The LBK theorem:
theory and phenomenology

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and ongoing work in collaboration with **Anna Kulesza** and **Roger Balsach**

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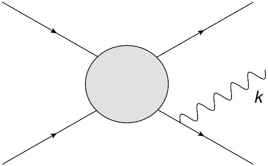
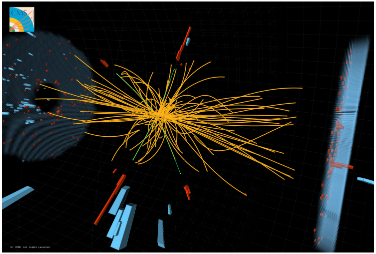
Introduction

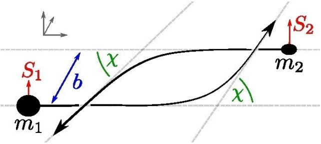
The study of the soft limit of gauge bosons radiation has a long history, dating back to the early days of QED [Bloch Nordsieck 1937, Low 1958, Burnett-Kroll 1967, Weinberg 1965].

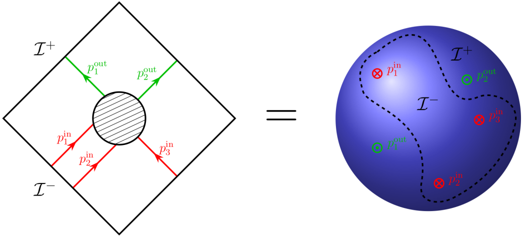


- Strict **soft limit** $k \to 0$ (Leading Power (**LP**) in the soft expansion) has been thoroughly investigated
- Subleading terms (**NLP**, **NNLP**, ...) less known, but recently a flurry of attention

An interdisciplinary area







[Image credits: CMS (cds.cern.ch/record/1406073), Antonelli, Kavanagh, Khalil, Steinhoff, Vines PRL 125, 011103, Strominger arXiv:1703.05448]

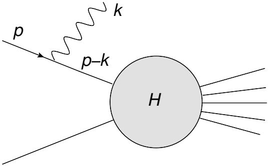
Outline

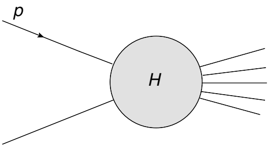
- LBK theorem: traditional form
- LBK theorem with shifted kinematics
- LBK theorem with modified shifted kinematics
- Results for $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$
- Loop corrections to LBK theorem

LBK theorem: traditional form

LBK theorem (LP)

Goal: $$\underbrace{\mathcal{A}(p_1,\ldots,p_n,k)}_{\text{radiative}} = \underbrace{\mathcal{S}(k)}_{\text{soft factor}} \times \underbrace{\mathcal{H}(p_1,\ldots,p_n)}_{\text{non-radiative}}$$

$\mathcal{A} =$ 

$\mathcal{H} =$ 

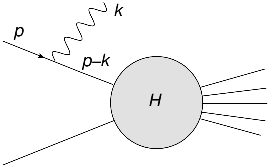
$$\mathcal{A} = H(p-k)\,\frac{(\not{p}-\not{k})}{(p-k)^2}\,(Q\,\epsilon^*(k)\cdot\gamma)\,u(p) \qquad\qquad \mathcal{H} = H\;u(p)$$

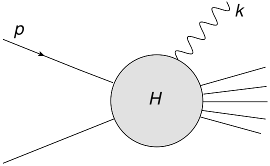
At **LP**, take the leading term for $k \to 0$ (**eikonal** approximation):

$$\mathcal{A} = \mathcal{S}_{LP}\,\mathcal{H}\,, \qquad \mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \eta_i \frac{\epsilon^*(k)\cdot p_i^\mu}{p_i\cdot k}$$

- insensitive to spin of hard emitter
- hard particles do not recoil ($k \to 0$)
- insensitive to the short distance physics i.e. non radiative amplitude $\mathcal{H}$

LBK theorem (NLP)





