Quantum Electrodynamics and Quantum Cyclotron Energy Levels

Ulrich D. Jentschura
Missouri University of Science and Technology Rolla, Missouri, USA

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Abstract

Relativistic and quantum electrodynamic corrections to the energy levels of quantum cyclotron states is are important for the determination of a number of fundamental constants, notably, for the g factor of the electron and positron, and atomic masses. We have recently analyzed the relativistic corrections in detail in [Phys. Rev. A **106***, 012816 (2022)] on the basis of higher-order Foldy-Wouthuysen transformations. Small modifications of literature values were found. The evaluation of quantum electrodynamic corrections requires the evaluation of bound-state Feynman diagrams with up to six magnetic vertices [Phys. Rev. D vol. 108, 036004 (2023)] and the use of fully relativistic Landau levels in the symmetric gauge, which were derived in [Phys. Rev. D vol. 108, 016016 (2023)]. Apparatus-dependent effects could limit the ultimate precision of the determination of the electron g factor [Phys. Rev. D vol. 107, 076014 (2023)], with the main apparatus-dependent effects impacting the so-called axial frequency. (As a supplement, a few other recent results such as those from Phys. Rev. D* **109***, 096020 (2024), will be briefly summarized.)*

What are Quantum Cyclotron States?

- Quantum cyclotron states are bound quantum states of an electron in a Penning trap.
- The (strong) magnetic field of the Penning trap is (by convention) assumed to be directed along the z axis.
- The confinement in the xy plane is mediated by the cyclotron orbits.
- The quadrupole (electric) field leads to confinement along the z axis.
- An "artificial atomic binding potential" is generated by the magnetic and electric fields of the Penning trap.
- Can define "cyclotron fine-structure constant" α_c and "axial fine-structure constant" α_z, which determine the spectrum of the quantum cyclotron states.

Brown and Gabrielse (Paper of 1986)

Gerald Gabrielse and Lowell S. Brown

The Rev. Mod. Phys. Article

Lowell S. Brown and Gerald Gabrielse

[Rev. Mod. Phys. **58**, 233–311 (1986)]

Geonium theory: Physics of a single electron or ion in a Penning trap

Lowell S. Brown and Gerald Gabrielse

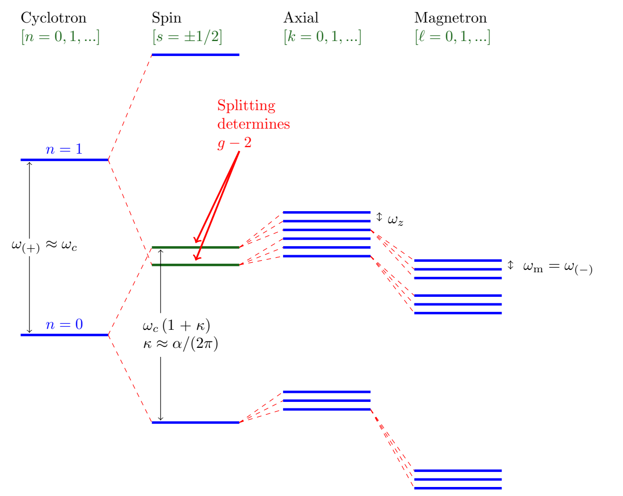
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195

A single charged particle in a Penning trap is a bound system that rivals the hydrogen atom in its simplicity and provides similar opportunities to calculate and measure physical quantities at very high precision. We review the theory of this bound system, beginning with the simple first-order orbits and progressively dealing with small corrections which must be considered owing to the experimental precision that is being achieved. Much of the discussion will also be useful for experiments with more particles in the trap, and several of the mathematical techniques have a wider applicability.

Quantum Cyclotron States and Bound–State QED

- Hydrogen Atom: The binding field is the Coulomb field.
- Quantum Cyclotron: The binding field is the magnetic field of the Penning trap (together with its quadrupole electric field).
- Hydrogen Atom: There is a particle at the center which can polarize the vacuum.
- Quantum Cyclotron: The closest particle which could polarize the vacuum sits in the walls of the Penning trap.
- Hydrogen Atom: The leading influence of the electron spin is at the level of the spin-orbit coupling terms.
- Quantum Cyclotron: The leading influence of the electron spin is of the same magnitude as the cyclotron motion itself.
- Hydrogen Atom: The main spectrum is characterized by the principal quantum number n; transition energies are of the order of the a few eV.
- Quantum Cyclotron: The main spectrum is characterized by the cyclotron quantum number n; transition energies are of the order of the a 10^{-4} eV.
- $\Rightarrow$ The quantum cyclotron is an weakly artificial "atom" (it is more weakly bound than a Coulomb system, i.e., more weakly than a hydrogenlike ion). Its spectrum exhibits the influence of the electron g factor (electron spin) very clearly.

