Quantum Simulation of Schwinger model & Energy Spectrum

Masazumi Honda (本多 正純)

YITP
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

CGPQI
Center for Gravitational Physics and Quantum Information
Yukawa Institute for Theoretical Physics, Kyoto University

KYOTO UNIVERSITY
KYOTO JAPAN
FOUNDED 1897

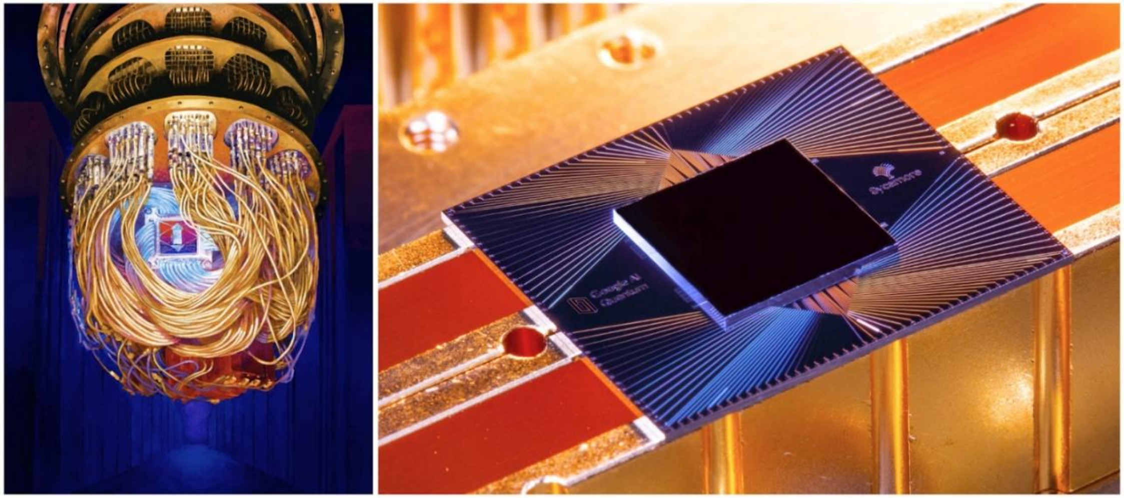
iTHEMS

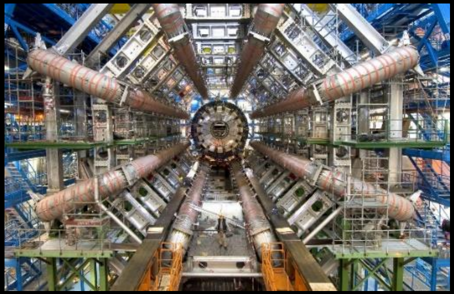
PRESTO
SAKIGAKE

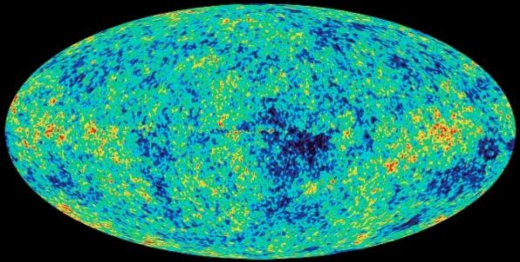
Refs:

[1] arXiv:2001.00485 [hep-lat],
w/ Bipasha Chakraborty (Southampton U.), Yuta Kikuchi (Quantinuum),
Taku Izubuchi (BNL-RIKEN BNL) & Akio Tomiya (International Prof. U. of Tech. in Osaka)
[2] arXiv:2105.03276 [hep-lat],
w/ Yuta Kikuchi, Etsuko Itou (YITP), Lento Nagano (Tokyo U.) & Takuya Okuda (Tokyo U.)
[3] arXiv:2110.14105 [hep-th], w/ Yuta Kikuchi, Etsuko Itou & Yuya Tanizaki (YITP)
[4] arXiv:2210.04237 [hep-th], w/ Etsuko Itou & Yuya Tanizaki
[5] work in progress, w/ Dongwook Ghim (YITP)

My long term goal







etc…

This talk is on

Application of Quantum Computation
to
Quantum Field Theory (QFT)

- Generic motivation:

simply would like to use powerful computers?

- Specific motivation:

Quantum computation is more natural in operator formalism

⟶ Liberation from infamous sign problem in Monte Carlo?

Our recent works

Charge-q Schwinger model with topological term

1+1d QED

$$\boxed{L = \frac{1}{2g^2} F_{01}^2 + \underbrace{\frac{\theta_0}{2\pi} F_{01}}_{\text{topological “theta term”}} + \overline{\psi}\,\mathrm{i}\,\gamma^\mu(\partial_\mu + \mathrm{i}\,q\,A_\mu)\psi - m\,\overline{\psi}\psi}$$

supposed to be difficult in the conventional approach:

- real time
- ∃ sign problem even in Euclidean case when θ isn’t small

Results:

[cf. Tensor Network approach:
Banuls-Cichy-Jansen-Saito ’16 ,
Funcke-Jansen-Kuhn ’19, etc.]

- Construction of the ground state
- Computation of $\langle \bar{\psi}\psi \rangle$ & consistency check/prediction

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]

- Exploration of the screening vs confinement problem & negative string tension behavior for some parameters

[MH-Itou-Kikuchi-Nagano-Okuda ’21] [MH-Itou-Kikuchi-Tanizaki ’21]

- energy spectrum by coherent imaging spectroscopy

[work in progress, MH-Ghim]

Plan

1. Introduction

2. Schwinger model as qubits

3. Algorithm to prepare ground state

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]

4. Screening, confinement & negative string tension

[MH-Itou-Kikuchi-Nagano-Okuda ’21, MH-Itou-Kikuchi-Tanizaki ’21]

5. Algorithm to compute energy spectrum

[work in progress, MH-Ghim]

6. Summary & Outlook

"Regularization" of Hilbert space

Hilbert space of QFT is typically ∞ dimensional

⟶ Make it finite dimensional!

- Fermion is easiest (up to doubling problem)
  - Putting on spatial lattice, Hilbert sp. is finite dimensional
- scalar
  - Hilbert sp. at each site is ∞ dimensional (need truncation or additional regularization)
- gauge field (w/ kinetic term)
  - no physical d.o.f. in 0+1D/1+1D (w/ open bdy. condition)
  - ∞ dimensional Hilbert sp. in higher dimensions

Charge-q Schwinger model

Continuum:

$$L = \frac{1}{2g^2}F_{01}^2 + \frac{\theta_0}{2\pi}F_{01} + \overline{\psi}\,\mathrm{i}\,\gamma^\mu(\partial_\mu + \mathrm{i}\,q\,A_\mu)\psi - m\,\overline{\psi}\psi$$

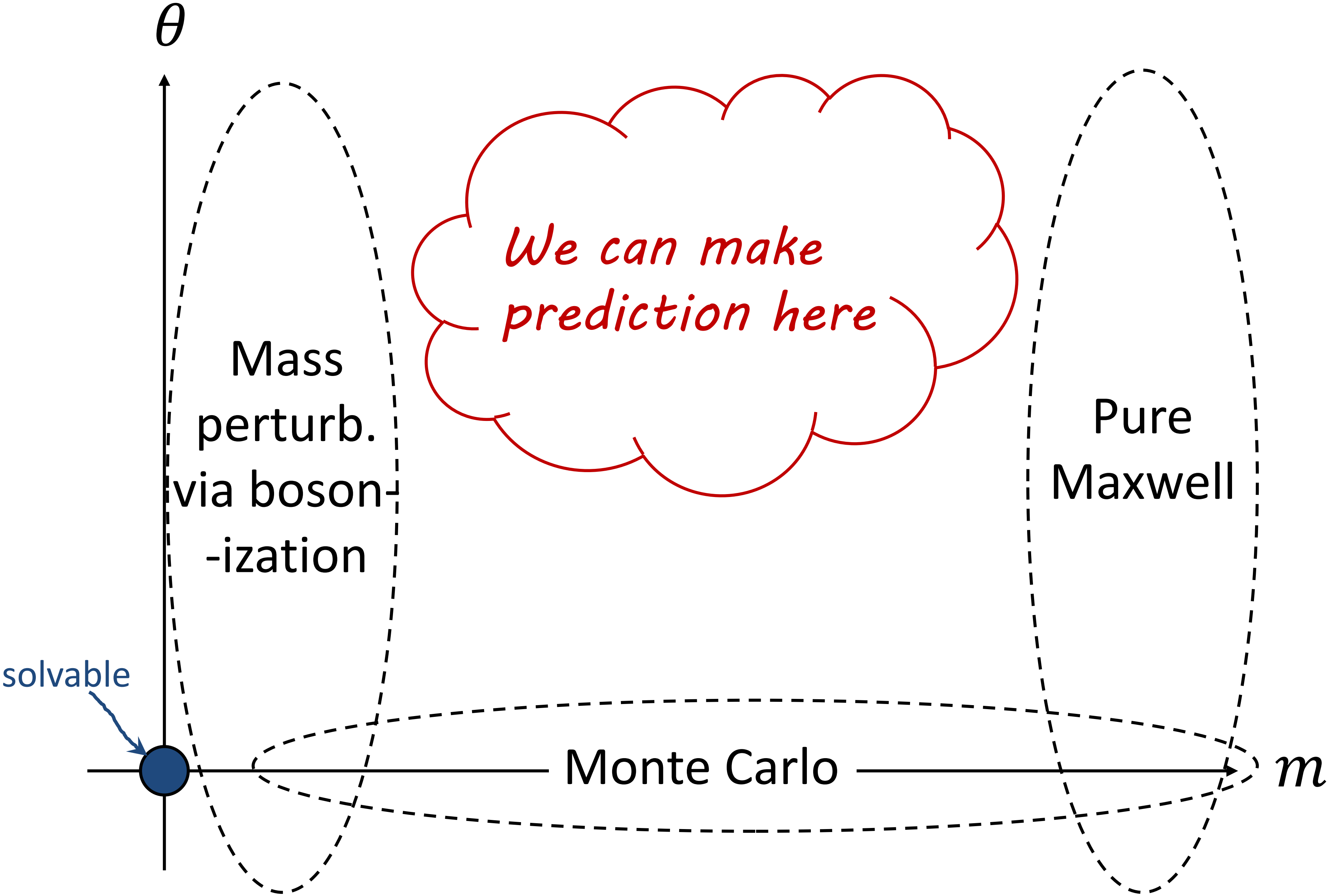
Taking temporal gauge $A_0 = 0$, (Π: conjugate momentum of A_1)

$$H(x) = \frac{g^2}{2}\left(\Pi - \frac{\theta_0}{2\pi}\right)^2 - \bar{\psi}\,\mathrm{i}\,\gamma^1(\partial_1 + \mathrm{i}\,qA_1)\psi + m\bar{\psi}\psi,$$

Physical states are constrained by Gauss law:

$$0 = -\partial_1\Pi - qg\bar{\psi}\gamma^0\psi$$

Map of accessibility/difficulty



We can make prediction here

Mass perturb. via boson-ization

Pure Maxwell

Monte Carlo

Put the theory on lattice

- **Fermion** (on site):

"Staggered fermion" [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{\underbrace{a}_{\text{lattice spacing}}{}^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \begin{matrix} \rightarrow \text{odd site} \\ \rightarrow \text{even site} \end{matrix}$$

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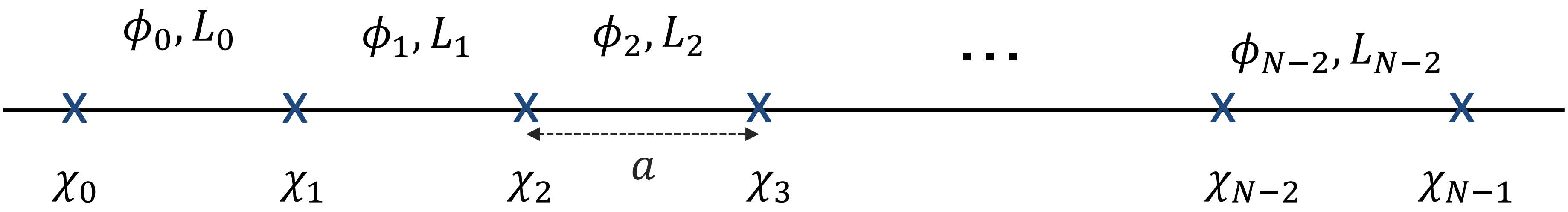
- **Gauge field** (on link):

$$\phi_n \leftrightarrow -agA^1(x), \qquad L_n \leftrightarrow -\frac{\Pi(x)}{g}$$



Lattice theory w/ staggered fermion

Hamiltonian:

$$H = J\sum_{n=0}^{N-2}\left(L_n + \frac{\theta_0}{2\pi}\right)^2 - \mathrm{i}w\sum_{n=0}^{N-2}\left[\chi_n^\dagger (U_n)^q \chi_{n+1} - \mathrm{h.c.}\right] + m\sum_{n=0}^{N-1}(-1)^n \chi_n^\dagger \chi_n$$

$$\left(w = \frac{1}{2a}, J = \frac{g^2 a}{2}\right)$$

Commutation relation:

$$[L_n, U_m] = U_m \delta_{nm}, \quad \{\chi_n, \chi_m^\dagger\} = \delta_{nm}$$

Gauss law:

$$L_n - L_{n-1} = q\left[\chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}\right]$$

Eliminate gauge d.o.f.