- External emission: expand up to $\mathcal{O}(k)$

$$
\begin{aligned}
\mathcal{A}^{\mu}_{\text{ext}}(p) &= H(p-k)\,\frac{(\not{p}-\not{k})}{(p-k)^2}(Q\,\gamma^{\mu})u(p) \\
&= Q\,H(p)\left(\frac{p^{\mu}}{p\cdot k}+\frac{k^{\mu}}{2p\cdot k}-\frac{k^2p^{\mu}}{2(p\cdot k)^2}-\frac{ik_{\nu}\sigma^{\mu\nu}}{p\cdot k}\right)u(p) \\
&\quad + \; Q\,\frac{p^{\mu}}{p\cdot k}k^{\nu}\underbrace{\frac{\partial H(p-k)}{\partial k_{\nu}}\bigg|_{k=0}}_{-\frac{\partial H(p)}{\partial p_{\nu}}}u(p) \; + \; \mathcal{O}(k)
\end{aligned}
$$

Here we used $\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu} - i\sigma^{\mu\nu}$ where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$ is the Lorentz generator for particles of spin $\frac{1}{2}$. Then sum over all external legs $\mathcal{A}^{\mu}_{\text{ext}} = \sum_i \mathcal{A}^{\mu}_{\text{ext}}(p_i)$

LBK THEOREM (NLP)

- Internal emission: use Ward identity $k_\mu(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{int}}) = 0$

$$\mathcal{A}^\mu_{\text{int}} = \sum_i Q_i \frac{\partial H(p^i)}{\partial p^i_\mu} u(p^i) + \underbrace{\Delta^\mu}_{\mathcal{O}(k)}$$

- Adding $\mathcal{A}^\mu_{\text{ext}}$ and $\mathcal{A}^\mu_{\text{int}}$:

$$\begin{aligned}
\mathcal{A}^\mu &= \sum_i Q_i \frac{p_i^\mu}{p_i \cdot k} \mathcal{H}(p_1 \ldots p_n) \\
&+ \sum_i Q_i \left(\frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{i k_\nu \sigma^{\mu\nu}}{p \cdot k} \right) \mathcal{H}(p_1 \ldots p_n) \\
&+ \sum_i Q_i \underbrace{\left(-\frac{p_i^\mu k^\nu}{p_i \cdot k} \frac{\partial}{\partial p_i^\nu} + \frac{\partial}{\partial p_i^\mu} \right)}_{-\frac{k_\nu}{p_i \cdot k} \underbrace{\left(p_i^\mu \frac{\partial}{\partial p_i^\nu} - p_i^\nu \frac{\partial}{\partial p_i^\mu} \right)}_{\equiv L^{\mu\nu}}} \mathcal{H}(p_1 \ldots p_n)
\end{aligned}$$

$L^{\mu\nu}$ is the angular momentum generator of the Lorentz group

LBK THEOREM (NLP)

This is the **sub-leading** soft theorem, known as **Low-Burnett-Kroll theorem**:
[Low 1958 (scalar emitters), Burnett-Kroll 1968 (spin $\frac{1}{2}$ emitters, conjecture for generic spin), Bell-VanRoyen 1969 (generic spin)]

$$\mathcal{A}(p_1, \ldots, p_n, k) = (\mathcal{S}_{LP} + \mathcal{S}_{NLP})\, \mathcal{H}(p_1, \ldots, p_n)\,,$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k}\,, \quad \mathcal{S}_{NLP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*_\mu(k) k_\nu (\sigma^{\mu\nu} + L^{\mu\nu})}{p_i \cdot k}$$

- corrections to the strict limit $k \to 0$: **small recoil** of the emitter taken into account
- sensitive to the **spin** of the emitter (e.g. $\sigma^{\mu\nu} = 0$ for scalars, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$for spin 1/2, etc.)
- orbital angular momentum $L^{\mu\nu}$ is sensitive to the short distance interactions in $\mathcal{H}$ (hard lines do not start from a pointlike vertex)
- NLP corrections here are valid only at the tree-level

From amplitudes to Cross-sections

At amplitude level two NLP contributions:

- Spin $\sigma^{\mu\nu}$
- Orbital $L^{\mu\nu}$ i.e. derivatives

Squaring and **summing over polarizations**, spin contribution becomes also a derivative. Crucial identity e.g. for leg p_1:

$$\frac{\not{k}\gamma^\mu}{p_1\cdot k}(\not{p}_1+m)+(\not{p}_1+m)\frac{\gamma^\mu\not{k}}{p_1\cdot k} \;=\; -\gamma^\mu+\frac{p_1^\mu}{p_1\cdot k}\not{k} \;=\; G_1^{\mu\nu}\frac{\partial}{\partial p_1^\nu}(\not{p}_1+m)$$