How is the Anomalous Magnetic Moment of the Electron Determined?



Quantum numbers for the quantum cyclotron (replacing hydrogen's $n\ell j\mu$):

$$k = 0, 1, 2, \ldots \quad \text{(axial)} \qquad \ell = 0, 1, 2, \ldots \quad \text{(magnetron)}\,,$$
$$n = 0, 1, 2, \ldots \quad \text{(cyclotron)}\,, \qquad s = \pm 1 \quad \text{(spin)}\,.$$

(Alphabetical sequence.) n and s are the "big ones". Parameters from Brown and Gabrielse (1986):

$$\omega_c = 2\pi \times 164.4\,\mathrm{GHz} \qquad \text{(cyclotron frequency)}\,,$$
$$\omega_z = 2\pi \times 64.42\,\mathrm{MHz} \qquad \text{(axial frequency)}\,, \qquad \omega_c \gg \omega_z\,.$$

Characteristic scale of the probability density in the xy plane and along z:

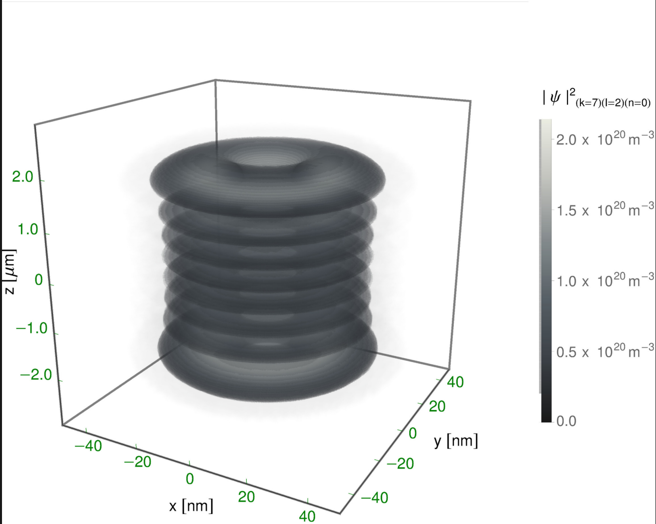
$$a_{0,c} = \sqrt{\frac{\hbar}{m\,\omega_c}} = 10.6\,\mathrm{nm}\,, \qquad \text{(cyclotron "Bohr" radius)}\,,$$
$$a_{0,z} = \sqrt{\frac{\hbar}{m\,\omega_z}} = 0.435\,\mu\mathrm{m}\,, \qquad \text{(axial "Bohr" radius)}\,, \qquad a_{0,z} \gg a_{0,c}\,.$$

Generalized fine-structure constants:

$$\alpha_c = \sqrt{\frac{\hbar\omega_z}{m\,c^2}} = 3.648 \times 10^{-5}\,, \qquad \alpha_z = \sqrt{\frac{\hbar\omega_z}{m\,c^2}} = 7.221 \times 10^{-7}\,.$$

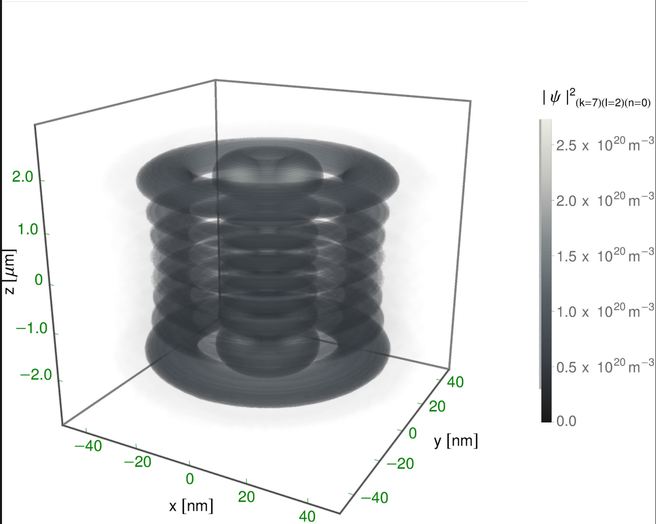
We observe: $\alpha_z \ll \alpha_c \ll \alpha_{\mathrm{QED}} \approx 1/137.036$.

