# Different versions of soft-photon theorems

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1 Introduction

2 Kinematics and phase space

3 Analysis of $\pi\pi \to \pi\pi$ and $\pi\pi \to \pi\pi\gamma$

4 Soft-photon theorem I

5 Soft-photon theorem II

6 Conclusions

# 1 Introduction

In this talk I shall mainly discuss the reactions

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2), \tag{1.1}$$

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon). \tag{1.2}$$

Let $\omega = k^0$ be the photon energy in the c.m. system.
We are interested in the limit $\underline{\omega \to 0}$. The classical papers dealing with this soft-photon limit are:

F. E. Low, "Bremsstrahlung of Very Low Energy Quanta in Elementary Particle Collisions", P. R. **110**, 974 (1958),
S. Weinberg, "Infrared Photons and Gravitons", P. R. **140** B516 (1965).

I hope to convince you in my talk that Low and Weinberg present quite <u>different versions</u> of soft-photon theorems. I shall show you their relation.

My talk is based on papers together with P. Lebiedowicz and A. Szczurek:

P. Lebiedowicz, O.N., A. Szczurek

PRD <u>105</u>, 014 022 (2022)

arXiv: 2307.12673 (2023)

arXiv: 2307.13291 (2023)

## Framework

We consider the reactions (1.1) and (1.2) in QCD plus leading order in electromagnetism. We use only exact QFT methods in this framework.

- energy-momentum conservation,
- gauge invariance,
- invariance under parity (P), charge conjugation (C), and time reversal (T),
- the generalised Ward identity for the pion fields, which in QCD are composite local fields,
- analyticity properties of amplitudes, in particular the Landau conditions.

## 2 Kinematics and phase space

We start with the elastic reaction

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2),$$

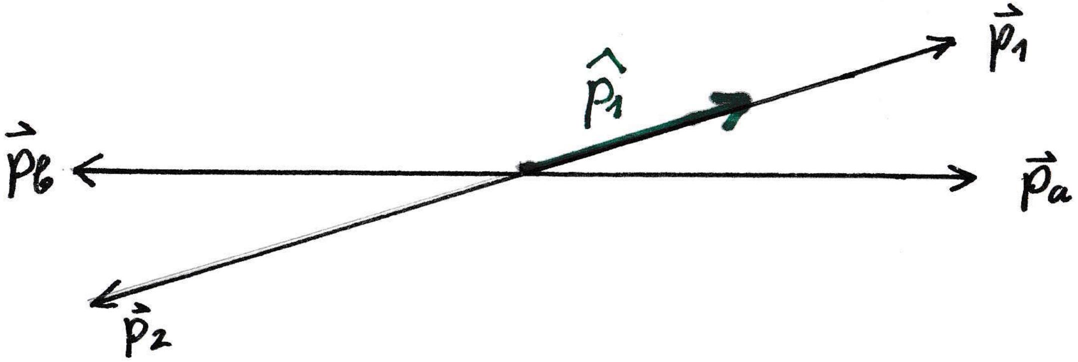
$$p_a + p_b = p_1 + p_2 . \tag{2.1}$$

We set as usual

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2,$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2. \tag{2.2}$$

We look at the reaction (2.1) in the c.m. system and consider a fixed value of the c.m. energy squared $s$. Then the energies and absolute values of the momenta are fixed. For a given initial configuration we can only vary $\hat{p}_1 = \vec{p}_1/|\vec{p}_1|$, the unit vector in direction of $\vec{p}_1$. The phase space is the unit sphere.




$$p_a^0 = p_b^0 = p_1^0 = p_2^0 = \frac{1}{2}\sqrt{s}\,,$$

$$|\vec{p}_a| = |\vec{p}_b| = |\vec{p}_1| = |\vec{p}_2| = \sqrt{\frac{s}{4} - m_\pi^2}\,,$$

$$\hat{p}_1 = \vec{p}_1 / |\vec{p}_1|\,. \qquad (2.3)$$

Now we go to the reaction with photon radiation

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon),$$

$$p_a + p_b = p_1' + p_2' + k. \tag{2.4}$$

Here we set

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_1' + p_2' + k)^2, \\ t_1 &= (p_a - p_1')^2 = (p_b - p_2' - k)^2, \\ t_2 &= (p_b - p_2')^2 = (p_a - p_1' - k)^2. \end{aligned} \tag{2.5}$$

We shall consider real and virtual photon emission and require

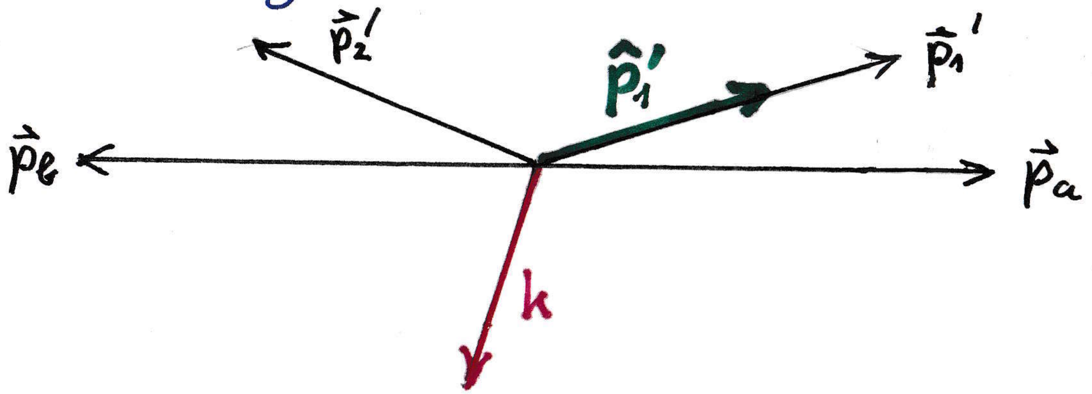
$$k^2 \geqslant 0, \quad k^0 \geqslant 0, \tag{2.6}$$