Then, traditional LBK with derivatives reads

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2=\sum_{ij}(-\eta_i\eta_jQ_iQ_j)\frac{p_i\cdot p_j}{p_i\cdot k\,p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad\to \textbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_jQ_iQ_j)\frac{p^i_\mu}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad\to \textbf{NLP}$$

Ambiguities at NLP

Problem: momentum conservation
l.h.s. $\sum_i p_i = k$ VS $\sum_i p_i = 0$ on the r.h.s. $\to$ difference for finite $k \neq 0$

One could restore $\sum_i p_i = k$ in $\mathcal{H}(p_1, \ldots, p_n)$ by replacing

$$\boxed{p_i \to \tilde{\mathbf{p}}_{\mathbf{i}}(\mathbf{k}) = p_i + \mathbf{c}_{\mathbf{i}} k + \mathcal{O}(k^2)}$$

$\mathbf{c}_{\mathbf{i}}$ are **arbitrary** coefficients $\implies$ LBK inconsistent at NLP?

$$\overline{|\mathcal{A}(p_1, \ldots, p_n, k)|}^2 = \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$
$$+ \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j \cdot k} \right) \frac{d}{dp_j^\nu} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$

LBK theorem is non-ambiguous if $\mathbf{c}_{\mathbf{i}}$ dependence **cancels up to NNLP** corrections.

Ambiguities at NLP

First Taylor expand in k

$$\overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\ldots,\tilde{\mathbf{p}}_n)|}^2 = \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + k^\mu \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k^2)$$

Then impose momentum conservation $k = \sum_i p_i$

$$\frac{d}{dp^\nu_j} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\ldots,\tilde{\mathbf{p}}_n)|}^2 = \frac{\partial}{\partial p^\nu_j} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + g^{\mu\nu}\xi_j \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(k)$$

Plug this into LBK with $\tilde{p}_i$
$\rightarrow$ we get original LBK (with p_i) + remainder term that depends on c_i

$$\begin{aligned} R(\mathbf{c_i}) &= \sum_{ij=1}^{n} (-\eta_i\eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k\, p_j \cdot k} k^\mu \sum_m \mathbf{c_m} \frac{\partial}{\partial p^m_\mu} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \\ &\quad + \sum_{ij=1}^{n} (-\eta_i\eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left(\eta^{\mu\nu} - \frac{p^\mu_j k^\nu}{p_j \cdot k} \right) \xi_j \sum_m \mathbf{c_m} \frac{\partial}{\partial p^\nu_m} \overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 + \mathcal{O}(1) \\ &= \mathbf{0} + \mathcal{O}(1) = \mathbf{NNLP} \end{aligned}$$

$\Longrightarrow$ NLP ambiguities cancel.

Ambiguities at NLP

- Traditional LBK is consistent at NLP
- Many forms of traditional LBK (all equivalent up to NNLP)
- consistent $\neq$ efficient. Some form of the theorem might be more efficient for a numerical implementation (NNLP effects can be visible in photon spectra since $k \neq 0$)
- in particular, is there a form where the non-radiative process can be computed with unambiguous physical momenta?

LBK theorem with shifted kinematics

From derivatives to shifts

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p^i_\mu}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \to \textbf{NLP}$$

Exploit the fact that derivatives are generators of translations:

$$f(x+\epsilon) = f(x) + \epsilon\,\frac{d}{dx}f(x)$$

$\rightarrow$ convert derivatives into **shifted momenta**

[DelDuca, Laenen, Magnea, Vernazza, White 2017, Bonocore, Kulesza 2021]

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{i,j=1}^{n} -\eta_i\eta_j\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\left(1-\sum_j \delta p_j^\nu\frac{\partial}{\partial p_j^\nu}\right)\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2$$

FROM DERIVATIVES TO SHIFTS

LBK with shifted kinematics:

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \underbrace{\sum_{i,j=1}^{n} -\eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}}_{\textbf{LP}\text{ factor!}} \overline{|\mathcal{H}(p_1+\delta p_1,\ldots,p_n+\delta p_n)|}^2$$

$$\delta p_j^\nu = \eta_j \xi_j Q_j \left(\sum_{k,l} \eta_k\eta_l Q_k Q_l \frac{p_k\cdot p_l}{(p_k\cdot k)(p_l\cdot k)}\right)^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{k\cdot p_i}\right)\left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)$$