1. Take open b.c. & solve Gauss law:

$$L_n = L_{-1} + q\sum_{j=1}^{n}\left(\chi_j^\dagger\chi_j - \frac{1-(-1)^j}{2}\right) \qquad \text{w/ } L_{-1} = 0$$

2. Take the gauge $U_n = 1$

Then,

$$H = -\mathrm{i}w\sum_{n=1}^{N-1}\left[\chi_n^\dagger\chi_{n+1} - \mathrm{h.c.}\right] + m\sum_{n=1}^{N}(-1)^n\chi_n^\dagger\chi_n$$
$$+J\sum_{n=1}^{N}\left[\frac{\theta_0}{2\pi} + q\sum_{j=1}^{n}\left(\chi_j^\dagger\chi_j - \frac{1-(-1)^j}{2}\right)\right]^2 .$$

This acts on finite dimensional Hilbert space

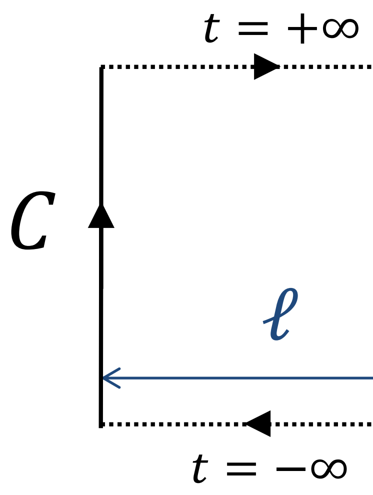
Insertion of the probe charges

① Introduce the probe charges $\pm q_p$:

$$e^{iq_p \int_C A}$$

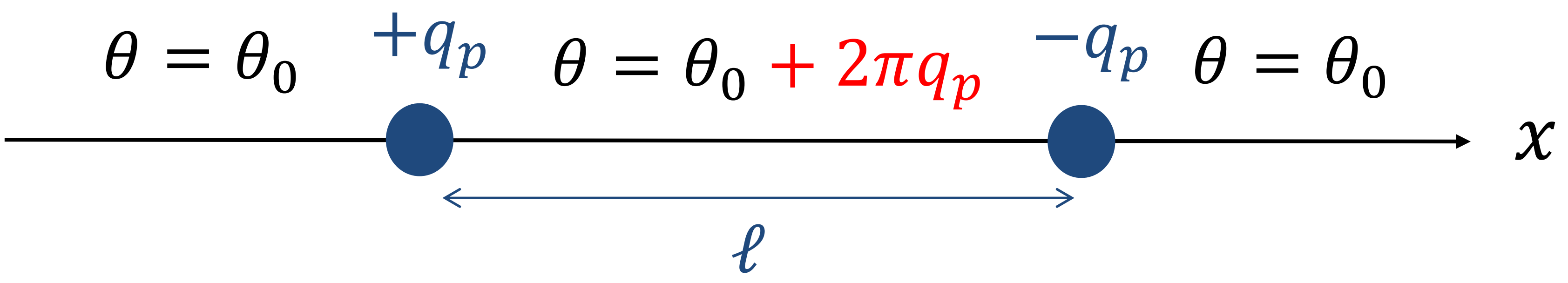
$$\|$$

$$e^{iq_p \int_{S,\partial S=C} F}$$



local θ-term w/ $\theta = 2\pi q_p$!!

② Include it to the action & switch to Hamilton formalism



③ Compute the ground state energy (in the presence of the probes)

Going to spin system

$$\{\chi_n^\dagger, \chi_m\} = \delta_{mn}, \quad \{\chi_n, \chi_m\} = 0$$

This is satisfied by the operator:

"Jordan-Wigner transformation"

[Jordan-Wigner'28]

$$\chi_n = \frac{X_n - \mathrm{i}Y_n}{2}\left(\prod_{i=1}^{n-1} -\mathrm{i}Z_i\right)$$

(X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

Now the system is purely a spin system:

$$H = -\mathrm{i}w\sum_{n=1}^{N-1}\left[\chi_n^\dagger\chi_{n+1} - \mathrm{h.c.}\right] + m\sum_{n=1}^{N}(-1)^n\chi_n^\dagger\chi_n + J\sum_{n=1}^{N}\left[\frac{\vartheta_n}{2\pi} + q\sum_{j=1}^{n}\left(\chi_j^\dagger\chi_j - \frac{1-(-1)^j}{2}\right)\right]^2$$

$$\Downarrow$$

$$H = J\sum_{n=0}^{N-2}\left[q\sum_{i=0}^{n}\frac{Z_i + (-1)^i}{2} + \frac{\vartheta_n}{2\pi}\right]^2 + \frac{w}{2}\sum_{n=0}^{N-2}\left[X_nX_{n+1} + Y_nY_{n+1}\right] + \frac{m}{2}\sum_{n=0}^{N-1}(-1)^nZ_n$$

Qubit description of the Schwinger model !!

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3. Algorithm to prepare ground state

   [Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]
4. Screening, confinement & negative string tension

   [MH-Itou-Kikuchi-Nagano-Okuda '21, MH-Itou-Kikuchi-Tanizaki '21]
5. Algorithm to compute energy spectrum

   [work in progress, MH-Ghim]
6. Summary & Outlook

Constructing ground state

$^{\exists}$various quantum algorithms to construct vacuum:

- adiabatic state preparation
- algorithms based on variational method
- imaginary time evolution

etc…

Here, let’s apply

adiabatic state preparation

Adiabatic state preparation

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\mathrm{vac}_0\rangle$ is known and unique

Step 2:

Step 3:

Adiabatic state preparation

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Step 2: Introduce adiabatic Hamiltonian $H_A(t)$ s.t.

$$\begin{cases} \bullet\ H_A(0) = H_0,\ H_A(T) = H_{\mathrm{target}} \\ \bullet\ \left|\frac{dH_A}{dt}\right| \ll 1 \text{ for } T \gg 1 \end{cases}$$

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Step 3: Use the adiabatic theorem

If $H_A(t)$ has a unique ground state w/ a finite gap for $\forall t$, then the ground state of H_{target} is obtained by

$$\boxed{|\mathrm{vac}\rangle = \lim_{T\to\infty} \mathcal{T} \exp\left(-i \int_0^T dt\, H_A(t)\right) |\mathrm{vac}_0\rangle}$$

Adiabatic state preparation (cont'd)

$$|\mathrm{vac}> = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\ H_A(t)\right)|\mathrm{vac}_0>$$

$$\simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\mathrm{vac}_0>$$

$$\left(U(t) = e^{-iH_A(t)\delta t}\right)$$

Here, we choose

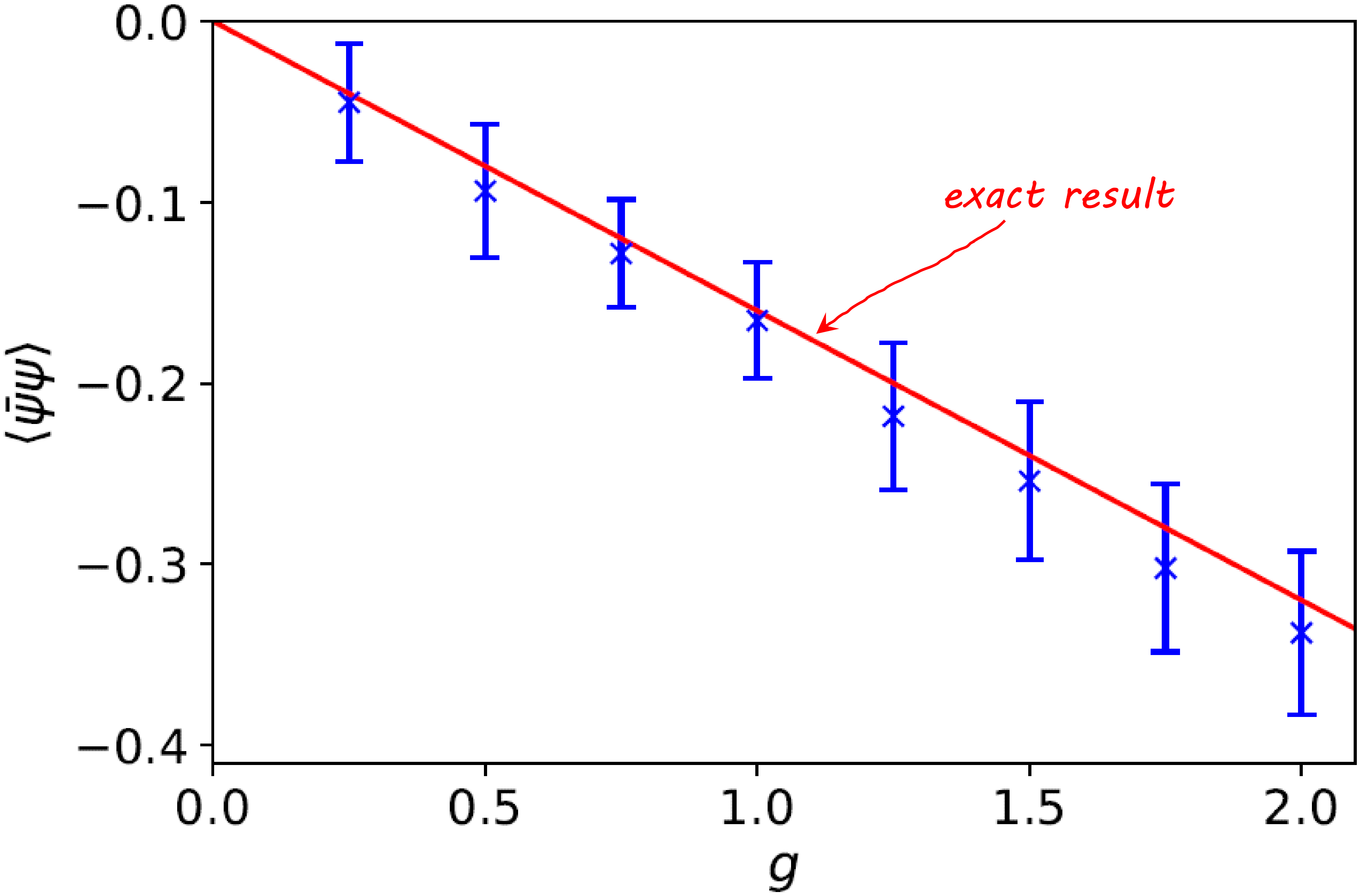
$$\begin{cases} H_0 = H\Big|_{w\to 0,\, \vartheta_n\to 0,\, m\to m_0} \quad \Longrightarrow \quad |\mathrm{vac}_0\rangle = |1010\cdots\rangle \\ H_A(t) = H\Big|_{w\to w(t), \vartheta_n\to\vartheta_n(t),\, m\to m(t)} \\ w(t) = f\left(\frac{t}{T}\right)w,\ \vartheta_n(t) = f\left(\frac{t}{T}\right)\vartheta_n, \quad m(t) = \left(1 - f\left(\frac{t}{T}\right)\right)m_0 + f\left(\frac{t}{T}\right)m \\ f(s)\text{: smooth function s.t. } f(0) = 0,\ f(1) = 1 \end{cases}$$

Demo: chiral condensate in massless case

$T = 100, \delta t = 0.1, N_{\text{max}} = 16, 1M$ shots

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

(after continuum limit)



exact result

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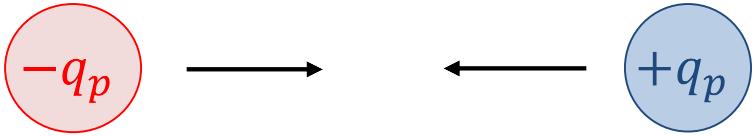
Screening versus Confinement

Let’s consider

potential between 2 heavy charged particles



Classical picture:

$$V(x) = \frac{q_p^2\, g^2}{2} x \;?$$

Coulomb law in 1+1d

||

confinement

too naive in the presence of dynamical fermions

Expectations from previous analyzes

Potential between probe charges $\pm q_p$ has been analytically computed

[Iso-Murayama '88, Gross-Klebanov-Matytsin-Smilga '95]

- massless case:

$$\mu \equiv g/\sqrt{\pi}$$

$$V(x) = \frac{q_p^2\, g^2}{2\mu}\left(1 - e^{-q\mu x}\right)$$

screening

- massive case:

Expectations from previous analyzes

Potential between probe charges $\pm q_p$ has been analytically computed

[Iso-Murayama '88, Gross-Klebanov-Matytsin-Smilga '95]

- massless case:

$$\mu \equiv g/\sqrt{\pi}$$

$$V(x) = \frac{q_p^2 g^2}{2\mu}(1 - e^{-q\mu x})$$ *screening*

- massive case: [cf. Misumi-Tanizaki-Unsal '19]

$$\Sigma \equiv g e^{\gamma}/2\pi^{3/2}$$

$$V(x) \sim mq\Sigma\left(\cos\left(\frac{\theta + 2\pi q_p}{q}\right) - \cos\left(\frac{\theta}{q}\right)\right) x \qquad (\mathrm{m} \ll g,\ |x| \gg 1/g\ )$$