For a given value of $s$ what are the free parameters of the reaction (2.4) for small $k$, say $|k^\mu| \ll \sqrt{s - 4m_\pi^2}$ ($\mu = 0,1,2,3$), in the c. m. system?

A convenient set of such parameters is given by the four-vector $k$ plus the unit vector $\hat{\vec{p}}_1{}' = \vec{p}_1{}'/|\vec{p}_1{}'|$.

$$\text{Phase space of } (2.4) = \{(k, \hat{\vec{p}}_1{}');\ k \in \text{part of } R_4,\ |\hat{\vec{p}}_1{}'| = 1\}. \tag{2.7}$$

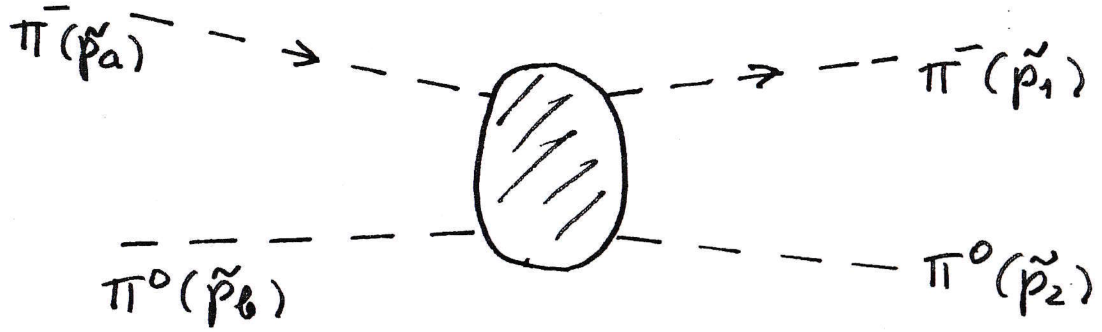
We see this easily by considering (2.4) for given $k$ in the rest system of $p_a + p_b - k = p_1' + p_2'$. In this system $\vec{p}_1{}'$ and $\vec{p}_2{}'$ are back to back with fixed $|\vec{p}_1{}'| = |\vec{p}_2{}'|$. The only freedom left is to vary $\vec{p}_1{}'$ in any direction. The same is then also true in the c.m. system if $k$ is small enough.



# 3 QFT analysis of $\pi\pi \longrightarrow \pi\pi$ and $\pi\pi \longrightarrow \pi\pi\gamma$

We consider the reaction, both on-shell and off shell,

$$\pi^-(\tilde{p}_a) + \pi^0(\tilde{p}_b) \longrightarrow \pi^-(\tilde{p}_1) + \pi^0(\tilde{p}_2) \qquad (3.1)$$




We have always energy - momentum conservation

$$\tilde{p}_a + \tilde{p}_b = \tilde{p}_1 + \tilde{p}_2 \qquad (3.2)$$

In relations which hold both on- and off shell we denote the momenta by $\tilde{p}_a, \ldots, \tilde{p}_2$.

As kinematic variables we have the masses of the, in general off shell, pions, an energy and a momentum transfer variable.

$$\tilde{\nu} = \tilde{p}_a \cdot \tilde{p}_b + \tilde{p}_1 \cdot \tilde{p}_2 \,, \qquad \tilde{t} = (\tilde{p}_a - \tilde{p}_1)^2 = (\tilde{p}_b - \tilde{p}_2)^2 \,,$$

$$m_a^2 = \tilde{p}_a^{\,2} \,, \quad m_b^2 = \tilde{p}_b^{\,2} \,, \quad m_1^2 = \tilde{p}_1^{\,2} \,, \quad m_2^2 = \tilde{p}_2^{\,2} . \tag{3.3}$$

Following Low we use here $\tilde{\nu}$ instead of Mandelstam's variable

$$\tilde{s} = \tilde{\nu} + \frac{1}{2}\left(m_a^2 + m_b^2 + m_1^2 + m_2^2\right). \tag{3.4}$$

The scattering amplitude for $\pi^- \pi^0 \longrightarrow \pi^- \pi^0$ can only depend on the above variables

$$T(\tilde{p}_a, \tilde{p}_b, \tilde{p}_1, \tilde{p}_2) = \mathcal{M}\,(\tilde{\nu}, \tilde{t}, m_a^2, m_b^2, m_1^2, m_2^2). \tag{3.5}$$

For the on-shell amplitude we have $\tilde{p}_a \to p_a, \dots, \tilde{p}_2 \to p_2$,

$$m_a^2 = m_b^2 = m_1^2 = m_2^2 = m_\pi^2, \quad \tilde{\nu} \to \nu \,, \ \tilde{t} \to t \,,$$

$$
\begin{aligned}
T(p_a, p_b, p_1, p_2)\Big|_{\text{on shell}} &= \mathcal{M}(\nu, t, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2) \\
&\equiv \mathcal{M}^{(on)}(\nu, t).
\end{aligned}
\tag{3.6}
$$

Now we come to the photon-emission reaction (on shell)

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k,\varepsilon) \qquad (3.7)$$

where we have from energy-momentum conservation

$$p_a + p_b = p_1' + p_2' + k. \qquad (3.8)$$

Note that for $k \neq 0$ we must have a change of $p_1'$, $p_2'$ from the values $p_1$, $p_2$ for $k=0$.