Note that

$$\delta p_i = \mathcal{O}(k) \qquad \sum_i \delta p_i = -k \qquad p_i\cdot\delta p_i = 0$$

FROM DERIVATIVES TO SHIFTS

Simple case: **2 charged particles**

$$|\mathcal{A}(p_1, p_2, k)|^2 = \left(\sum_{i,j=1}^{2} -\eta_i \eta_j Q_i Q_j \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \right) |\mathcal{H}(p_1 + \delta p_1, p_2 + \delta p_2)|^2 \qquad (1)$$

where

$$\delta p_1^\mu = \frac{1}{2} \left(-\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu + \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$
$$\delta p_2^\mu = \frac{1}{2} \left(\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$

Immediate to see

$$\delta p_i = \mathcal{O}(k)$$
$$\delta p_1 + \delta p_2 = -k$$
$$p_i \cdot \delta p_i = 0$$

From derivatives to shifts

$$p_i \cdot \delta p_i = 0 \implies (p_i + \delta p_i)^2 = m^2 + \mathcal{O}(k^2) = m^2 + \text{NNLP}$$

Momenta are on-shell at NLP, hence theorem consistent at NLP

However, **masses do get shifted** by a NNLP amount!

$$(\delta p_j)^2 = Q_j^2 \left(\sum_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k \cdot p_l}{(p_k \cdot k)(p_l \cdot k)} \right)^{-1} \neq 0 \,,$$

i.e. with shifts we recovered momentum conservation, but momenta are off-shell at NNLP $\rightarrow$ problem for numerical implementations, where $k \neq 0$

Is there a LBK formulation fulfilling both **momentum conservation** AND **on-shell** condition exactly (i.e. not just at NLP)?

LBK theorem with modified shifted kinematics

Modified shifts

LBK theorem works at NLP
$\implies$ freedom to introduce **spurious NNLP** terms in the shifts.

We would like shifts δp_i to

(i) conserve momentum exactly, i.e.

$$\sum_i \xi_i \delta p_i + k = 0 \, ,$$

(ii) not shift the masses exactly, i.e.

$$(p_i + \delta p_i)^2 = m_i^2 \, ,$$

(iii) reduce to old shifts up to NNLP corrections, i.e.

$$\delta p_j^\nu = \eta_j \xi_j Q_j \left(|\mathcal{S}_{\mathrm{LP}}|^2 \right)^{-1} \sum_i \left(\frac{\eta_i Q_i}{k \cdot p_i} \right) \left(p_i^\nu - \frac{p_i \cdot p_j}{p_j \cdot k} k^\nu \right) + \mathcal{O}\left(k^2\right) .$$

Is this possible?

Modified shifts

Consider the ansatz

$$\delta p_i^\mu = \sum_j A_{ij}^{\mu\nu} p_{j\nu} + B_i^{\mu\nu} k_\nu \ ,$$

and determine coefficients $A_{ij}^{\mu\nu}$ and B_i by imposing conditions (i)-(iii).

$\rightarrow$ conditions not too constraining, many solutions for δp_i. But we seek a single solution!

- restrict our ansatz

$$\delta p_i^\mu = \sum_j A\eta_i \xi_i Q_i \frac{\eta_j Q_j}{k \cdot p_j} p_{j\nu} + B_i^{\mu\nu} k_\nu$$

- impose $p_{j\nu}$ and k_ν to be linear independent
- verify that solution has correct behaviour for $k \rightarrow 0$

Modified shifts

Result: [Balsach, DB, Kulesza]

$$\delta p_i^\mu = A\eta_i\xi_i Q_i \sum_j \frac{\eta_j Q_j}{k\cdot p_j} p_{j\nu} G_i^{\nu\mu} - \frac{1}{2}\frac{A^2 Q_i^2 |\mathcal{S}_{\rm LP}|^2}{p_i\cdot k} k^\mu ,$$

with

$$A = \frac{1}{\chi}\left(\sqrt{1+\frac{2\chi}{|\mathcal{S}_{\rm LP}|^2}} - 1\right) \qquad \chi = \sum_i \frac{\xi_i Q_i^2}{p_i\cdot k} .$$

$$|\mathcal{S}_{\rm LP}|^2 = \sum_{i,j} \eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}$$

- Momentum is **conserved** (exactly)
- Momenta are **on-shell** (exaclty)
- Shifts are $\mathcal{O}(k) \implies$ equivalent to traditional LBK at NLP

