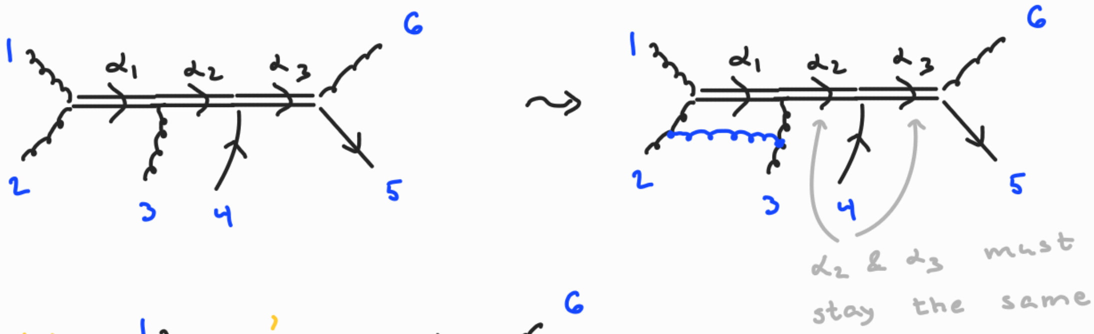
just 3 real numbers

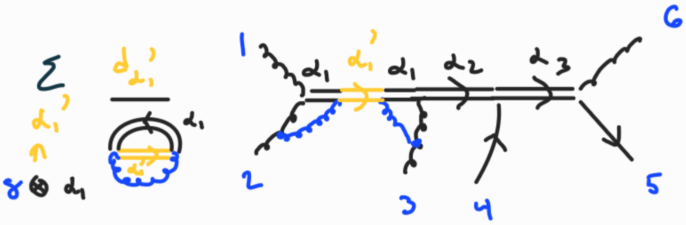
In general:

- Decomposition much more efficient than when expanding out in bases with symmetrizers and anti-symmetrizers.
- Generally the completeness relation has to be used ~>
- Topolog generally puts constraints on contributing vectors ~> only some basis vectors contribute.
- Also gluon exchange and $\underbrace{\text{emission}}_{\text{see } 1503.00530}$ can be treated

  This is <u>not</u> as simple as in the trace bases
- The $\sim (N_{g+q\bar{q}}!)^2$ in the squaring step is avoided, instead $\sim N_{\text{basis vectors}}$ $< (N_{g+q\bar{q}})!$

Ex Gluon exchange between 2 & 3 in





- Exchange between 1, 2 or 5 6 trivial
  __________ " __________ (1 or 2) and (5 or 6) most complicated as all reps can change but not "too much" since

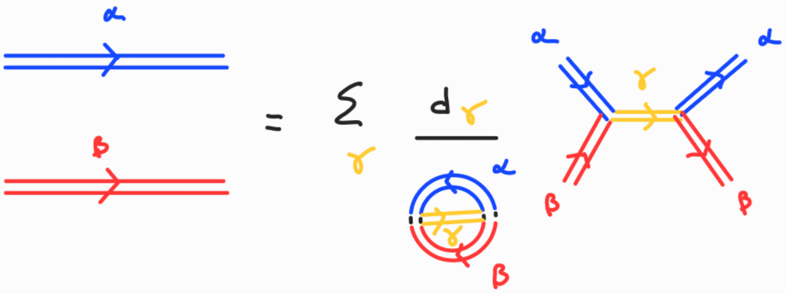
$$\alpha_1' \in \left(\alpha_1 \otimes 8 \cap 8 \otimes 8 \cap \alpha_2 \otimes 8\right) \text{ etc.}$$

L5: Exercises

1) Calculate the 6j 

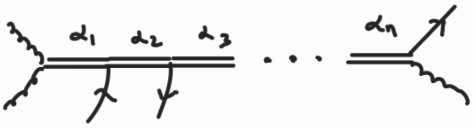
2) Knowing this 6j (re-)calculate the vertex correction

3) Take the trace of the completeness relation



and interpret the result

4) Prove that the multiplet basis vectors



are orthogonal.