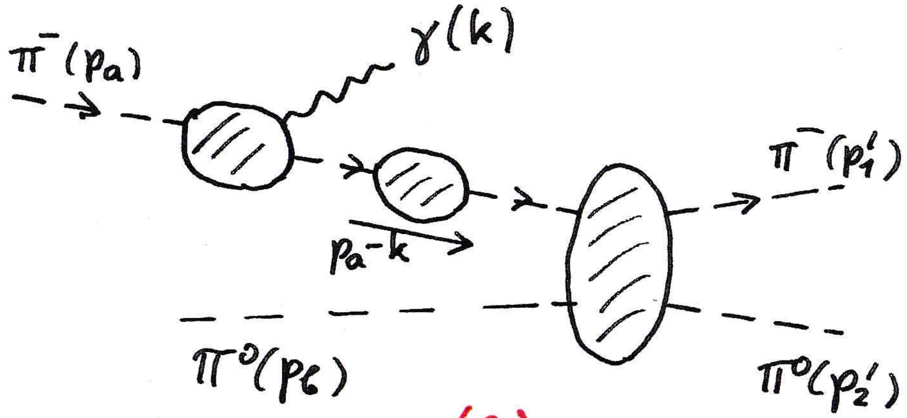
The amplitude for the above reaction is

$$\langle \gamma(k,\varepsilon), \pi^-(p_1'), \pi^0(p_2') | T | \pi^-(p_a), \pi^0(p_b) \rangle = \varepsilon^{\lambda *} \mathcal{M}_\lambda . \qquad (3.9)$$

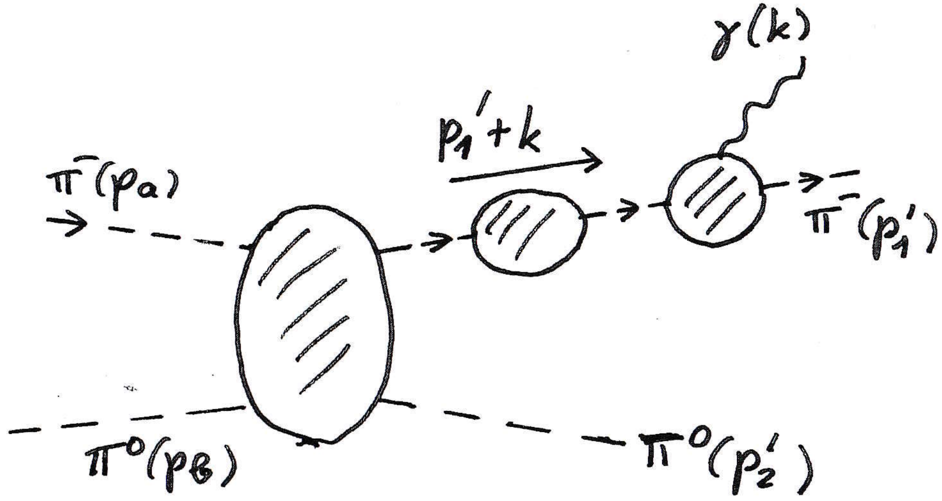
In the following we consider $\mathcal{M}_\lambda$ for real and timelike virtual photons

$$\mathcal{M}_\lambda = \mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k), \quad k^2 \geq 0, \; k^0 \geq 0. \qquad (3.10)$$

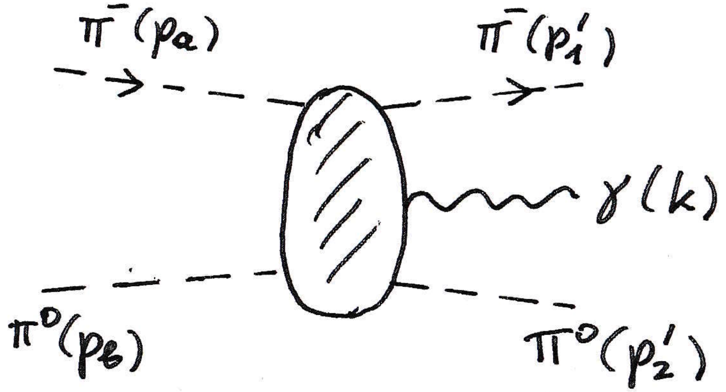
# Diagrams for $\mathcal{M}_\lambda$




(a)




(b)

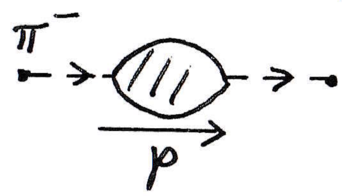



(c)