What Do Quantum Cyclotron States Look Like?



$[k = 7,\ n = 0,\ \ell = 2]$ [From Phys. Rev. D **107**, 076014 (2023)]
Probability density is independent of the spin quantum number s.

What Do Quantum Cyclotron States Look Like?



$[k = 7,\ n = 1,\ \ell = 2]$ [From Phys. Rev. D **107**, 076014 (2023)]
Probability density is independent of the spin quantum number s.

Characteristic Frequencies

Corrected cyclotron frequency:

$$\omega_{(+)} = \frac{1}{2}\left(\omega_c + \sqrt{\omega_c^2 - 2\omega_z^2}\right) \approx \alpha_c^2 m\,.$$

Corrected magnetron frequency ($\vec{E} \times \vec{B}$ drift):

$$\omega_{(-)} = \frac{1}{2}\left(\omega_c - \sqrt{\omega_c^2 - 2\omega_z^2}\right) \approx \frac{\omega_z^2}{2\omega_c} = \alpha_c^2 \xi_z^4\, m\,.$$

Convenient to introduce the ratio of axial and cyclotron coupling parameters:

$$\xi_z = \frac{\alpha_z}{\alpha_c} = \sqrt{\frac{\omega_z}{\omega_c}}\,.$$

Algebra of operators according to Brown and Gabrielse!
Main frequency: $\omega_{(+)} \approx \omega_c$. Main coupling parameter:

$\alpha_c = 3.648 \times 10^{-5}$ (corresponds to $|\vec{B}| = 5.87\,\mathrm{T}$ and $\omega_c = 2\pi \times 164.4\,\mathrm{GHz}$)

Algebra of the Nonrelativistic Hamiltonian

Magnetic trap field in the z direction.
Magnetic spin coupling H_σ.
Axial confinement Hamiltonian H_z.
Radial variable: $\vec{\rho} = x\,\hat{\mathrm{e}}_x + y\,\hat{\mathrm{e}}_y$.
Momentum in the xy plane: $\vec{p}_\parallel = -\mathrm{i}(\hat{\mathrm{e}}_x\partial_x + \hat{\mathrm{e}}_y\partial_y)$.
Electron charge e. Electron mass m. Nonrelativistic Hamiltonian H_0:

$$H_0 = H_\parallel + H_\sigma + H_z \quad \text{(radial+spin+axial)},$$

$$H_\parallel = \frac{\vec{p}_\parallel^{\,2}}{2m} - \frac{e}{2m}\vec{L}\cdot\vec{B}_\mathrm{T} + \frac{1}{8}\,m\omega_c^2\,\rho^2 - \frac{1}{4}m\omega_z^2\rho^2$$

$$H_\sigma = -\frac{e}{2m}\,(1+\kappa)\,\vec{\sigma}\cdot\vec{B}_\mathrm{T}$$

$$H_z = \frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2 z^2$$

Algebraic decomposition for H_ξ and H_z:

$$H_\parallel = \omega_{(+)}\left(a^\dagger_{(+)}\,a_{(+)} + \frac{1}{2}\right) - \omega_{(-)}\left(a^\dagger_{(-)}\,a_{(-)} + \frac{1}{2}\right)$$

$$H_z = \omega_z\left(a_z^\dagger\,a_z + \frac{1}{2}\right)$$

Diagonalization of the Nonrelativistic Hamiltonian

Nonrelativistic Hamiltonian H_0:

$$H_0 = H_{\|} + H_\sigma + H_z \quad \text{(radial+spin+axial)}$$

$$H_{\|} = \omega_{(+)} \left(a^\dagger_{(+)}\, a_{(+)} + \frac{1}{2} \right) - \omega_{(-)} \left(a^\dagger_{(-)}\, a_{(-)} + \frac{1}{2} \right)$$

$$H_\sigma = -\frac{e}{2m}\,(1+\kappa)\,\vec{\sigma}\cdot\vec{B}_{\mathrm{T}}$$

$$H_z = \omega_z \left(a_z^\dagger\, a_z + \frac{1}{2} \right)$$

Energy eigenvalues:

$$\begin{aligned} E_{k\ell n s} &= \omega_c(1+\kappa)\,\frac{s}{2} + \omega_{(+)}\left(n+\frac{1}{2}\right) + \omega_z\left(k+\frac{1}{2}\right) - \omega_{(-)}\left(\ell+\frac{1}{2}\right) \\ &\approx \omega_c(1+\kappa)\,\frac{s}{2} + \omega_c\left(n+\frac{1}{2}\right) \approx \omega_c\left(n+\frac{s}{2}+\frac{1}{2}\right) \end{aligned}$$

This is the exact nonrelativistic energy eigenvalue!