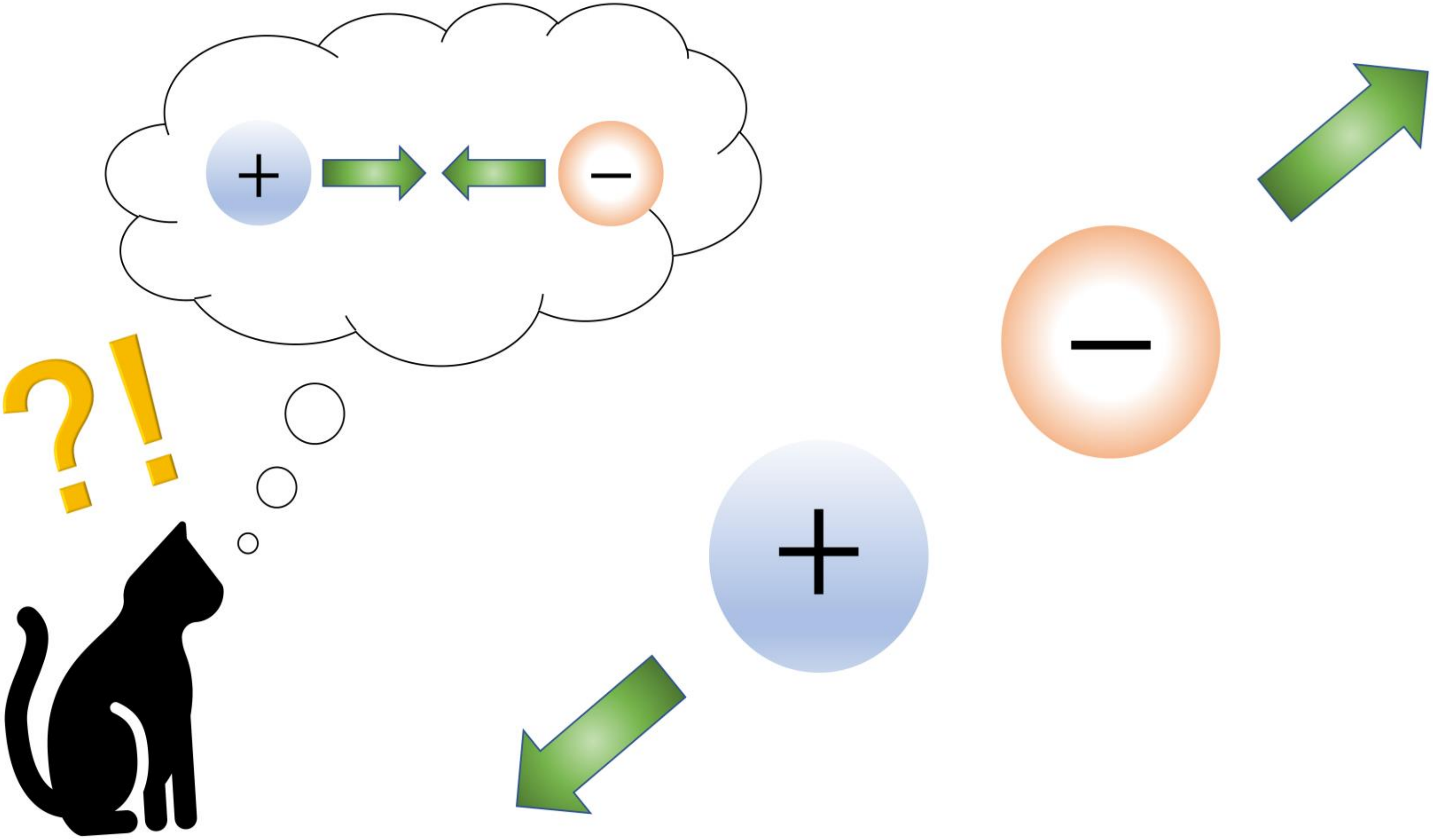
$$\begin{cases} = \mathrm{Const.} & \text{for } \mathrm{q_p/q} = \mathbf{Z} \\ \propto x & \text{for } \mathrm{q_p/q} \neq \mathbf{Z} \end{cases}$$

screening

confinement?

*but sometimes **negative** slope!*

That is, as changing the parameters…



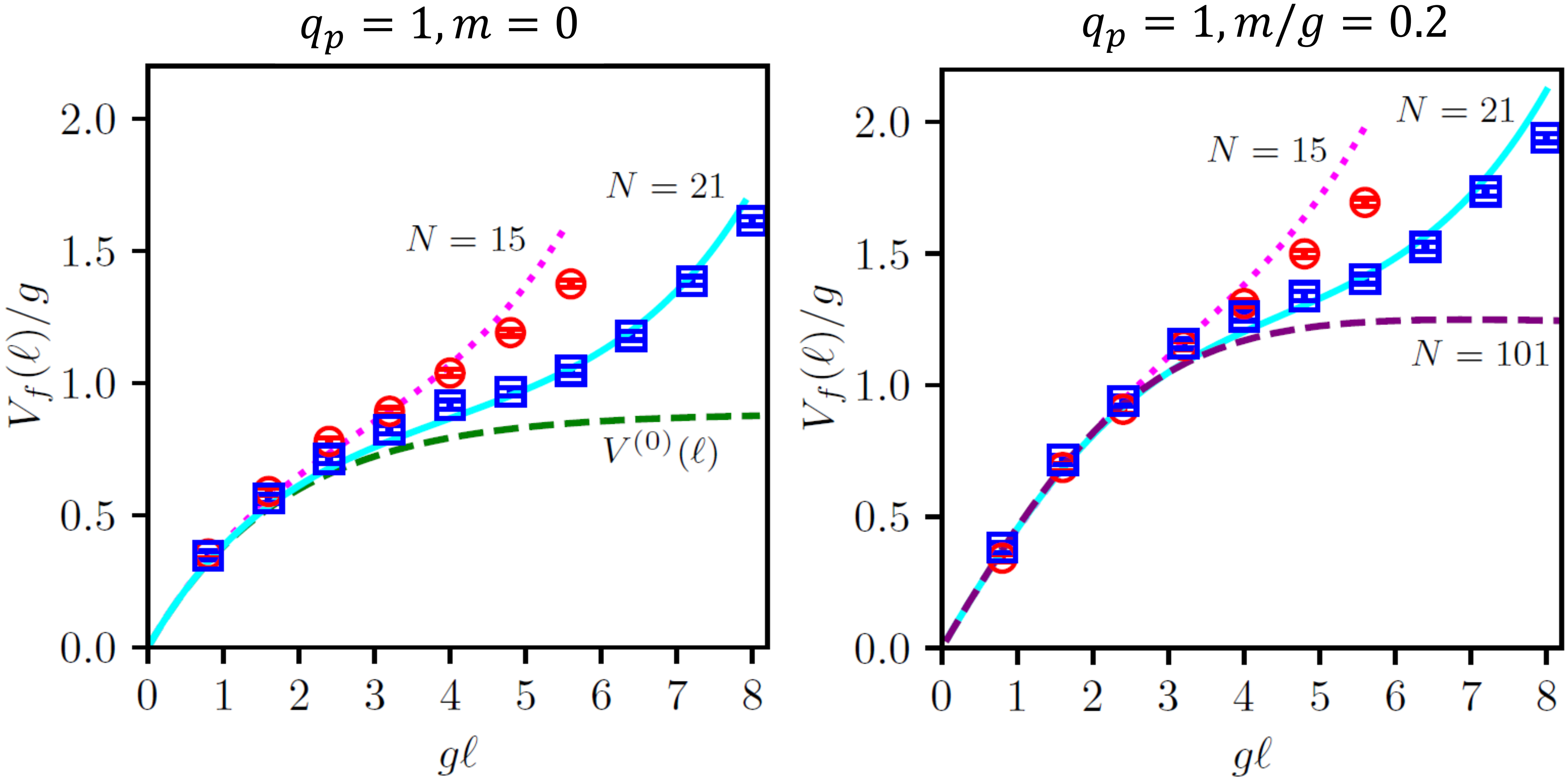
Let’s explore this aspect by quantum simulation!

Massless vs massive for $\theta_0 = 0$ & $q_p/q \in \boldsymbol{Z}$

[MH-Itou-Kikuchi-Nagano-Okuda '21]

Parameters: $g = 1, a = 0.4, N = 15$ & $21, T = 99, q_p/q = 1$

Lines: analytical results in the continuum limit (finite & ∞ vols.)



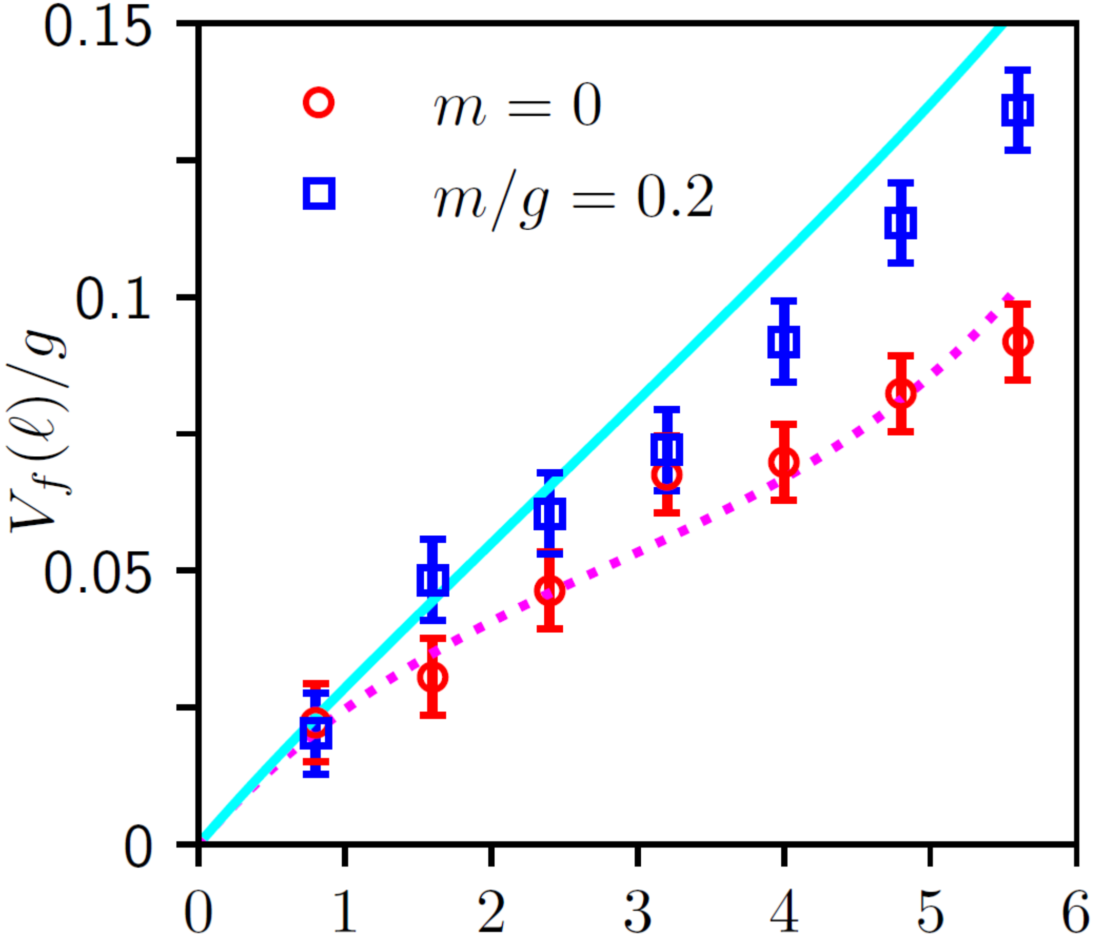
Consistent w/ expected screening behavior

Results for $\theta_0 = 0$ & $q_p/q \notin \mathbf{Z}$

[MH-Itou-Kikuchi-Nagano-Okuda '21]

Parameters: $g = 1, a = 0.4, N = 15, T = 99, q_p/q = 1/4, m = 0$ & 0.2

Lines: analytical results in the continuum limit (finite & ∞ vol.)

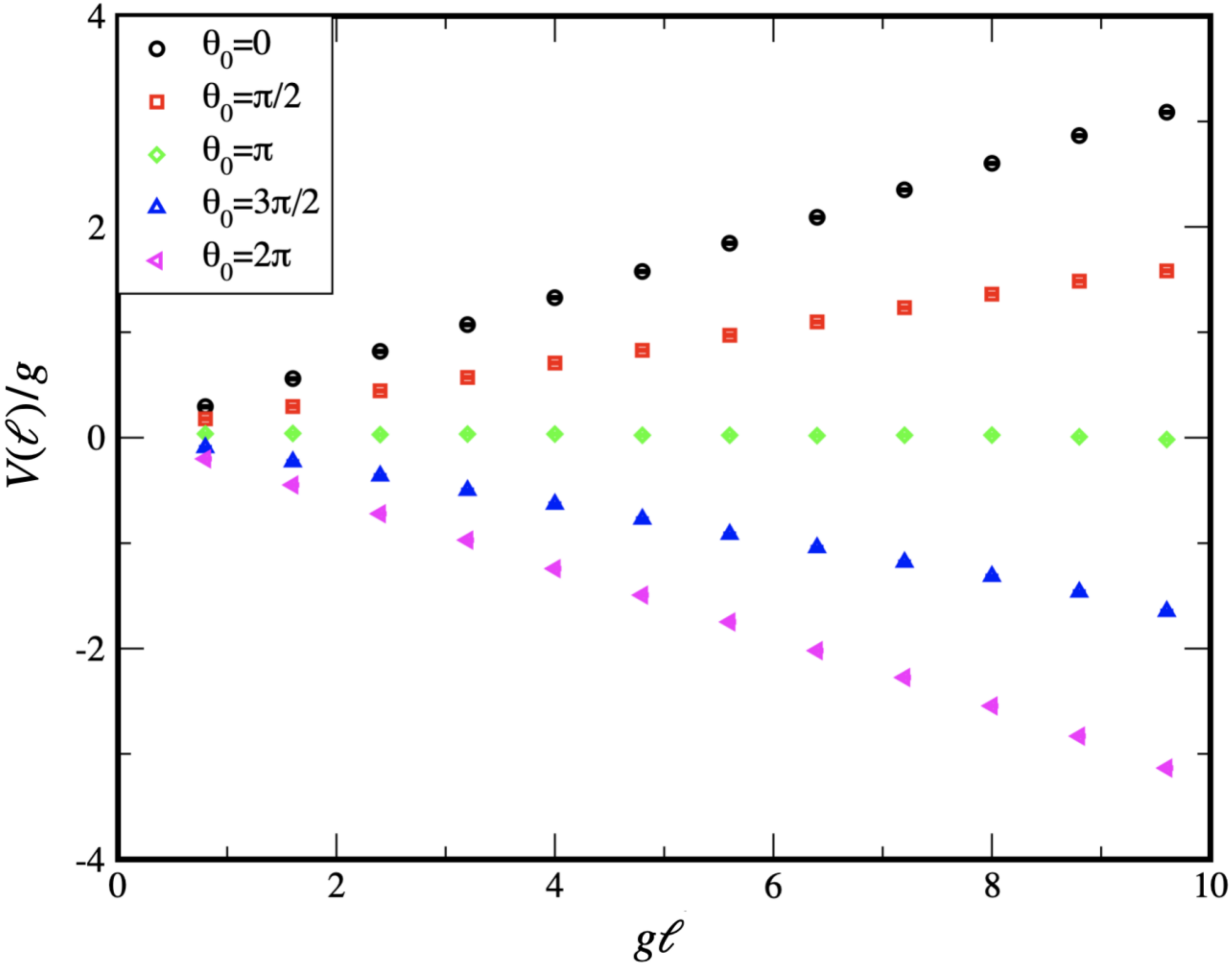


Consistent w/ expected confinement behavior

Positive / negative string tension

[MH-Itou-Kikuchi-Tanizaki ’21]