(a), (b) 1 particle reducible

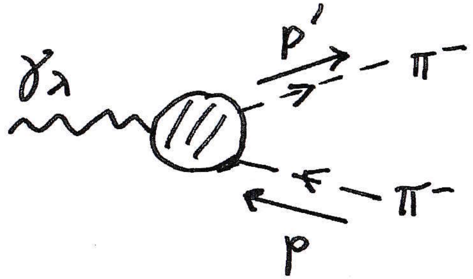
(c) 1 particle irreducible

pion propagator:




$i\Delta(p^2)$

vertex $\gamma\pi\pi$:




$ie\Gamma_\lambda(p',p)$

(3.11)

With the $\pi\pi \to \pi\pi$ off-shell amplitude, the pion propagator, and the $\gamma\pi\pi$ vertex we get $\mathcal{M}_\lambda = \mathcal{M}_\lambda^{(a)} + \mathcal{M}_\lambda^{(b)} + \mathcal{M}_\lambda^{(c)}$, where

$$\mathcal{M}_\lambda^{(a)} = -e\, \mathcal{M}^{(a)} \quad \Delta[(p_a-k)^2]\, \Gamma_\lambda(p_a-k, p_a),$$

$$\mathcal{M}^{(a)} = T(p_a-k, p_b, p_1', p_2')\big|_{\text{off shell}}$$

$$= \mathcal{M}\ \left[(p_a-k, p_b)+p_1'\cdot p_2',\ (p_b-p_2')^2,\ (p_a-k)^2,\ m_\pi^2,\ m_\pi^2,\ m_\pi^2\right],$$

$$\mathcal{M}_\lambda^{(b)} = -e\, \Gamma_\lambda(p_1', p_1'+k)\, \Delta[(p_1'+k)^2]\ \mathcal{M}^{(b)}, \tag{3.12}$$

$$\mathcal{M}^{(b)} = T(p_a, p_b, p_1'+k, p_2')\big|_{\text{off shell}}$$

$$= \mathcal{M}\ \left[p_a\cdot p_b + (p_1'+k, p_2'),\ (p_b-p_2')^2,\ m_\pi^2,\ m_\pi^2,\ (p_1'+k)^2,\ m_\pi^2\right]. \tag{3.13}$$

We shall now use the best tool from QFT: gauge invariance.

With this we get the Ward-Takahashi identity

$$(p'-p)^\lambda \Gamma_\lambda(p',p) = \Delta^{-1}(p'^2) - \Delta^{-1}(p^2) \tag{3.14}$$

and the condition

$$k^\lambda \left[ m_\lambda^{(a)} + m_\lambda^{(b)} + m_\lambda^{(c)} \right] = 0. \tag{3.15}$$

As a consequence of these we find

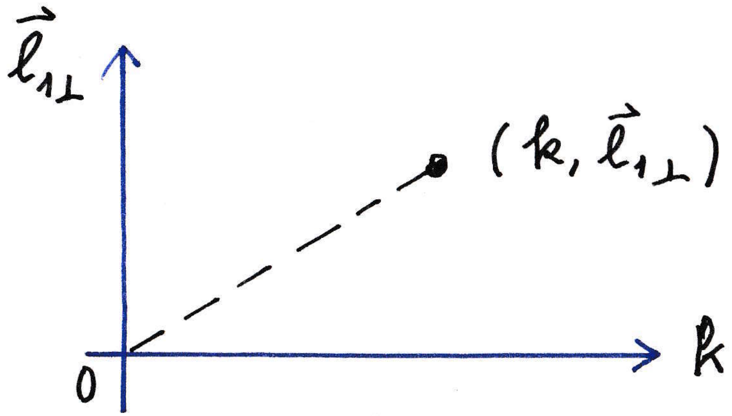
$$k^\lambda m_\lambda^{(c)} = -e\, m^{(a)} + e\, m^{(b)}. \tag{3.16}$$

# 4 Soft photon theorem I

In this section we shall give the expansion of the amplitude $\mathcal{M}_\lambda$ around the phase-space point $(k=0,\ \hat{p}_1' = \hat{p}_1)$. In a small neighbourhood of this phase-space point we set:

$$\hat{p}_1' = \hat{p}_1 - \vec{\ell}_{1\perp}/|\vec{p}_1| \,, \quad \vec{\ell}_{1\perp}\cdot\hat{p}_1 = 0 \,, \quad k, \vec{\ell}_{1\perp} = \mathcal{O}(\omega) \,. \tag{4.1}$$

This neighbourhood has 6 dimensions. Schematically we represent it as follows:




Below we shall study $\mathcal{M}_\lambda$ on such a ray starting from the origin.