$\implies$ This form of LBK allow computation of non-radiative process $\mathcal{H}$ with most general-purpose event generators

Price to pay: **spurious NNLP** terms in the shifts

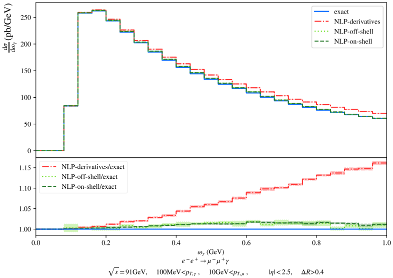
THREE VERSIONS OF (TREE-LEVEL) LBK

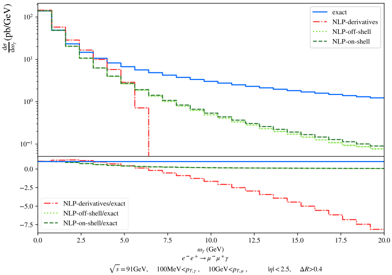
- All theoretically consistent at NLP
- NNLP ambiguities contained in all three versions ("scheme" dependence)
- When spectra are computed numerically, NNLP effects are visible
- Which version is more efficient and versatile? Which has more predictive power?
- Once we select the best NLP method, what is resolution in momentum we need for NLP to be measurable?

Results for $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$

THREE VERSIONS OF (TREE-LEVEL) LBK

Results for $e^+e^- \to \mu^+\mu^-\gamma$ [Balsach, DB, Kulesza]





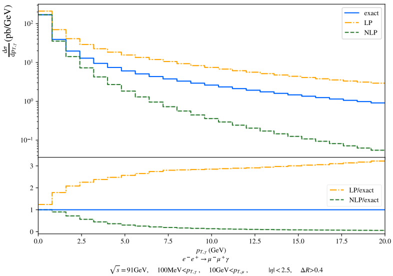
Note

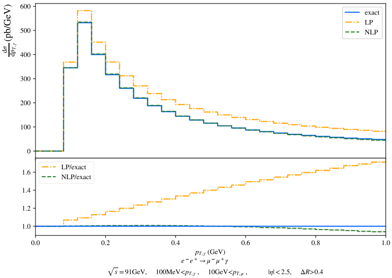
- non-radiative amplitude can be computed analytically (used here for derivatives and off-shell shifts)
- exact means tree-level with no soft expansion

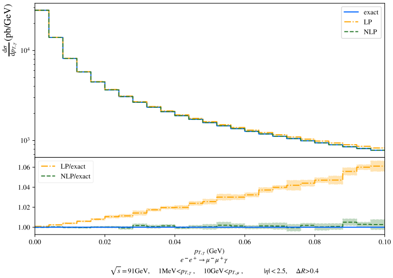
On-shell shifts work better. Used later as NLP

LP VS NLP

$e^+e^- \to \mu^+\mu^-\gamma$: p_t distributions [Balsach, DB, Kulesza]

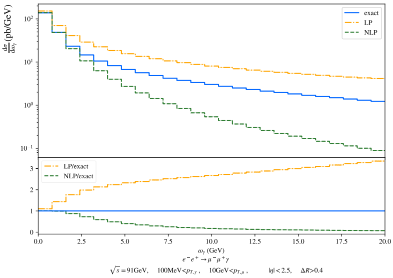


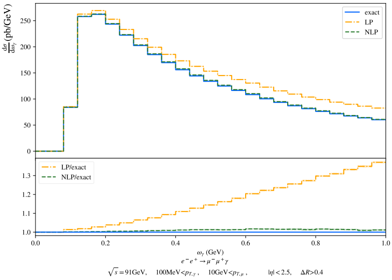




LP VS NLP

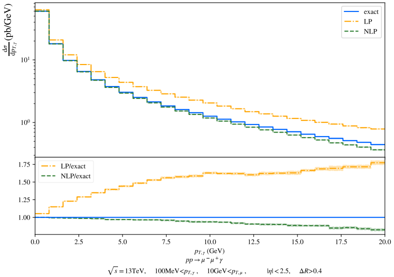
$e^+e^- \to \mu^+\mu^-\gamma$: (c.m.) ω distributions [Balsach, DB, Kulesza]