Wave Functions of the Nonrelativistic Hamiltonian

Wave function by creation operators:

$$\psi_{k\ell ns}(\vec{r}) = \frac{\left(a_z^\dagger\right)^k}{\sqrt{k!}} \frac{\left(a_{(-)}^\dagger\right)^\ell}{\sqrt{\ell!}} \frac{\left(a_{(+)}^\dagger\right)^n}{\sqrt{n!}} \psi_0(\vec{r})\, \chi_{s/2}$$

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The $\chi_{s/2}$ denote fundamental spinors.
The orbital part of the ground-state wave function is

$$\psi_0(\vec{r}) = \sqrt{\frac{m\sqrt{\omega_c^2 - 2\omega_z^2}}{2\pi}} \exp\left(-\frac{m}{4}\sqrt{\omega_c^2 - 2\omega_z^2}\,\rho^2\right) \times \left(\frac{m\omega_z}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2} m\omega_z z^2\right)$$

where

$$\omega_{(+)} - \omega_{(-)} = \sqrt{\omega_c^2 - 2\omega_z^2}$$

Fourth–Order Corrections to Quantum Cyclotron Energies

Recall the nonrelativistic energy eigenvalue for vanishing axial frequency:

$$E_{\mathrm{NR}} = \omega_c \left(n + \frac{s}{2} + \frac{1}{2} \right) = \alpha_c^2\, m \left(n + \frac{s}{2} + \frac{1}{2} \right) = E^{[2]}$$

$E^{[2]}$ is proportional to $\alpha_c^2 m$
$E^{[3]}$ is proportional to $\alpha\, \alpha_c^2\, m$
$E^{[4]}$ is proportional to $\alpha_c^4\, m$
$E^{[5]}$ is proportional to $\alpha\, \alpha_c^4\, m$ [self energy]
$E^{[7]}$ is proportional to $\alpha\, \alpha_c^6\, m$ [self energy]
[treat α and α_c on the same footing]
The multiplying coefficients are functions of $\xi_z = \alpha_z/\alpha_c$. Recall:

$$E_{k\ell ns} = E^{[2]} + E^{[3]} = \omega_c(1+\kappa)\,\frac{s}{2} + \omega_{(+)} \left(n + \frac{1}{2} \right) + \omega_z \left(k + \frac{1}{2} \right) - \omega_{(-)} \left(\ell + \frac{1}{2} \right)$$

$$\approx \alpha_c^2 m \left(n + \frac{s}{2} + \frac{1}{2} \right) + \frac{\alpha}{2\pi} \alpha_c^2 m \frac{s}{2}$$

Fourth–Order Corrections to Quantum Cyclotron Energies

Fourth–Order Corrections [Phys. Rev. A **106**, 012816 (2022)]:

$$E^{(4)} = E_1 + E_2 \propto \alpha_c^4\, m \qquad [\hbar = c = \epsilon_0 = 1]$$

where E_1 is due to relativistic kinematics, and E_2 is due to spin-orbit coupling:

$$E_1 = -\frac{1}{2m}\left[\frac{\omega_{(+)}^2\left(n+\frac{1}{2}\right)+\omega_{(-)}^2\left(\ell+\frac{1}{2}\right)}{\omega_{(+)}-\omega_{(-)}} + \frac{\omega_z}{2}(k+\tfrac{1}{2}) + \frac{\omega_c s}{2}\right]^2 - \frac{\omega_z^4\left[(n+\frac{1}{2})(\ell+\frac{1}{2})+\frac{1}{4}\right]}{4m(\omega_{(+)}-\omega_{(-)})^2} - \frac{\omega_z^2}{16m}\left[\left(k+\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right]$$

Brown and Gabrielse (1986) in their equation (7.48) have a minus sign. Small update here. Formula for E_2:

$$E_2 = -\frac{\omega_z^2 s}{4m}\,\frac{\omega_{(+)}(n+\frac{1}{2})+\omega_{(-)}(\ell+\frac{1}{2})}{\omega_{(+)}-\omega_{(-)}}$$

For $\alpha_c^6 m$ corrections see [Phys. Rev. A **106**, 012816 (2022)].