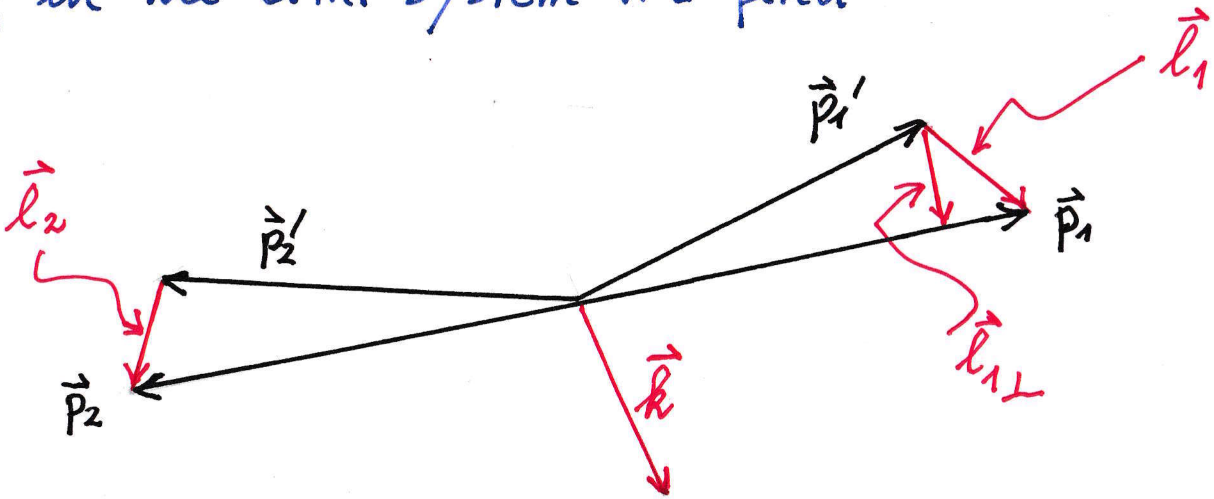
For $(k=0, \vec{l}_{1\perp}=0)$ we have the kinematics of the reaction without radiation, $\pi^-(p_a)+\pi^0(p_b)\to\pi^-(p_1)+\pi^0(p_2)$. For $(k, \vec{l}_{1\perp})$ we have the kinematics of photon radiation with

$$p_1' = p_1 - l_1 \;, \quad p_2' = p_2 - l_2 , \tag{4.2}$$
$$l_1 + l_2 = k ,$$

as required by energy-momentum conservation

$$p_a + p_b = p_1' + p_2' + k = p_1 + p_2 .$$

Working in the c.m. system we find



$$p_1 = \begin{pmatrix} p_1^0 \\ |\vec{p}_1|\,\hat{p}_1 \end{pmatrix} , \qquad p_2 = \begin{pmatrix} p_1^0 \\ -|\vec{p}_1|\,\hat{p}_1 \end{pmatrix}$$

$$s = (p_1 + p_2)^2 , \quad p_1^0 = \frac{1}{2}\sqrt{s} , \quad |\vec{p}_1| = \sqrt{\frac{s}{4} - m_\pi^2} , \quad |\hat{p}_1| = 1 , \tag{4.3}$$

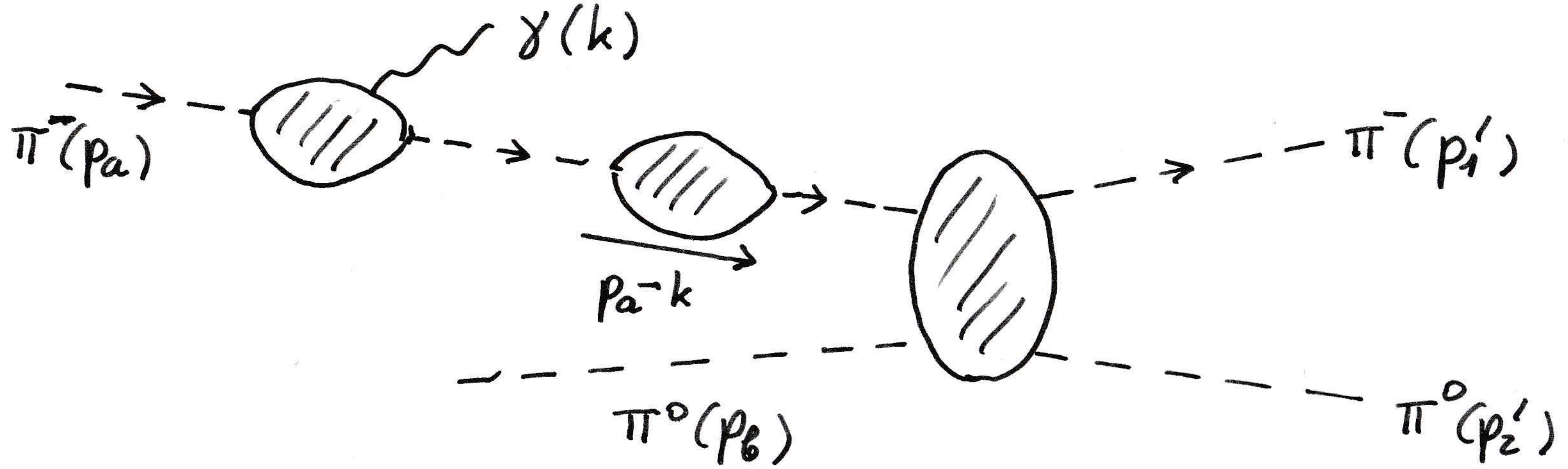
$$l_1 = \begin{pmatrix} \frac{p_2 \cdot k}{\sqrt{s}} \\ \frac{p_1^0}{|\vec{p}_1|\sqrt{s}}(p_2 \cdot k)\,\hat{p}_1 + \vec{l}_{1\perp} \end{pmatrix} + \mathcal{O}(\omega^2) , \quad l_2 = \begin{pmatrix} \frac{p_1 \cdot k}{\sqrt{s}} \\ \vec{k} - \frac{p_1^0\,\hat{p}_1}{|\vec{p}_1|\sqrt{s}}(p_2 \cdot k) - \vec{l}_{1\perp} \end{pmatrix} + \mathcal{O}(\omega^2)$$

$$p_1 \cdot l_1 = 0 + \mathcal{O}(\omega^2) , \qquad p_2 \cdot l_2 = 0 + \mathcal{O}(\omega^2) . \tag{4.4}$$