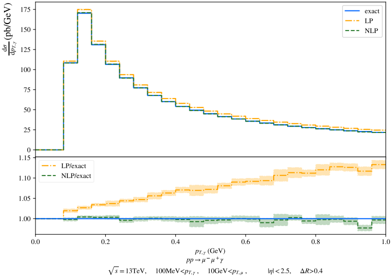


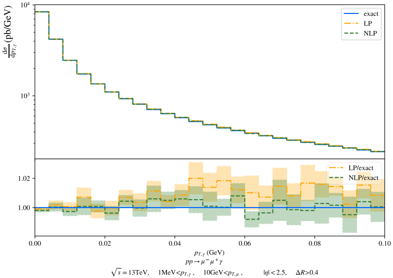


LP VS NLP

$pp \to \mu^+\mu^-\gamma$: p_t distributions [Balsach, DB, Kulesza]

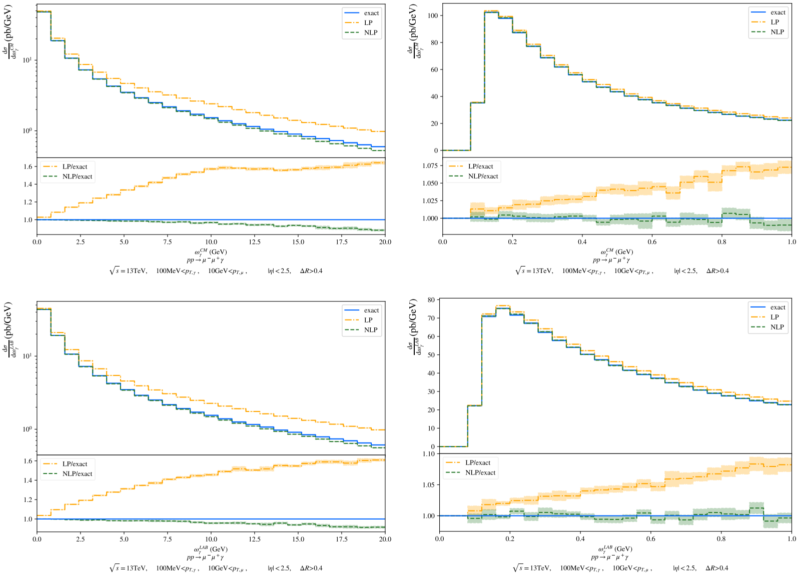






LP VS NLP

$pp \to \mu^+\mu^-\gamma$: (c.m. and lab) ω distributions [Balsach, DB, Kulesza]



Loop corrections to LBK theorem

NLP bremsstrahlung with QCD corrections

- at LP, soft theorems **do not** receive **loop corrections**.

$$\epsilon^*_\mu(k)\mathcal{A}^\mu = \mathcal{S}_{LP}\ \mathcal{A}_n\ , \qquad \mathcal{A}_n = \mathcal{A}_n^{(0)}, \mathcal{A}_n^{(1)}, \mathcal{A}_n^{(2)}, \ldots$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ ,$$

- at NLP, soft theorems **do** receive **loop corrections**.[Bern,Davies,Nohle 2014, He,Huang,Wen 2014, Larkoski,Neill, Stewart 2014, DB,Laenen,Magnea,Vernazza,White 2014]

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(0)}\ ,$$

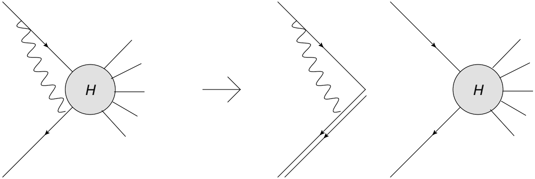
$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(1)} + \mathbf{?}\ ,$$

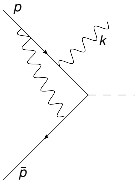
$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ , \qquad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k)k_\nu(\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

Various sources of correction. E.g. soft region in the massive case [Engel,Signer,Ulrich 2021]. In the high energy limit, it is interesting to look at the **massless limit** (crucial for the massless parton model) and the **collinear region**

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Virtual collinear effects are captured by **radiative jet functions J^μ** [DelDuca 1990, DB, Laenen, Magnea, Vernazza, White 2014, Gervais 2017, Beneke, Garny, Szafron, Wang 2018, Laenen, Damste, Vernazza, Waalewijn, Zoppi 2020, Liu, Neubert, Schnubel, Wang 2021].