Parameters: $g = 1, a = 0.4, N = 25, T = 99, q_p/q = -1/3, m = 0.15$

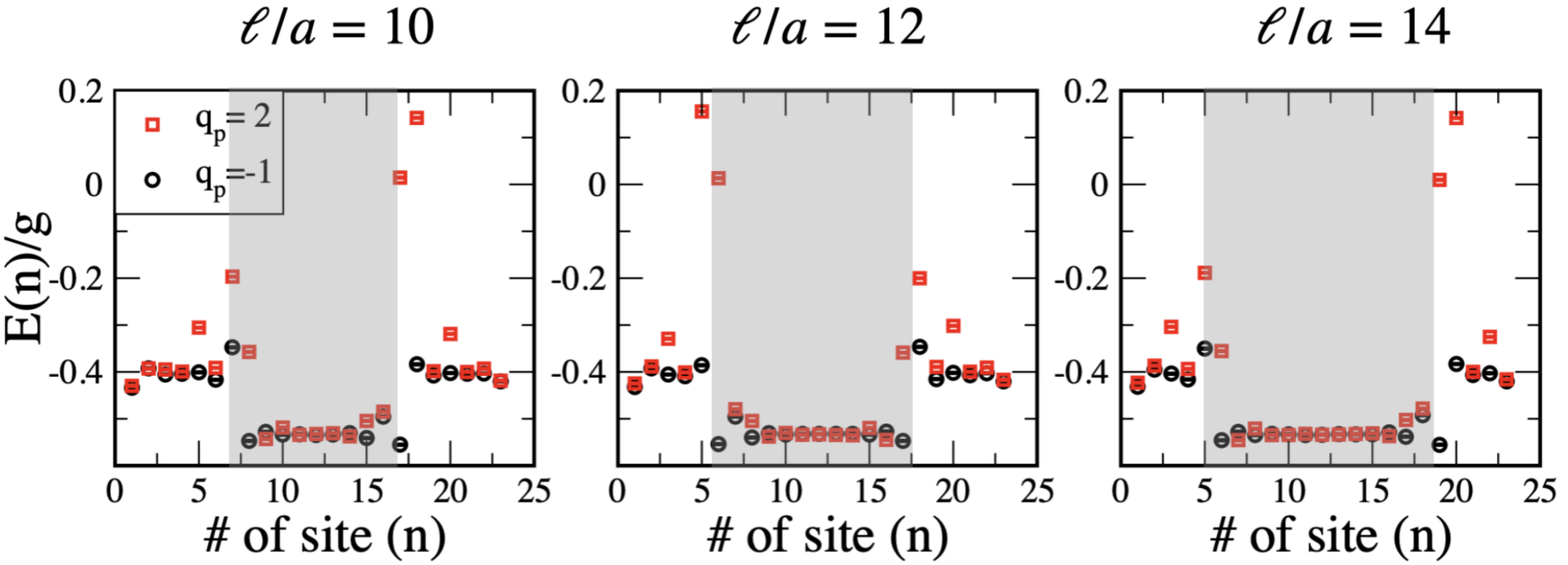


Sign(tension) changes as changing θ-angle!!

Energy density @ negative tension regime

[MH-Itou-Kikuchi-Tanizaki ’21]

$$g = 1, a = 0.4, N = 25, T = 99,\ q_p/q = -1/3, m = 0.15, \theta_0 = 2\pi$$



Lower energy inside the probes!!

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   [work in progress, MH-Ghim]
6. Summary & Outlook

Energy spectrum in quantum field theory

Information in energy spectrum:

- degeneracy of ground states
- energy gap between ground & 1st excited states
- distribution of excited states at low levels

⇨ phase structure, mass spectrum of particles

Energy spectrum in quantum field theory

Information in energy spectrum:

- degeneracy of ground states
- energy gap between ground & 1st excited states
- distribution of excited states at low levels

⇒ phase structure, mass spectrum of particles

Desired algorithm:

efficient computation of spectrum at low levels

(doesn’t need ground state energy itself)

For this purpose, it seems inefficient to explicitly construct energy eigenstates one by one and measure their energies

Algorithm: coherent imaging spectroscopy

[Senko-Smith-Richerme-Lee-Campbell-Monroe ’14]
[working in progress, MH-Ghim]

We’d like to know spectrum of excited energies:

$$\hat{H}_{\mathrm{target}}\,|n\rangle = E_n|n\rangle$$

Time dependent Hamiltonian:

$$\hat{H}(t;\nu) = \hat{H}_{\mathrm{target}} + B\sin(\nu t)\cdot\hat{O}$$

Survival probability of ground state after some time:

$$P(\nu) := |\langle 0|\mathcal{T}e^{-i\int dt\hat{H}\,(t;\nu)}|0\rangle|^2$$

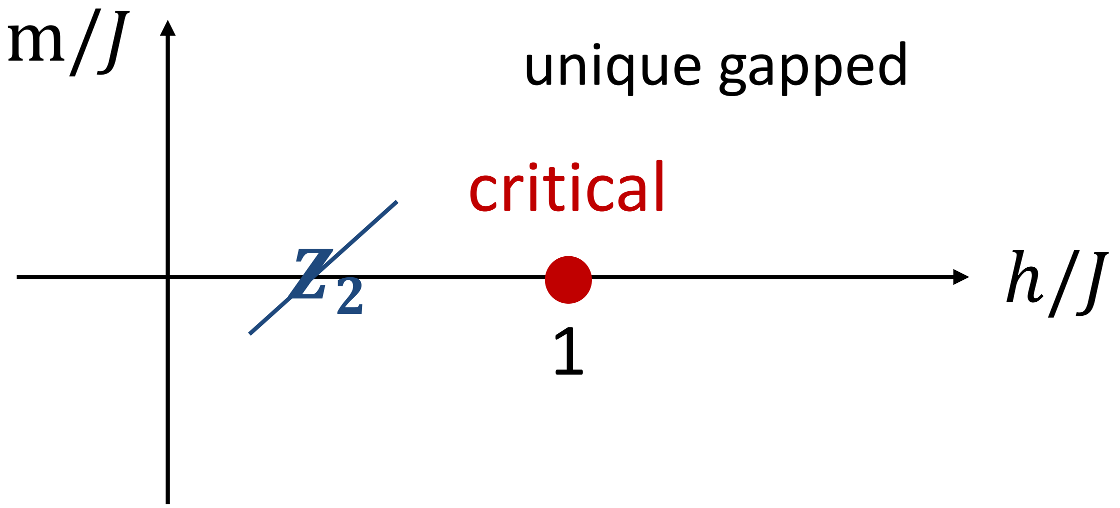
becomes small when $\nu \sim E_n$

Coherent imaging spectroscopy in Ising model

[working in progress, MH-Ghim]

$$\hat{H}_{\text{Ising}} = -J \sum_{n=1}^{N-1} Z_n Z_{n+1} - h \sum_{n=1}^{N} X_n - m \sum_{n=1}^{N} Z_n$$

Known phase diagram:



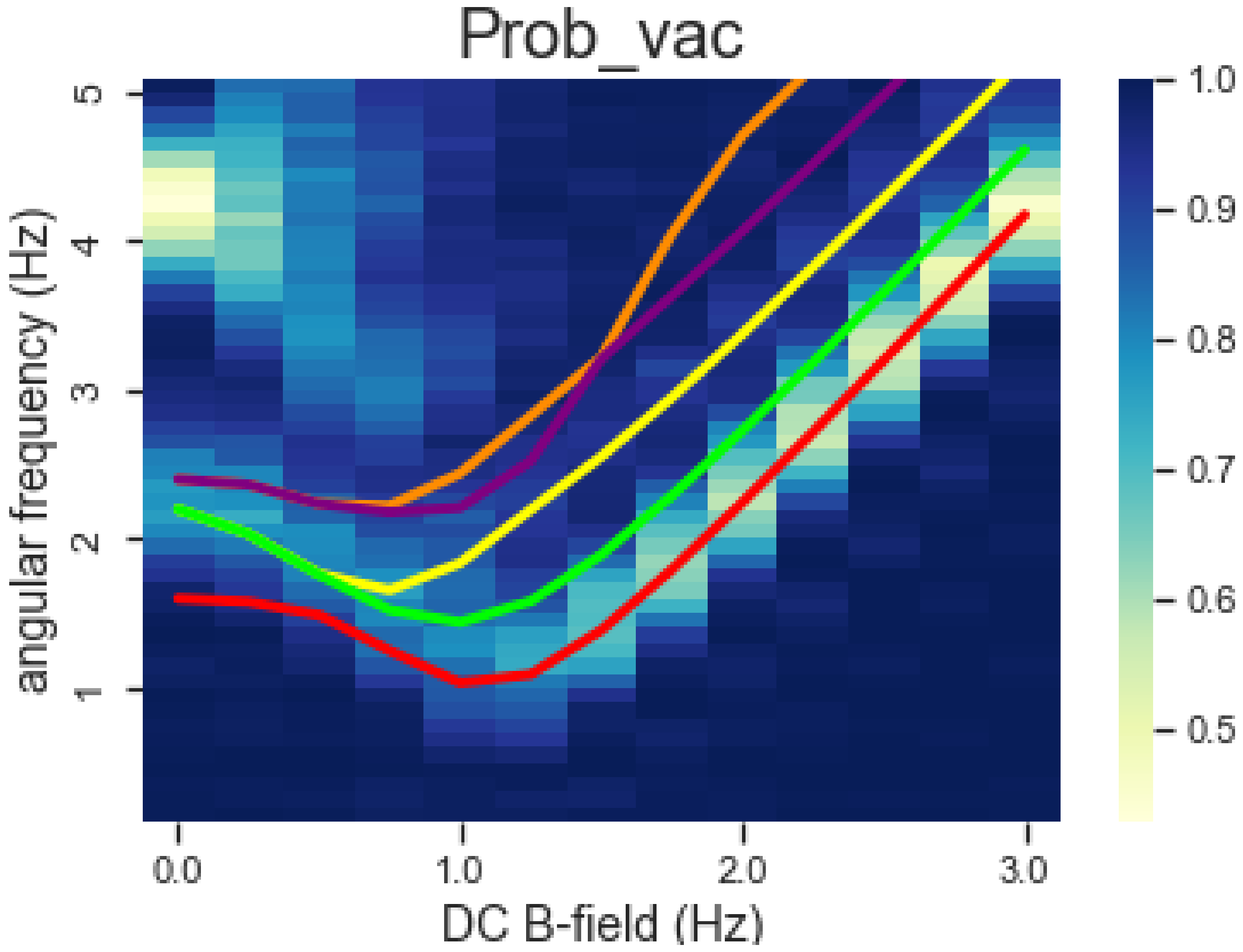
Let’s consider time evolution by

$$\hat{H}_{\text{Ising}} + B\sin(\nu t) \sum_{n=1}^{N} Y_n$$

Coherent imaging spectroscopy in Ising model (cont'd)

[working in progress, MH-Ghim]

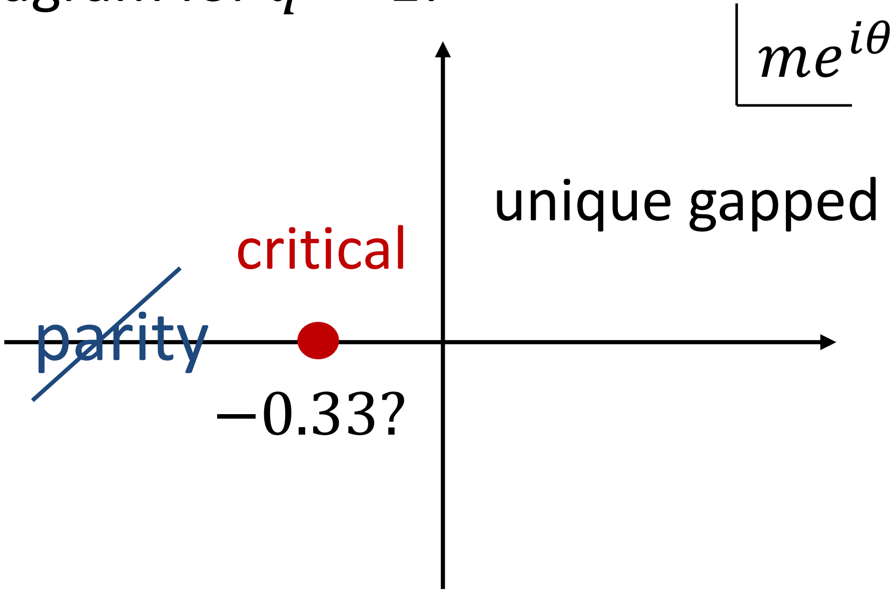
$N = 8, m/J = 0.1$ （$|0\rangle$ by adiabatic state preparation)



Coherent imaging spectroscopy in Schwinger model

$$H = J\sum_{n=0}^{N-2}\left(L_n + \frac{\theta_0}{2\pi}\right)^2 - \mathrm{i}w\sum_{n=0}^{N-2}\left[\chi_n^\dagger (U_n)^q \chi_{n+1} - \mathrm{h.c.}\right] + m\sum_{n=0}^{N-1}(-1)^n \chi_n^\dagger \chi_n$$

Expected phase diagram for $q = 1$:

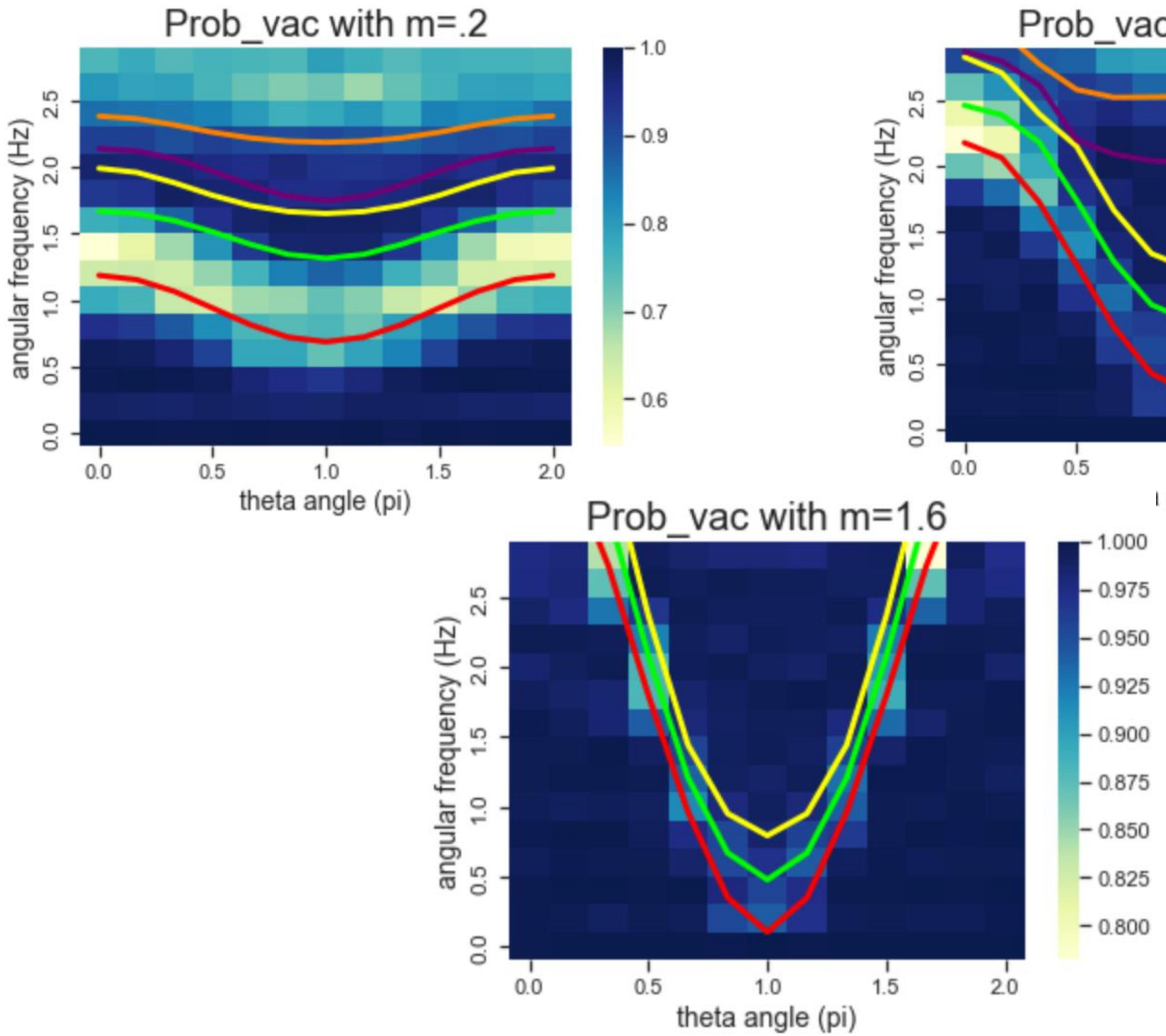


Let's consider time evolution by (perturbed by "$\bar{\psi}\gamma_5\psi$")

$$\widehat{H} + B\sin(\nu t)\sum_{n=0}^{N-1}(-1)^n\left(\chi_n^\dagger \chi_{n+1} - \chi_{n+1}^\dagger \chi_n\right)$$

Coherent imaging spectroscopy in Schwinger model (cont'd)

($N = 13, g = 1, w = 1, |0\rangle$ by adiabatic state preparation)



preliminary

Summary & Outlook

Summary

- Quantum computation is suitable for operator formalism that is free from sign problem

- Instead we have to deal with huge vector space. Quantum computers in future may do this job.

- constructed the ground state of the Schwinger model w/ the topological term by adiabatic state preparation

- found agreement in the chiral condensate with the exact result for $m = 0$ & mass perturbation theory for small m

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

- explored the screening vs confinement problem & negative string tension behavior

[MH-Itou-Kikuchi-Nagano-Okuda '21]

[MH-Itou-Kikuchi-Tanizaki '21]

- energy spectrum by coherent imaging spectroscopy

[work in progress, MH-Ghim]

Towards “quantum supremacy”?

The problems in this talk involve only ground state in 1+1D

→ Tensor Network is better → able to take $N = \mathcal{O}(100)$

[MH-Itou-Tanizaki ’22]

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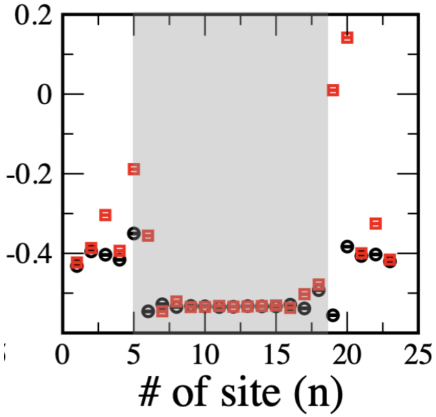
→ Tensor Network is better → able to take $N = \mathcal{O}(100)$

[MH-Itou-Tanizaki ’22]

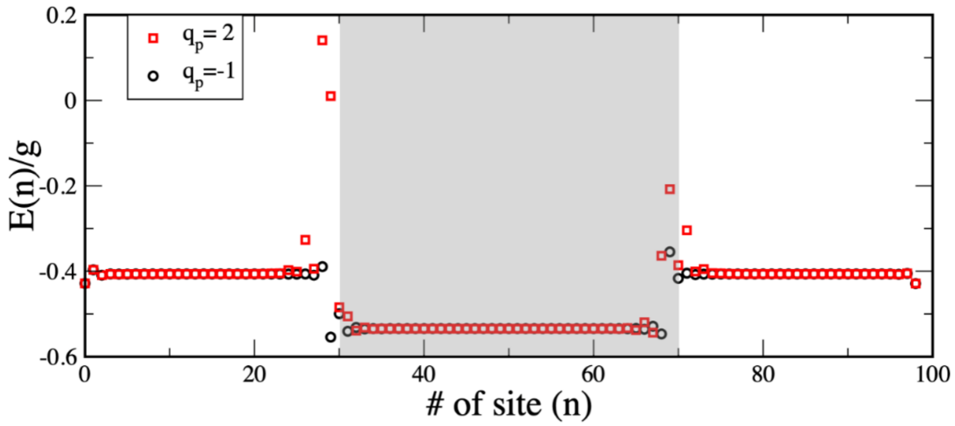
Adiabatic state preparation:

$\ell/a = 14$



Tensor Network (DMRG):

$\ell/a = 40,\ N = 101$



Towards “quantum supremacy”?

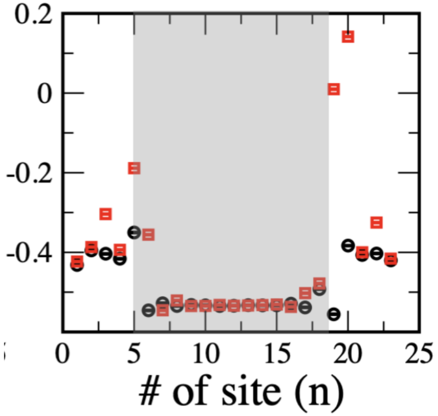
The problems in this talk involve only ground state in 1+1D

→ Tensor Network is better → able to take $N = \mathcal{O}(100)$

[MH-Itou-Tanizaki ’22]

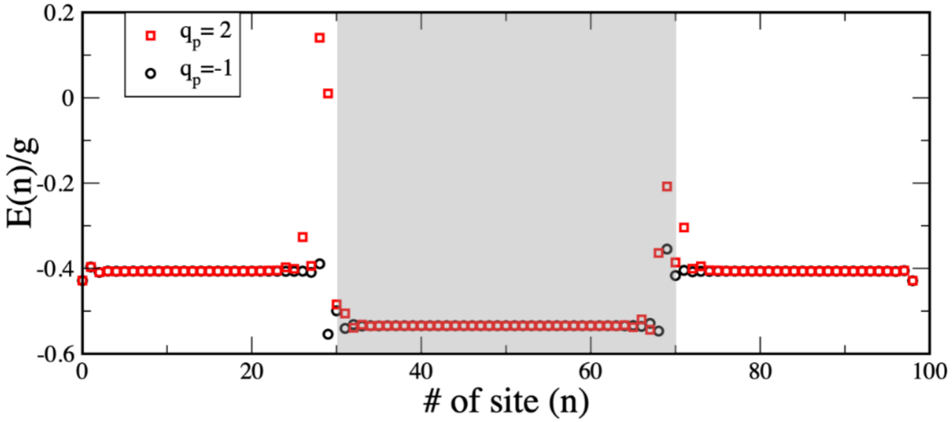
Adiabatic state preparation:

$\ell/a = 14$



Tensor Network (DMRG):

$\ell/a = 40, \quad N = 101$



should study problems not efficiently simulated by MC & TN

- long time evolution, many pt. function, non-local op.
- system w/ strong entanglement (matrix models?)

Thanks!

Appendix

Symmetries in charge-q Schwinger model

$$L = \frac{1}{2g^2}F_{01}^2 + \frac{\theta_0}{2\pi}F_{01} + \overline{\psi}\,\mathrm{i}\,\gamma^\mu(\partial_\mu + \mathrm{i}\,q\,A_\mu)\psi - m\,\overline{\psi}\psi$$

- $\boldsymbol{Z_q}$ chiral symmetry for $m = 0$

  —— ABJ anomaly: $U(1)_A \to \boldsymbol{Z_q}$

  —— known to be spontaneously broken

- $\boldsymbol{Z_q}$ 1-form symmetry

  —— remnant of $U(1)$ 1-form sym. in pure Maxwell

  —— Hilbert sp. is decomposed into q sectors *"universe"*

  (cf. common for $(d-1)$-form sym. in d dimensions)

Accessible region by analytic computation

- **Massive limit:**

The fermion can be integrated out

&

the theory becomes effectively pure Maxwell theory w/ θ

- **Bosonization:**

[Coleman '76]

$$\mathcal{L} = \frac{1}{8\pi}\left(\partial_\mu \phi\right)^2 - \frac{(qg)^2}{8\pi^2}\phi^2 + \frac{e^\gamma qg}{2\pi^{3/2}} m \cos(\phi + \theta/q)$$

exactly solvable for $m = 0$

&

small m regime is approximated by perturbation

Sign problem in path integral formalism

In Minkowski space,

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right] + \frac{g\theta}{4\pi}\int F \in \boldsymbol{R}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \ \mathcal{O}\ e^{iS}}{\int DAD\psi D\bar{\psi}\ e^{iS}}$$

highly oscillating

In Euclidean space,

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right] + i\,\frac{g\theta}{4\pi}\int F \in \boldsymbol{C}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \ \mathcal{O}\ e^{-S}}{\int DAD\psi D\bar{\psi}\ e^{-S}}$$

highly oscillating for non-small θ

Comments on choices of setup

There were many choices of setup to come here…

- Formulation of continuum theory?
- Type of lattice fermion?
- Boundary condition?
- Impose Gauss law?
- How to map fermion to spin system?
- Even N or odd N?

Choice of continuum theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

(used for the case w/ probes)

$\Updownarrow$ “chiral anomaly” [cf. Fujikawa’79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

(used for the case w/o probes)

- Equivalent for continuum theory w/o bdy.

—— (generically) inequivalent for theory on lattice or w/ bdy.

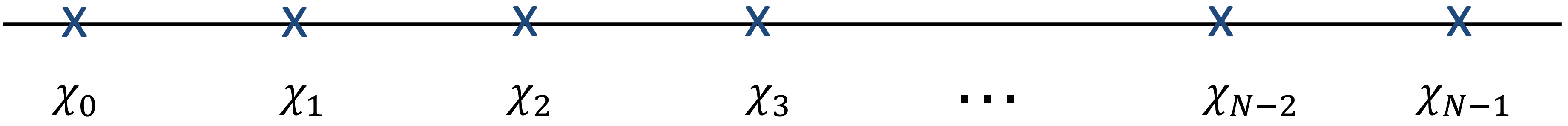
- The latter doesn’t violate θ-periodicity even for open b.c.

Choice of boundary conditions

Gauss law: $L_n - L_{n-1} = q\left[\chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}\right]$

Open b.c.	Periodic b.c.
▪ $L_n =$ (fermion op.)	▪ one of L_n's remains
$\longrightarrow \quad \dim(\mathcal{H}_{\mathrm{phys}}) < \infty$	$\longrightarrow \quad \dim(\mathcal{H}_{\mathrm{phys}}) = \infty$ *additional truncation needed*
▪ θ-periodicity is lost	▪ ${}^\exists\theta$-periodicity
▪ momentum not conserved	▪ momentum conserved

Even N or odd N?



Staggered fermion: $\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \begin{array}{l} \longrightarrow \text{odd site} \\ \longrightarrow \text{even site} \end{array}$

- Usually even N is taken (p.b.c. allows only even N)
- Open b.c. allows both but parity is different: $\chi_n \to i(-1)^n \chi_{N-n-1}$

	n mod 2	$\bar{\psi}\psi \sim \sum_n (-1)^n \chi_n^\dagger \chi_n$	$\bar{\psi}\gamma^5\psi \sim \sum_n (-1)^n (\chi_n^\dagger \chi_{n+1} - \mathrm{h.c.})$
even N	changes	flipped	invariant
odd N	invariant	invariant	flipped

Odd N seems more like the continuum theory?

Improvement of Suzuki-Trotter decomposition

The leading order decomposition:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\delta t}e^{-iH_2\delta t} + \mathcal{O}(\delta t^2)$$

The 2nd order improvement:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\frac{\delta t}{2}}e^{-iH_2\delta t}e^{-iH_1\frac{\delta t}{2}} + \mathcal{O}(\delta t^3)$$

cf. Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}$$

This increases the number of gates at each time step
but we can take larger δt (smaller M) to achieve similar accuracy.
Totally we save the number of gates.