Recent Paper

Sixth-order corrections and formulas for sixth-order Foldy–Wouthuysen transformation in general electric and magnetic fields can be found in a recent paper:

PHYSICAL REVIEW A **106**, 012816 (2022)

Foldy-Wouthuysen transformation in strong magnetic fields and relativistic corrections for quantum cyclotron energy levels

Albert Wienczek,[1,2] Christopher Moore,[1] and Ulrich D. Jentschura[1,*]
[1]*Department of Physics and LAMOR, Missouri University of Science and Technology, Rolla, Missouri 65409, USA*
[2]*Faculty of Physics, University of Warsaw, Pasteura 5, 02–093 Warsaw, Poland*

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We carry out a direct iterative Foldy-Wouthuysen transformation of a general Dirac Hamiltonian coupled to an electromagnetic field, including the anomalous magnetic moment. The transformation is carried out through an iterative disentangling of the particle and antiparticle Hamiltonians in the expansion for higher orders of the momenta. The time-derivative term from the unitary transformation is found to be crucial in supplementing the transverse component of the electric field in higher orders. Final expressions are obtained for general combined electric and magnetic fields, including strong magnetic fields. The time derivative of the electric field is shown to enter only in the seventh order of the fine-structure constant if the transformation is carried out in the standard fashion. We put special emphasis on the case of strong fields, which are important for a number of applications, such as electrons bound in Penning traps.

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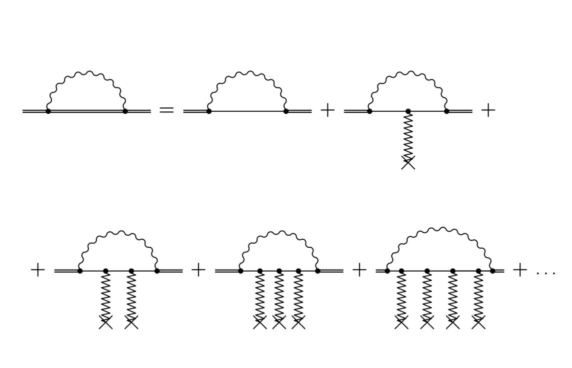
[... also contains formulas for the anomalous-magnetic-moment terms of seventh order (“seventh” = six momenta and one $\kappa = (g-2)/2$).]

What about the Self Energy?

Interactions with the external magnetic field (Feynman diagrams):
(Replacement $\gamma^0 V \to -e\vec{\alpha}\cdot\vec{A}_{\mathrm{T}} = -\frac{e}{2}(\vec{B}_{\mathrm{T}} \times \vec{r})$, couples upper and lower components of the Dirac spinor, need more vertices)



The self-energy of an electron bound in a Coulomb field was calculated by Bethe, Kroll and Lamb, and Feynman. The treatment been generalized to quantum cyclotron states very recently. High-energy part: normally treated using form factors, but this approach is not suitable for quantum cyclotron states. Low-energy part: “trapped” Bethe logarithm.

Self–Energy for a Bound Electron

Hydrogen:

$$\Delta E_{\mathrm{SE}} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} mc^2 \, F(Z\alpha)$$

$$F(Z\alpha) = A_{41} \ln(Z\alpha)^{-2} + A_{40} + (Z\alpha) A_{50} + (Z\alpha)^2 \left[A_{62} \ln^2(Z\alpha)^{-2} + A_{61} \ln(Z\alpha)^{-2} + A_{60}\right] + \dots$$

Quantum cyclotron:

$$\Delta E_{\mathrm{SE}} = \omega_c \, \kappa \, \frac{s}{2} + \frac{\alpha}{\pi} \alpha_c^4 \, mc^2 \, \mathcal{F}(\alpha_c)$$

$$\mathcal{F}(\alpha_c) = \mathcal{A}_{41} \ln(\alpha_c^{-2}) + \mathcal{A}_{40} + \alpha_c^2 \left[\mathcal{A}_{61} \ln(\alpha_c^{-2}) + \mathcal{A}_{60}\right] + \dots$$

Changes:
$F \to \mathcal{F}$, $A \to \mathcal{A}$, and no n^3 in the denominator!
In a quantum cyclotron, the characteristic momentum grows with the cyclotron quantum number n, while in a a hydrogen atom, the characteristic momentum scale decreases with increasing principal quantum number n.