In particular, the one-loop quark radiative jet function in dimensional regularization (with $d = 4 - 2\epsilon$ and $\bar{\mu}$ the $\overline{\text{MS}}$ scale) reads
[DB,Laenen,Magnea,Melville,Vernazza,White,2015]

$$J^{\mu(1)} = \left(\frac{\bar{\mu}^2}{2p\cdot k}\right)^{\epsilon} \Bigg[\left(\frac{2}{\epsilon} + 4 + 8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\frac{p^\mu}{p\cdot n} - \frac{n^\mu}{p\cdot n}\right) - (1+2\epsilon)\frac{i k_\alpha S^{\alpha\mu}}{p\cdot k}$$

$$+ \left(\frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon\right)\frac{k^\mu}{p\cdot k} + (1+3\epsilon)\left(\frac{\gamma^\mu \not{n}}{p\cdot n} - \frac{p^\mu}{p\cdot k}\frac{\not{k}\not{n}}{p\cdot n}\right)\Bigg] + \mathcal{O}(\epsilon^2, k)$$

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Thus, the next-to-soft theorem (i.e. LBK theorem) receives a **logarithmic correction**:

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(0)} \ ,$$

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(1)} + \left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} \ ,$$

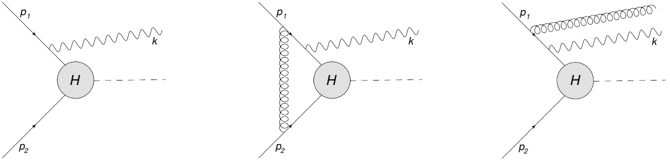
$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k} \ , \quad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k)k_\nu(\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

$$\left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} = \frac{2}{p_1\cdot p_2}\left[\sum_{ij}\left(\frac{1}{\epsilon} + \log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)\right) q_j\, p_i\cdot k\, \frac{p_j\cdot\epsilon}{p_j\cdot k}\right]\mathcal{A}_n^{(0)}$$

- Note that amplitude is IR divergent $\epsilon \to 0$
- $\log(\omega_k)$ corrections to soft theorems in QED also discussed (mainly classically) by Laddha-Sahoo-Sen. Here however more standard approach (i.e. dim.reg.) to regularization of soft and collinear divergences, which allows implementation in the massless limit required in QCD partonic calculations.

NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

IR divergences ($1/\epsilon$) cancel by adding real emission diagram:



The **soft photon** emission from the loop with a **collinear gluon** is captured by the radiative jet function J^μ (note here the **mixed QED-QCD** effect)
The corresponding contribution is what is needed for a process with a single quark-antiquark pair in the **massless limit** such as

- $e^+e^- \to q\,\bar{q}\,\gamma$
- $p\,p \to \mu^+\mu^-\gamma$
- ...

For processes with more than two colored particles situation more subtle (but structure is similar)

NLP bremsstrahlung with QCD corrections

The soft photon bremsstrahlung at $\mathcal{O}(\alpha_s)$ becomes

$$\boxed{\frac{d\sigma_{\text{NLP}}}{d^3k} = \frac{d\sigma_{\text{LP+(NLP-tree)}}}{d^3k} + \frac{\alpha_s}{4\pi}\frac{d\sigma_{\text{NLP-}J}}{d^3k}\,,}$$

where

$$\boxed{\frac{d\sigma_{\text{NLP-}J}}{d^3k} = \frac{\alpha}{(2\pi)^2}\frac{1}{\omega_k}\int d^3p_3 \ldots d^3p_n \left(\sum_{i=1}^{2} \eta_i \frac{8\log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}{p_i \cdot k}\right) d\sigma_H(p_1, ..., p_n)}$$

- **Correction of order** $\boxed{\alpha_s \log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}$ **to LP spectrum** $\frac{d\sigma}{d\omega_k}$ **hence particularly enhanced for small** ω_k **and small** k_t
- **expecially relevant for hadrons (since for leptons -** $\alpha \ll \alpha_s, m \to 0$**)**

Conclusions

- Three different formulations of (tree-level) LBK theorem (derivatives, off-shell shifts, on-shell shifts) are all theoretically consistent at NLP
- Different formulations correspond to reshuffling of NNLP effects, which might be numerically relevant (scheme choice) $\Longrightarrow$ not all formulations equally efficient
- New LBK formulation with on-shell shifted kinematics allows standard event generation for non-radiative process
- Numerical results show resolution in energy/momentum for NLP effects to be visible