Time evolution operator

Suzuki-Trotter decomposition:

(more precisely, we actually use its improvement but I skip it)

$$e^{-i\widehat{H}t} = \left(e^{-i\widehat{H}\frac{t}{M}}\right)^M \qquad (M \in \boldsymbol{Z}, M \gg 1)$$

$$\simeq \left(e^{-iH_Z\frac{t}{M}} e^{-iH_{ZZ}\frac{t}{M}} e^{-iH_{XX}\frac{t}{M}} e^{-iH_{YY}\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\begin{cases} H_Z = \dfrac{m\cos\theta}{2}\displaystyle\sum_{n=1}^{N}(-1)^n Z_n - \dfrac{J}{2}\displaystyle\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell=1}^{n} Z_\ell \\ H_{ZZ} = \dfrac{J}{2}\displaystyle\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell, \\ H_{XX} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\dfrac{m}{2}\sin\theta\right) X_n X_{n+1} \\ H_{YY} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\dfrac{m}{2}\sin\theta\right) Y_n Y_{n+1} \end{cases}$$

Can we express it in terms of elementary gates?

Time evolution operator (cont'd)

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_Z\frac{t}{M}}e^{-iH_{ZZ}\frac{t}{M}}e^{-iH_{XX}\frac{t}{M}}e^{-iH_{YY}\frac{t}{M}}\right)^M$$

- The 1st one is trivial:

$$e^{-icZ} = R_Z(2c)$$

- For the others, use the identities: (proof skipped)

$$\begin{cases} e^{-icZ_1Z_2} = CXR_Z^{(2)}(2c)CX \\ e^{-icX_1X_2} = CXR_X^{(1)}(2c)CX \\ e^{-icY_1Y_2} = R_Z^{(1)}\left(-\frac{\pi}{2}\right)R_Z^{(2)}\left(-\frac{\pi}{2}\right)e^{-icX_1X_2}R_Z^{(2)}\left(\frac{\pi}{2}\right)R_Z^{(1)}\left(\frac{\pi}{2}\right) \end{cases}$$

Only elementary gates !!

Tradeoff of symmetries in Suzuki-Trotter dec.

Suzuki-Trotter decomposition:

(more precisely, we actually use its improvement but I skip it)

$$(M \in \boldsymbol{Z}, M \gg 1)$$

$$e^{-iHt} = \left(e^{-iH\frac{t}{M}}\right)^M \simeq \left(e^{-iH_1\frac{t}{M}} e^{-iH_2\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\Rightarrow \quad H_{\text{eff}} = \frac{1}{-it} \log\left(e^{-iH_1\frac{t}{M}} e^{-iH_2\frac{t}{M}}\right)^M$$

Symmetries may be broken by decomposition

Tradeoff:

- Parity friendly (& translation if p.b.c.)

$$H = H_{XX} + H_{YY} + H_{ZZ} + H_Z \qquad \cancel{U(1)}$$

- $U(1)$ friendly

$$H = H_{XX+YY}^{(\text{even})} + H_{XX+YY}^{(\text{odd})} + H_{ZZ} + H_Z \qquad \cancel{P}$$

Comment on adiabatic state preparation

$$\boxed{(\text{"systematic error"}) \sim \frac{1}{T\,(\text{gap})^2}}$$

😄 Advantage:

- guaranteed to be correct for $T \gg 1$ & $\delta t \ll 1$ if $H_A(t)$ has a unique gapped vacuum
- can directly get excited states under some conditions

☹️ Disadvantage:

- doesn’t work for degenerate vacua
- costly — likely requires many gates

➡ more appropriate for FTQC than NISQ

Without probes

VEV of mass operator (chiral condensation)

$$\langle\bar{\psi}(x)\psi(x)\rangle = \langle\text{vac}|\bar{\psi}(x)\psi(x)|\text{vac}\rangle$$

Instead of the local op., we analyze the average over the space:

$$\frac{1}{2Na}\langle\text{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\text{vac}\rangle$$

Once we get the vacuum, we can compute the VEV as

$$\frac{1}{2Na}\langle\text{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\text{vac}\rangle = \frac{1}{2Na}\sum_{n=1}^{N}(-1)^n\sum_{i_1\cdots i_N=0,1}\langle\text{vac}|Z_n|i_1\cdots i_N\rangle\langle i_1\cdots i_N|\text{vac}\rangle$$

$$= \frac{1}{2Na}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{n+i_n}\,|\langle i_1\cdots i_N|\text{vac}\rangle|^2$$

How can we obtain the vacuum?

Massless case

For massless case,

θ is absorbed by chiral rotation ➡ $\theta = 0$ w/o loss of generality

No sign problem

Nevertheless,

it's difficult in conventional approach because computation of fermion determinant becomes very heavy

$\exists$ Exact result:

[Hetrick-Hosotani '88]

$$\langle \bar{\psi}(x)\psi(x) \rangle = -\frac{e^{\gamma}}{2\pi^{3/2}} g \simeq -0.160 g$$

Can we reproduce it?

Expectation value of mass op. (chiral condensation)

$$\langle \bar{\psi}(x)\psi(x)\rangle = \langle \mathrm{vac}|\bar{\psi}(x)\psi(x)|\mathrm{vac}\rangle$$

Instead of the local op., we analyze the average over the space:

$$\frac{1}{2Na}\langle \mathrm{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\mathrm{vac}\rangle$$

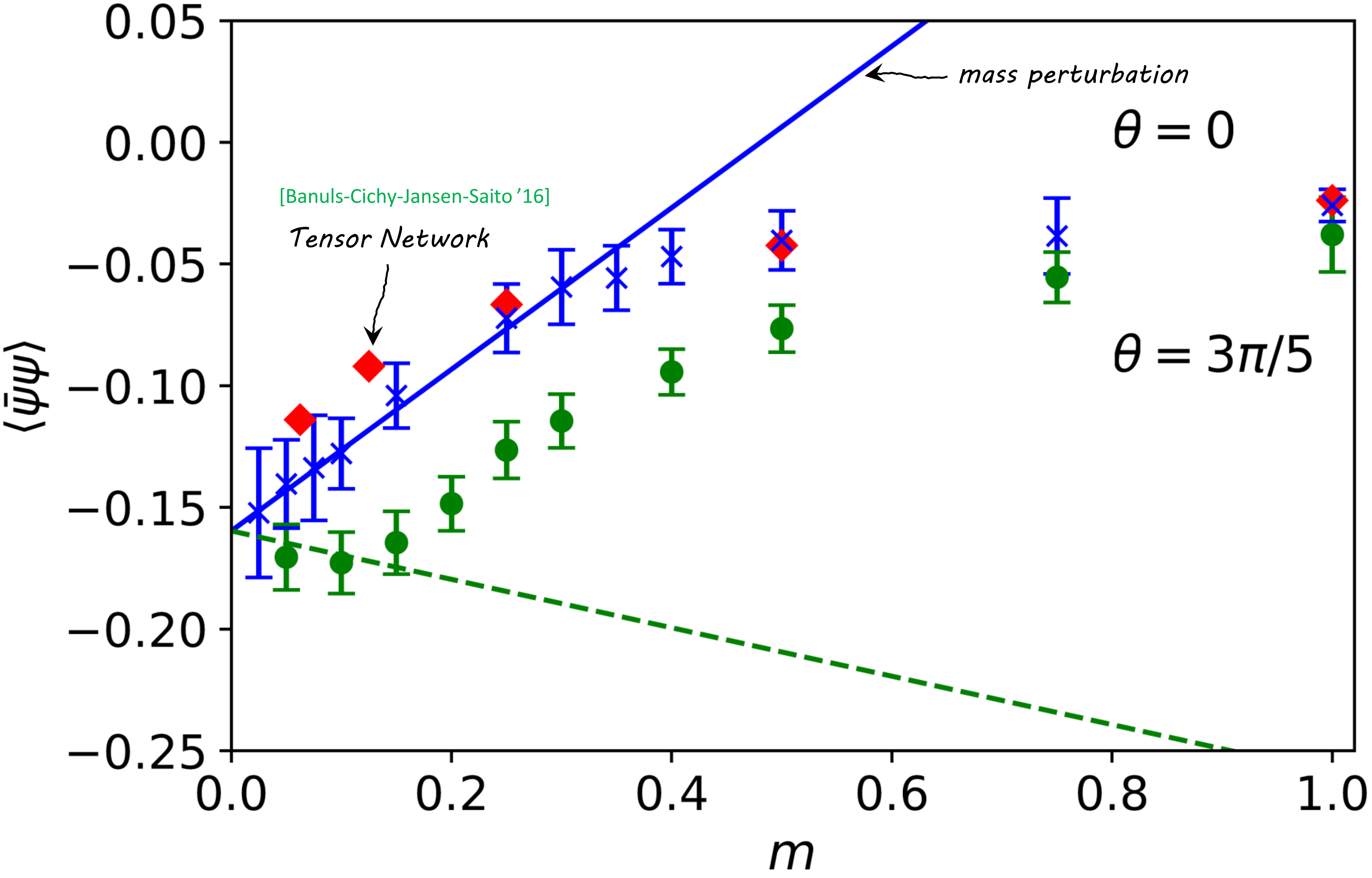
Once we get the vacuum, we can compute the VEV as

$$\frac{1}{2Na}\langle \mathrm{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\mathrm{vac}\rangle = \frac{1}{2Na}\sum_{n=1}^{N}(-1)^n \sum_{i_1\cdots i_N=0,1}\langle \mathrm{vac}|Z_n|i_1\cdots i_N\rangle\langle i_1\cdots i_N|\mathrm{vac}\rangle$$

$$= \frac{1}{2Na}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{n+i_n}\,|\langle i_1\cdots i_N|\mathrm{vac}\rangle|^2$$

Chiral condens. for massive case at g=1

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]



Estimation of systematic errors

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

Approximation of vacuum:

$$|\text{vac} > \simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\text{vac}_0 > \equiv |\text{vac}_A\rangle$$

Approximation of VEV:

$$\langle \mathcal{O} \rangle \equiv \langle \text{vac}|\mathcal{O}|\text{vac}\rangle \simeq \langle \text{vac}_A|\mathcal{O}|\text{vac}_A\rangle$$

Introduce the quantity

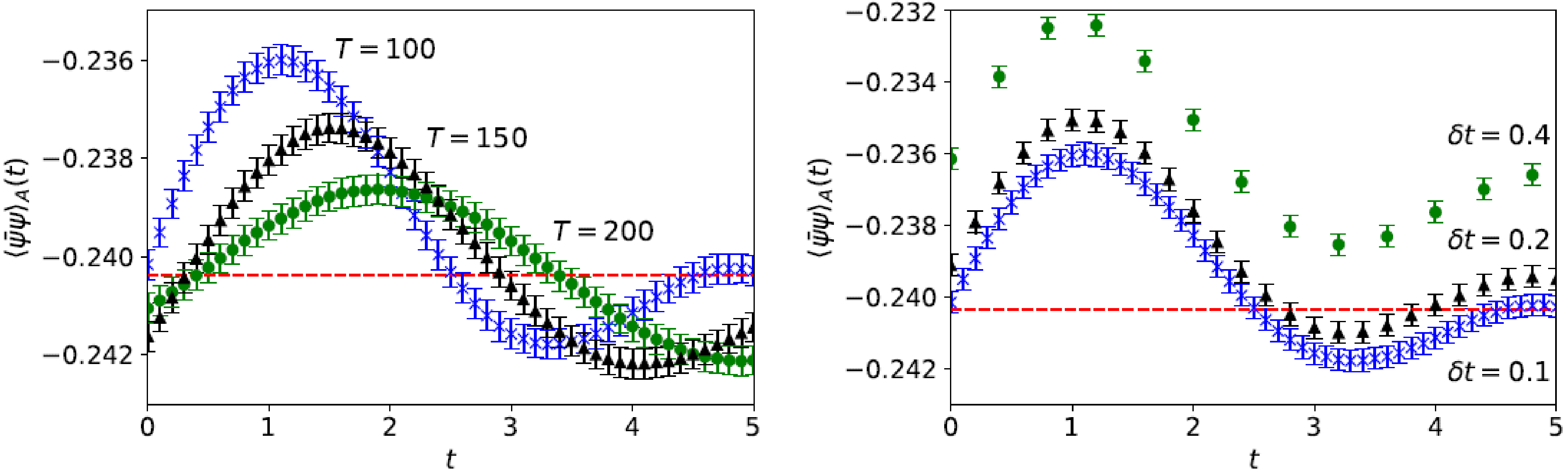
$$\langle \mathcal{O} \rangle_A(t) \equiv \langle \text{vac}_A| e^{i\hat{H}t} \mathcal{O} e^{-i\hat{H}t} |\text{vac}_A\rangle$$

$$\left\{\begin{array}{l} \text{independent of t if } |\text{vac}_A\rangle = |\text{vac}\rangle \\ \text{dependent on t if } |\text{vac}_A\rangle \neq |\text{vac}\rangle \end{array}\right.$$

This quantity describes intrinsic ambiguities in prediction

➡ Useful to estimate systematic errors

Estimation of systematic errors (Cont'd)



Oscillating around the correct value

➡ Define central value & error as

$\frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) + \min\langle\mathcal{O}\rangle_A(t)\right)$ & $\frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) - \min\langle\mathcal{O}\rangle_A(t)\right)$

Massive case

Result of mass perturbation theory: [Adam '98]

$$\langle\, \bar{\psi}(x)\psi(x) \,\rangle \simeq -0.160g + 0.322m \cos\theta + \mathcal{O}(m^2)$$

However,

$\exists$ subtlety in comparison: this quantity is UV divergent

$(\sim m \log \Lambda)$

⇨ Use a regularization scheme to have the same finite part

Here we subtract free theory result before taking continuum limit:

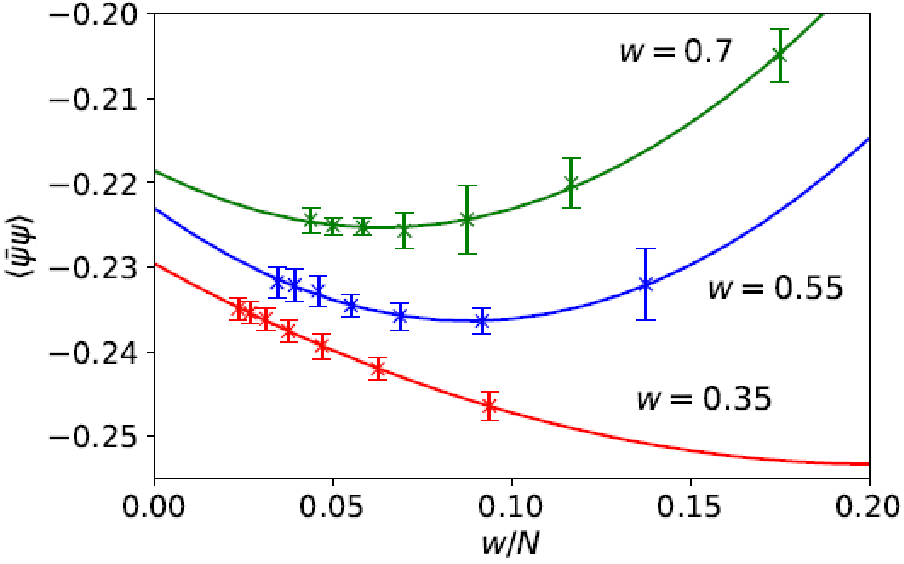
$$\lim_{a\to 0}\left[\langle\bar{\psi}\psi\rangle - \langle\bar{\psi}\psi\rangle_{\mathrm{free}}\right]$$

Thermodynamic & Continuum limit

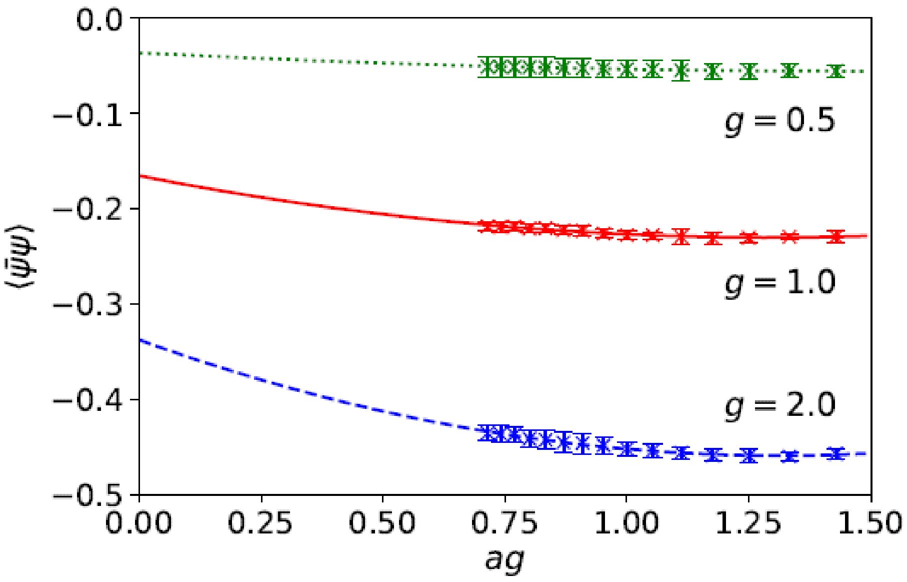
$g = 1, m = 0, N_{\text{max}} = 16, T = 100, \delta t = 0.1, 1M$ shots

#(measurements)

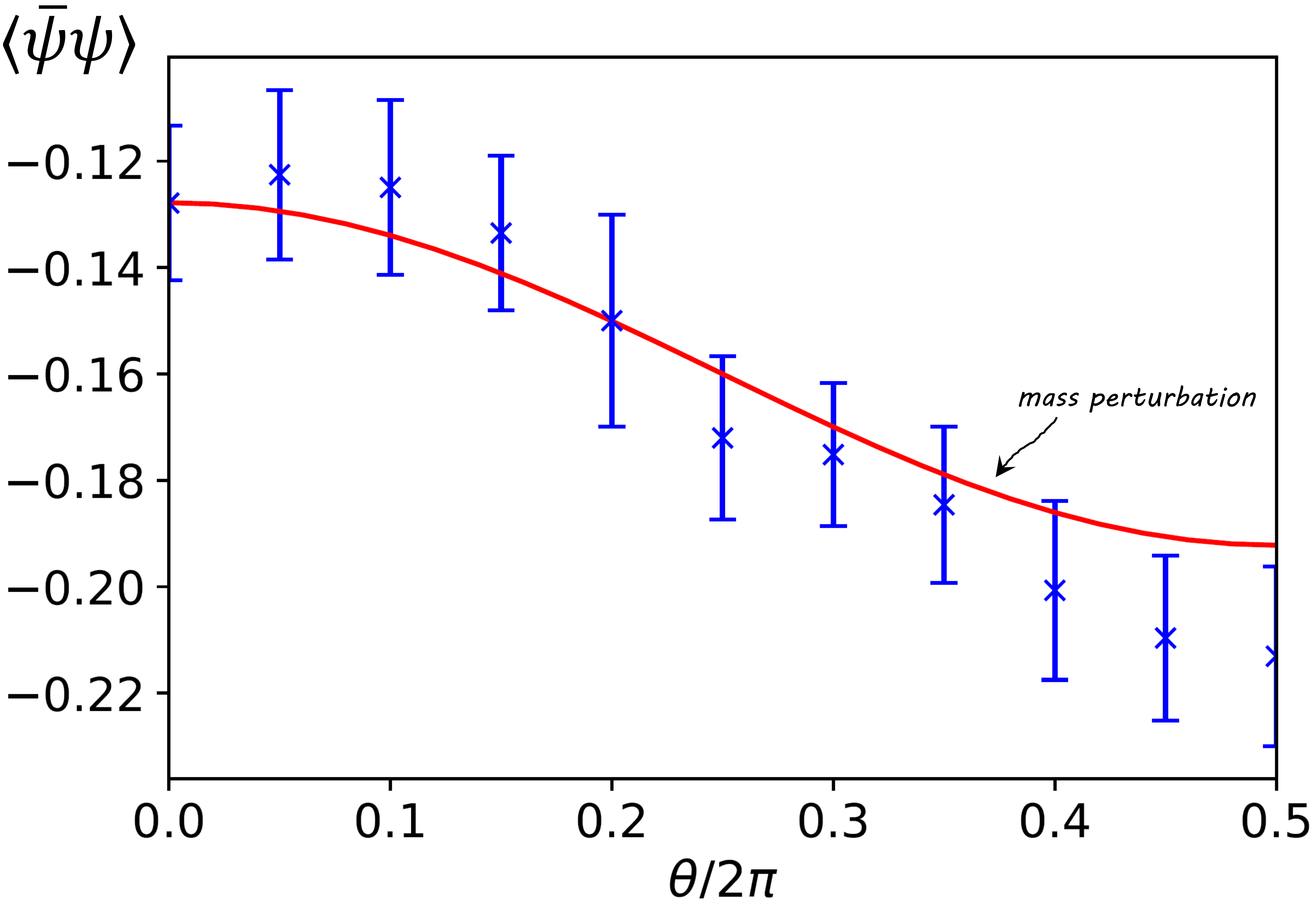
Thermodynamic limit (w/ fixed a)



Continuum limit (after $V \to \infty$)



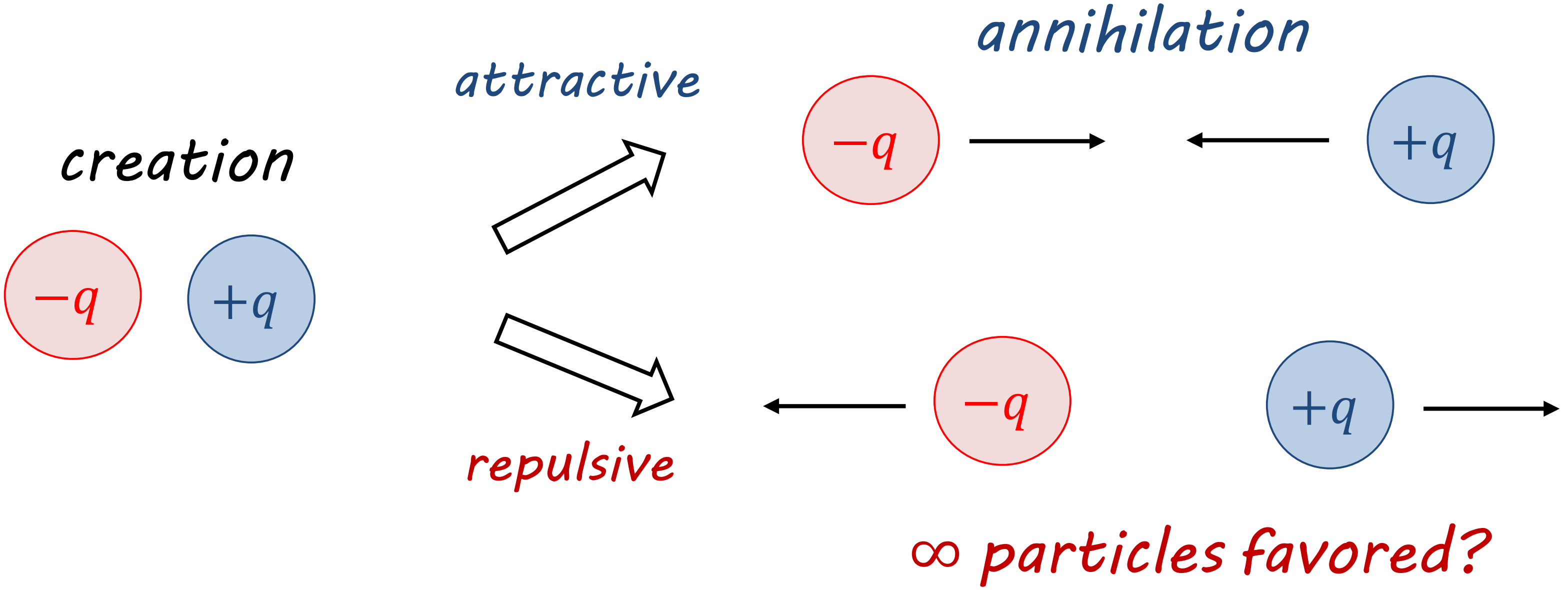
θ dependence at $m = 0.1$ & $g = 1$



With probes

FAQs on negative tension behavior

Q1. It sounds that many pair creations are favored.
Is the theory unstable?

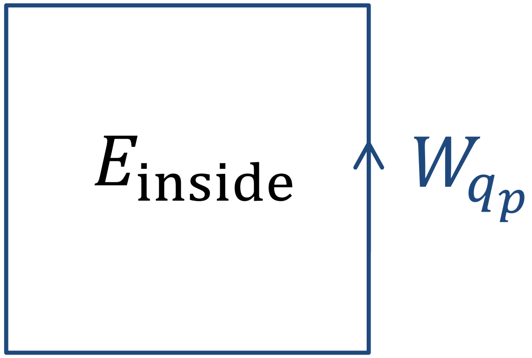


— **No.** Negative tension appears only for $q_p \neq q\boldsymbol{Z}$.

So, such unstable pair creations do not occur.

FAQs on negative tension behavior (cont’d)

[cf. MH-Itou-Kikuchi-Tanizaki ’21]



E_{inside} W_{qp}

$E_{\text{outside}}\ (= E_0 ?\,)$

Q2. It sounds $E_{\text{inside}} < E_{\text{outside}}$. Strange?

—— Inside & outside are in different “superselect. sectors” decomposed by Z_q 1-form sym.

$$\mathcal{H} = \bigoplus_{\ell=0}^{q-1} \mathcal{H}_\ell$$ *“universe”*

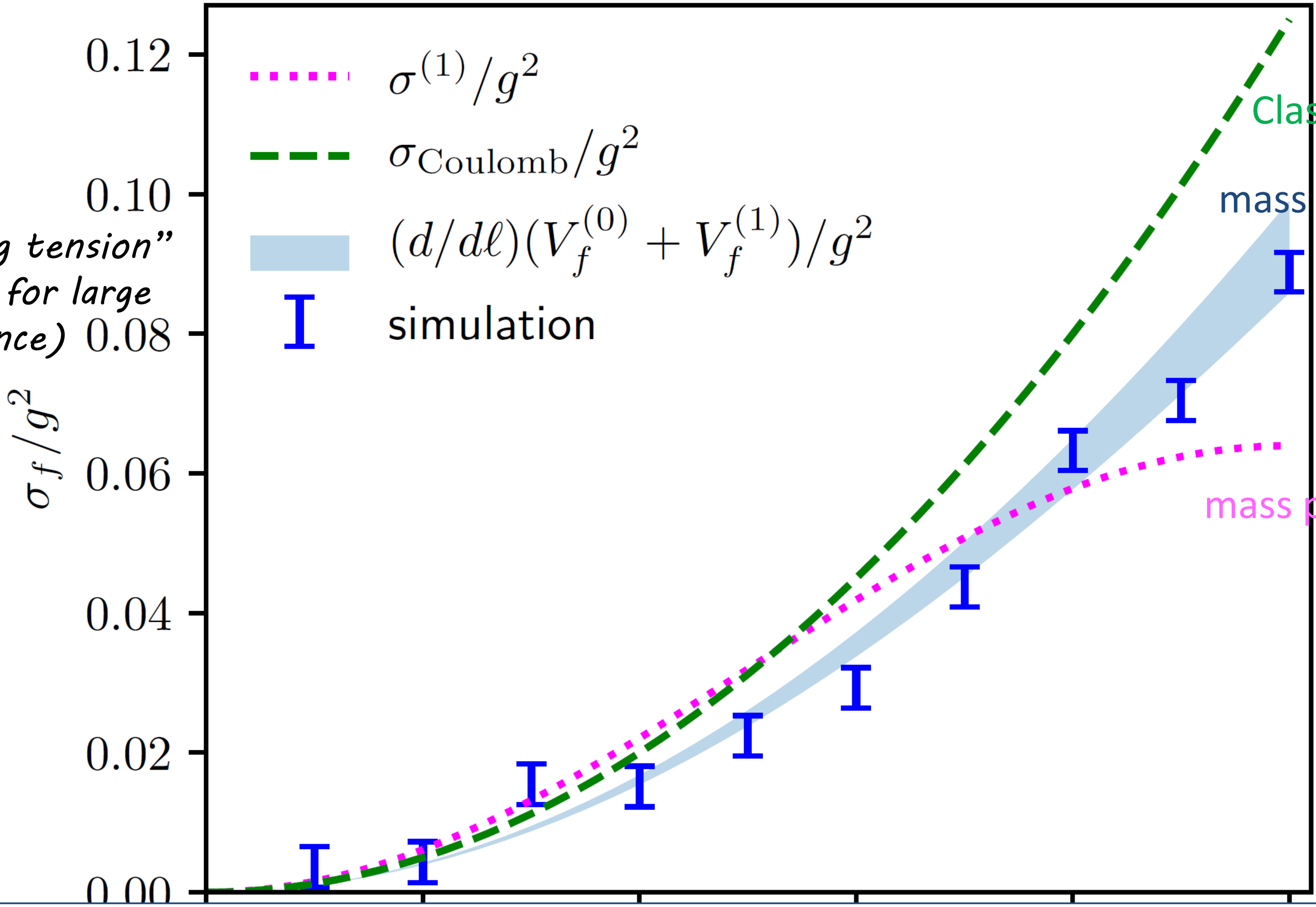
E_{inside} & E_{outside} are lowest in each universe:

$$E_{\text{inside}} = \min_{\mathcal{H}_{\ell+qp}} (E)\,, \qquad E_{\text{outside}} = \min_{\mathcal{H}_\ell} (E)$$