Now we come to the expansion of the amplitude $\mathcal{M}_\lambda$ for $\omega \to 0$.
To be precise we set, in the c.m. system with $\omega \geqslant 0$:

$$k = \omega \begin{pmatrix} 1 \\ \vec{\tilde{k}} \end{pmatrix}, \quad |\vec{\tilde{k}}| \leqslant 1, \quad \vec{l}_{1\perp} = \omega \, \vec{\tilde{l}}_{1\perp}, \quad |\vec{\tilde{l}}_{1\perp}| = \mathcal{O}(1). \tag{4.5}$$

We keep $\vec{\tilde{k}}$ and $\vec{\tilde{l}}_{1\perp}$ fixed and consider the expansion of the radiative amplitude for $\omega \to 0$. That is, we consider $\mathcal{M}_\lambda$ on a ray starting at the origin in the phase space $\{(k, \vec{l}_{1\perp})\}$.
Of course, we shall get a Laurent expansion for $\mathcal{M}_\lambda$.
We illustrate this for the term $\mathcal{M}_\lambda^{(a)}$.




$$\mathcal{M}_\lambda^{(a)} = -e\, T(p_a - k, p_b, p_1', p_2')\Big|_{\text{off shell}} \Delta[(p_a-k)^2]\, \Gamma_\lambda(p_a - k, p_a), \tag{4.6}$$

$$T(p_a - k, p_b, p_1', p_2')\Big|_{\text{off shell}} = \mathcal{M}\big[(p_a - k, p_b) + (p_1 - l_1, p_2 - l_2),\ (p_b - p_2 + l_2)^2,$$
$$(p_a - k)^2,\ m_\pi^2,\ m_\pi^2,\ m_\pi^2\big]. \tag{4.7}$$

From the generalised Ward identity we find for $\omega \to 0$

$$\Delta[(p_a - k)^2]\, \Gamma_\lambda(p_a - k, p_a) = \frac{(2p_a - k)_\lambda}{-2p_a \cdot k + k^2} + \mathcal{O}(\omega). \tag{4.8}$$

To get $\mathcal{M}_\lambda^{(a)}$ to the orders $\omega^{-1}$ and $\omega^0$ we need $\mathcal{M}$ to the orders $\omega^0$ and $\omega$.

This expansion of $\mathcal{M}[\ \ ]$ is straightforward, remembering that $k$, $l_1$, $l_2$ are all of order $\omega$.

Treating in this way $\mathcal{M}_\lambda^{(a)}$ and $\mathcal{M}_\lambda^{(b)}$ and determining $\mathcal{M}_\lambda^{(c)}$ to the order $\omega^0$ from the gauge invariance condition (3.16) we get for the case of real photon emission, $k^2=0$, the following

$$
\begin{aligned}
\mathcal{M}_\lambda(p_a,p_b,p_1',p_2',k) &= e\left[\frac{p_{a\lambda}}{p_a\cdot k}-\frac{p_{1\lambda}'}{p_1'\cdot k}\right]\mathcal{M}^{(on)}(\nu,t)\\
&-2e\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k}-p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(on)}(\nu,t)\\
&-2e\left[\frac{p_{a\lambda}}{p_a\cdot k}-\frac{p_{1\lambda}}{p_1\cdot k}\right]\left[(p_a-p_1,k)-(p_a\cdot l_1)\right]\frac{\partial}{\partial t}\mathcal{M}^{(on)}(\nu,t)+\mathcal{O}(\omega).
\end{aligned}
\tag{4.9}
$$

We can, for consistency, still expand

$$
\frac{p_{1\lambda}'}{p_1'\cdot k}=\frac{(p_1-l_1)_\lambda}{(p_1-l_1,k)}=\frac{p_{1\lambda}}{p_1\cdot k}+\frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l_1\cdot k)-l_{1\lambda}(p_1\cdot k)\right].
\tag{4.10}
$$

This gives us our final result for real photon emission

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\mathcal{M}^{(on)}(\nu, t) \qquad \Big|\ \omega^{-1}$$

$$- e\,\frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l_1\cdot k) - l_{1\lambda}(p_1\cdot k)\right]\mathcal{M}^{(on)}(\nu, t)$$

$$- 2e\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial \nu}\mathcal{M}^{(on)}(\nu, t) \qquad \Big|\ \omega^{0}$$

$$- 2e\left[(p_a - p_1, k) - (p_a\cdot l_1)\right]\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\frac{\partial}{\partial t}\mathcal{M}^{(on)}(\nu, t)$$

$$+ \mathcal{O}(\omega). \tag{4.11}$$

$$\nu = s - 2m_\pi^2, \quad t = (p_a - p_1)^2 = (p_b - p_2)^2.$$

With (4.11) we have given the Laurent expansion of $\mathcal{M}_\lambda$ to the orders $\omega^{-1}$ and $\omega^0$ around the phase space point $(k=0, \vec{l}_{1\perp}=0)$ corresponding to $k=0, p_1'=p_1, p_2'=p_2$.

The amplitude $\mathcal{M}^{(on)}(\nu, t)$ corresponds to the basic process

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2).$$

The pole term $\propto \omega^{-1}$ in (4.11) is exactly Weinberg's soft-photon term. He writes

"Hence the effect of attaching one soft-photon line to an arbitrary diagram is simply to supply an extra factor,

$$\sum_n e_n \eta_n p_n{}^\mu / [p_n \cdot q - i\eta_n \varepsilon], \qquad (4.12)$$

the sum running over all external lines in the original diagram."

Here $\eta_n = +1$ $(-1)$ for an outgoing (incoming) charged particle.

In our work we have given the next to leading term, $\mathcal{O}(\omega^0)$, to Weinberg's pole term.

# 5 Soft photon theorem II

Now we want to discuss Low's version of soft-photon theorem. Of course, as starting point he considers the diagrams (a), (b), (c) for $\mathcal{M}_\lambda$. He also uses the generalised Ward identity which gave us (4.8) for $\Delta[(p_a-k)^2]\,\Gamma_\lambda(p_a-k, p_a)$. Considering only real photon emission we have then for $\mathcal{M}_\lambda^{(a)}$ (see eq. (2.11) of Low)

$$\mathcal{M}_\lambda^{(a)} = e\,\mathcal{M}\left[(p_a-k, p_b) + p_1'\cdot p_2', (p_b-p_2')^2, m_a^2=(p_a-k)^2, m_\pi^2, m_\pi^2, m_\pi^2\right]\frac{p_{a\lambda}}{p_a\cdot k}\,. \tag{5.1}$$

Now Low expands $\mathcal{M}$ with respect to $k$ keeping $p_1'$ and $p_2'$ fixed. That is, he expands with respect to $k$ which is explicit in the chosen parametrisation. In this way we get:

$$\mathcal{M}_\lambda^{(a)}(p_a, p_b, p_1', p_2', k) = e\,\frac{p_{a\lambda}}{(p_a\cdot k)}\Big\{\mathcal{M}^{(on)}\left[p_a\cdot p_b + p_1'\cdot p_2',\ (p_b - p_2')^2\right]$$

$$- (p_b\cdot k)\frac{\partial}{\partial\nu}\mathcal{M}^{(on)}\left[p_a\cdot p_b + p_1'\cdot p_2',\ (p_b - p_2')^2\right]$$

$$- 2(p_a\cdot k)\,\frac{\partial}{\partial m_a^2}\mathcal{M}\left[p_a\cdot p_b + p_1'\cdot p_2',\ m_a^2, m_\pi^2, m_\pi^2, m_\pi^2\right]\Big|_{m_a^2 = m_\pi^2}\Big\} + \mathcal{O}(k)\,. \tag{5.2}$$

Note an important point: whereas the expansion of the scalar function $\mathcal{M}[\ ]$ with respect to $k$ keeping $p_1'$ and $p_2'$ fixed is completely legitimate and a usual expansion, this is not the case for $\mathcal{M}_\lambda^{(a)}$. We must remember the energy-momentum conservation:

$$p_a + p_b = p_1' + p_2' + k. \tag{5.3}$$

Keeping $p_1'$ and $p_2'$ fixed, also $k$ is fixed and (5.2) makes sense only for this one value of $k$.

Now we can treat $\mathcal{M}_\lambda^{(b)}$ in a similar way and then determine $\mathcal{M}_\lambda^{(c)}$ approximately from the gauge-invariance condition (3.16). The result is Low's formula (eq. (2.16) of Low):

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p'_{1\lambda}}{p'_1\cdot k}\right]\mathcal{M}^{(on)}(\nu_L, t_2)$$
$$- e\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} + p'_{1\lambda}\frac{p'_2\cdot k}{p'_1\cdot k} - p_{b\lambda} - p'_{2\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(on)}(\nu_L, t_2) + \mathcal{O}(k), \tag{5.4}$$

where

$$\nu_L = p_a\cdot p_b + p_1'\cdot p_2' = s - 2m_\pi^2 - (p_a + p_b, k),$$
$$t_2 = (p_b - p_2')^2 = (p_a - p_1' - k)^2. \tag{5.5}$$

We emphasise again that (5.4) is not an expansion of $\mathcal{M}_\lambda$ around some phase-space point. The r.h.s of (5.4) gives an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point $p_1'$, $p_2'$, $k$.

We can use (5.4) only for one value of $k = p_a + p_b - p_1' - p_2'$. If we use (5.4) at a different value of $k$ we violate energy-momentum conservation.