“String tension” for $\theta_0 = 0$

Parameters: $g = 1, a = 0.4, N = 15, T = 99, m/g = 0.2$

[MH-Itou-Kikuchi-Nagano-Okuda ’21]



“string tension” (slope for large distance)

Classical Coulomb

mass pert. (finite V)

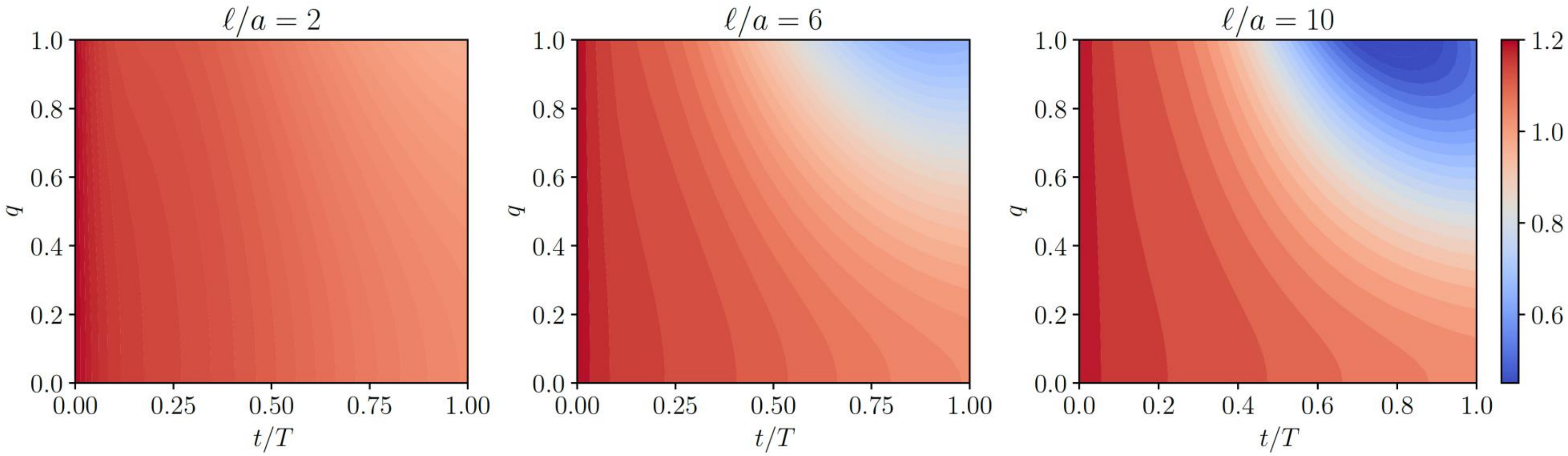
mass pert. (∞-vol.)

confinement by nontrivial dynamics!

Comment: density plots of energy gap

(known as “Tuna slice plot” inside the collaboration) [MH-Itou-Kikuchi-Nagano-Okuda ’21]

Parameters: $g = 1, a = 0.4, N = 15, q_p/q = 1, m/g = 0.15$



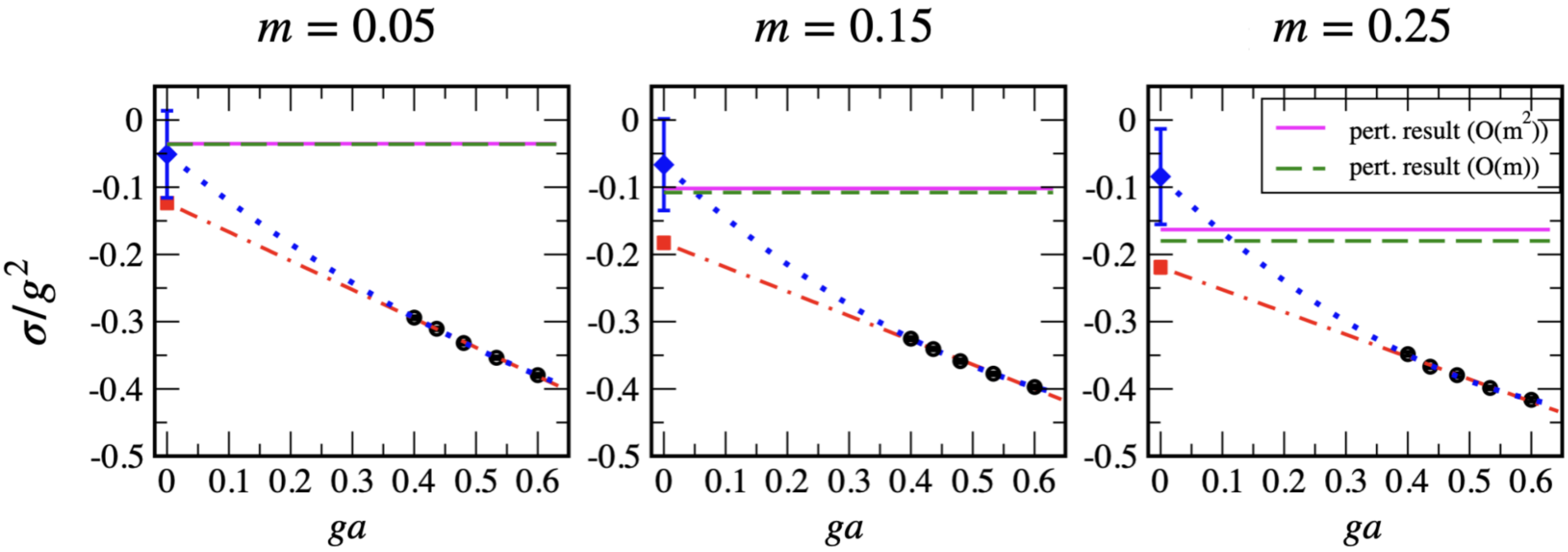
smaller gap for larger ℓ

⇨ larger systematic error for larger ℓ

Continuum limit of string tension

[MH-Itou-Kikuchi-Tanizaki '21]

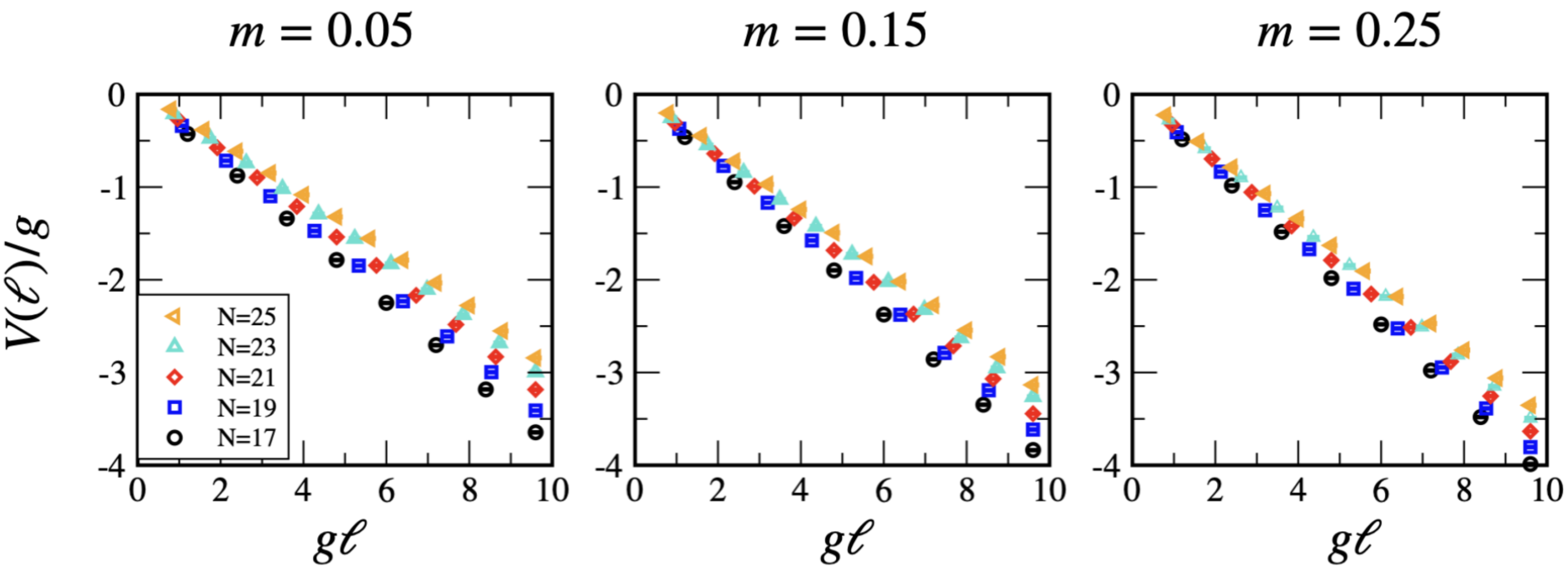
$g = 1,\ (\mathrm{Vol.}) = 9.6/g, T = 99,\ q_p/q = -1/3, m = 0.15, \theta_0 = 2\pi$



basically agrees with mass perturbation theory

N-dependence of V w/ fixed physical volume

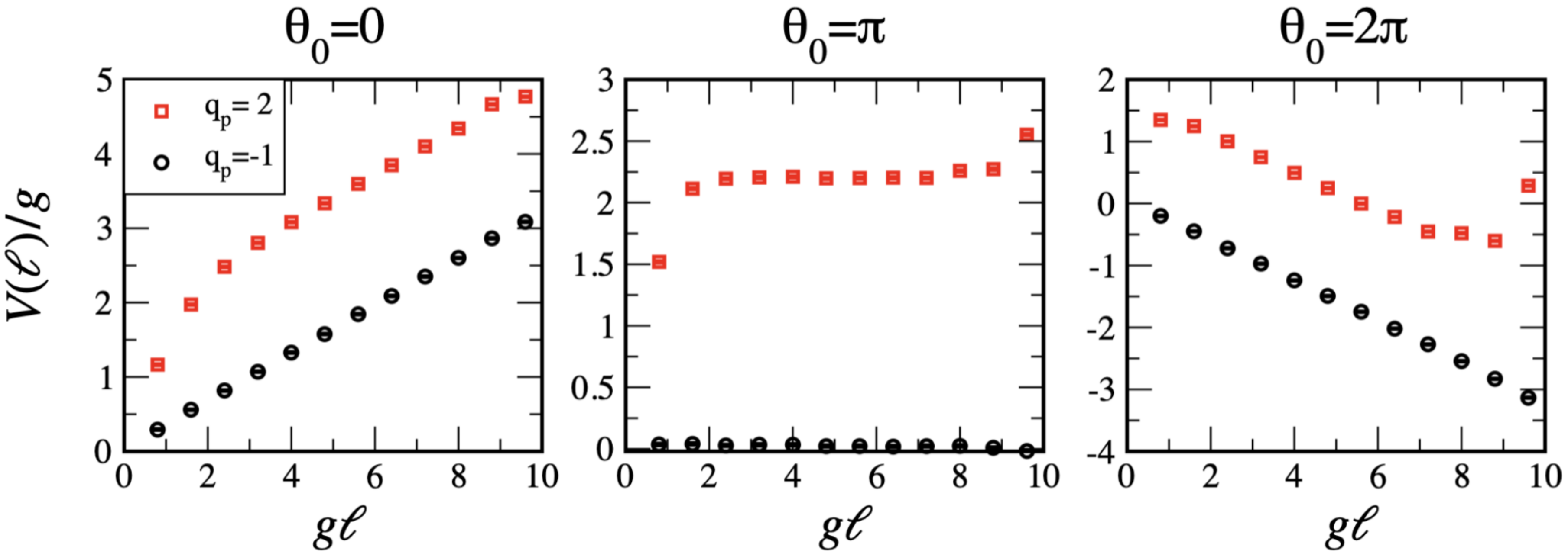
[MH-Itou-Kikuchi-Tanizaki ’21]



Comparison of $q_p/q = -1/3$ & $q_p/q = 2/3$

[MH-Itou-Kikuchi-Tanizaki ’21]

Parameters: q $= 3, g = 1, a = 0.4, N = 25, T = 99, m = 0.15$



Similar slopes → (approximate) $\boldsymbol{Z}_3$ symmetry

Adiabatic scheduling

[MH-Itou-Kikuchi-Tanizaki ’21]

$$N = 17, ga = 0.40, m = 0.20, q_p = 2, \theta_0 = 2\pi,$$