Furthermore, the leading approximation in (5.4) does not give what is frequently called Low's theorem. We see this best by considering the reactions

$$\pi^-(p_a) + \pi^+(p_b) \longrightarrow \pi^-(p_1') + \pi^+(p_2') + \gamma(k, \varepsilon),$$

$$\pi^-(p_a) + \pi^+(p_b) \longrightarrow \pi^-(p_1) + \pi^+(p_2). \tag{5.6}$$

According to Low we get

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}'}{p_1'\cdot k}\right]\mathcal{M}^{(on)}(\nu_L, t_2)$$

$$+ e\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}'}{p_2'\cdot k}\right]\mathcal{M}^{(on)}(\nu_L, t_1) + \mathcal{O}(\omega^0),$$

$$\nu_L = s - 2m_\pi^2 - (p_a + p_b, k), \quad t_1 = (p_a - p_1')^2, \quad t_2 = (p_b - p_2')^2, \qquad (5.7)$$

According to Weinberg we have, see (4.12),

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2', k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k} - \frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right]\mathcal{M}^{(on)}(\nu, t) + \mathcal{O}(\omega^0)$$

$$\nu = s - 2m_\pi^2, \quad t = (p_a - p_1)^2 = (p_b - p_2)^2. \qquad (5.8).$$

Low's formula (5.7) gives an approximate expression for $\mathcal{M}_\lambda$ at one given phase-space point. Weinberg's formula (which is frequently but incorrectly called Low's theorem) gives the pole term of the Laurent expansion of $\mathcal{M}_\lambda$ around the point $p_i' = p_i$, $k = 0$.

Let us go back to $\pi^- + \pi^0 \to \pi^- + \pi^0 + \gamma$. In (5,4) we have Low's formula which gives us an approximate expression for $m_\lambda$ at a given phase-space point. We can construct, as we did in Sec. 4, the corresponding expansion of this approximate expression around the phase-space point $(k=0,\ \hat{p}_1)$. Inserting in (5.4) $p_1' = p_1 - l_1$, $p_2' = p_2 - l_2$ from (4.2) we get

$$
\begin{aligned}
m^{(on)}(\nu_L, t_2) &= m^{(on)}\left[\nu - (p_a + p_b, k),\ t - 2\left((p_a - p_1, k) - p_a \cdot l_1\right)\right] + \mathcal{O}(\omega^2) \\
&= m^{(on)}(\nu, t) - (p_a + p_b, k)\frac{\partial}{\partial \nu} m^{(on)}(\nu, t) - 2\left[(p_a - p_1, k) - p_a \cdot l_1\right]\frac{\partial}{\partial t} m^{(on)}(\nu, t) \\
&\quad + \mathcal{O}(\omega^2),
\end{aligned}
\qquad (5.9)
$$

$$\mathcal{M}_\lambda(p_a, p_b, p_1', p_2'; k) = e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}'}{p_1'\cdot k}\right]\mathcal{M}^{(on)}(\nu, t)$$

$$-2e\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(on)}(\nu, t)$$

$$-2e\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\left[(p_a - p_1, k) - (p_a\cdot l_1)\right]\frac{\partial}{\partial t}\mathcal{M}^{(on)}(\nu, t) + \mathcal{O}(\omega). \tag{5.10}$$

This is identical to our result (4.9) and from there we can go on to (4.11) using (4.10).

In this way we have given the relation between Low's formula and the Laurent series (4.11) where the pole term $\propto \omega^{-1}$ is given by Weinberg's soft-photon theorem and the next to leading term $\propto \omega^0$ by our calculation.

# 6 Conclusions

We have discussed the reactions

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1) + \pi^0(p_2), \tag{6.1}$$

and

$$\pi^-(p_a) + \pi^0(p_b) \longrightarrow \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon) \tag{6.2}$$

for $k \to 0$. For small enough $k$ the phase-space of (6.2) is given by $\{(k, \hat{p}_1')\}$, where we consider the c.m. system and $\hat{p}_1' = \vec{p}_1{}'/|\vec{p}_1{}'|$ varies over the unit sphere. We set $k^0 = \omega$.

(i) In Sec. 4 we have given the expansion of the amplitude for (6.2), $\mathcal{M}_\lambda$, around the phase-space point $(k=0, \hat{p}_1)$ corresponding to the basic elastic process (6.1). This expansion is a Laurent series. The pole term $\propto \omega^{-1}$ is given by Weinberg's version of the soft-photon theorem. We have calculated the next to leading term $\propto \omega^0$ of this series.

(ii) Low's version of the soft-photon theorem is not an expansion of this kind but gives an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point. What is frequently called Low's theorem is in reality the soft photon theorem in the version of Weinberg; see (4.12).

(iii) In our previous work we took it as premiss that Low's formula corresponded to an expansion of $\mathcal{M}_\lambda$ as explained in (i). Then the inevitable conclusion was that Low's result violated energy-momentum conservation. We have seen now that this premiss does not hold and, therefore, also this conclusion does not hold. We shall clarify all this in our next paper following the reasoning of my present talk.

(iv) We have shown how we get from Low's formula, giving an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point, to the expansion of $\mathcal{M}_\lambda$ around $(k=0, \hat{p}_1)$ in agreement with (i).

(v) We have discussed in our papers also the expansion in $\omega$ of the cross section for $\pi\pi \to \pi\pi\gamma$. In the same spirit as summarised in (i) we have treated the reactions

$\pi p \to \pi p$ and $\pi p \to \pi p \gamma$.

(vi) We hope that with this investigation we could clarify the meaning of Low's and Weinberg's versions of soft-photon theorems and their relation, in particular, concerning the next to leading terms.

Thank you for your attention!