Result for the Self Energy

In the leading order in α_c, one obtains the following results [see Phys. Rev. D **108**, 016016 (2023) and Phys. Rev. D **108**, 036014 (2023)]. High–energy part and low–energy part:

$$E_{\rm HEP} \approx \frac{\alpha}{4\pi}\, s\,\omega_c + \frac{2\alpha}{3\,\pi}\,\alpha_c^4 m \left[\ln\left(\frac{m}{2\,\epsilon}\right) - \frac{13}{72}\right]$$

$$E_{\rm LEP} \approx \frac{2\,\alpha}{3\,\pi}\,\alpha_c^4 m\,\ln\left(\frac{\epsilon}{\alpha_c^2\,m}\right) - \frac{2\,\alpha}{3\,\pi}\alpha_c^4 m\ln(k_{0\rm T})$$

where $\ln k_{0\rm T}$ is the "trapped Bethe logarithm":

$$\ln(k_{0\rm T}) = \frac{\omega_{(+)}^3\,\ln\left(\frac{\omega_{(+)}}{\omega_c}\right) - \omega_{(-)}^3\,\ln\left(\frac{\omega_{(-)}}{\omega_c}\right)}{\omega_c^2\,(\omega_{(+)} - \omega_{(-)})} + \frac{\omega_z^2}{2\omega_c^2}\,\ln\left(\frac{\omega_z}{\omega_c}\right)$$

Focus on the cancelation of the photon energy cutoff ϵ:

$$E_{\rm SE} = E_{\rm HEP} + E_{\rm LEP} \sim \frac{2\alpha}{3\,\pi}\,\alpha_c^4 m\,\ln(\alpha_c^{-2})\,, \qquad \mathcal{A}_{41} = \frac{2}{3}\,,$$

$$E_{\rm HEP} \sim \frac{2\alpha}{3\,\pi}\,\alpha_c^4 m\,\ln\left(\frac{m}{2\,\epsilon}\right), \qquad E_{\rm LEP} \sim \frac{2\,\alpha}{3\,\pi}\,\alpha_c^4 m\,\ln\left(\frac{\epsilon}{\alpha_c^2\,m}\right)$$

Almost ironic: Almost no dependence on k, ℓ, n and s!

Self–Energy: Contributions in Higher Order in ξ_z

$$E_{\rm SE} = E^{[1]}_{\rm HEP} + E^{[2]}_{\rm HEP} + E_{\rm LEP} = \sum_{i=1}^{6} \mathcal{T}_i$$

Six individual contributions have been calculated:

$$\mathcal{T}_1 = \omega_c \kappa \frac{s}{2}$$

$$\mathcal{T}_2 = \frac{\alpha}{\pi}\, \alpha_c^4 m \left[\frac{2}{3} \ln\left(\alpha_c^{-2}\right) - \frac{2}{3}\, \ln(2) - \frac{13}{108}\right]$$

$$\mathcal{T}_3 = -\frac{\alpha}{8\pi} \frac{s\omega_c\omega_z}{m} (k + \tfrac{1}{2}) = -\frac{\alpha}{8\pi} \alpha_c^4 m s\, \xi_z^2\, (k + \tfrac{1}{2})$$

$$\mathcal{T}_4 = -\frac{\alpha}{4\pi} \frac{\omega_z^2 s}{m} \frac{\omega_{(+)}(n + \frac{1}{2}) + \omega_{(-)}(\ell + \frac{1}{2})}{\omega_{(+)} - \omega_{(-)}} = \frac{\alpha}{\pi} \alpha_c^4 m \left[-\frac{1}{8}(2n+1) s\, \xi_z^4\right] + \mathcal{O}(\xi_z^6)$$

$$\mathcal{T}_5 = -\frac{\alpha}{3\,\pi} \frac{\omega_z^2}{m} \ln\left(\frac{\omega_z}{\omega_c}\right) = \frac{\alpha}{\pi} \alpha_c^4 m \left[-\frac{2}{3}\, \xi_z^4\, \ln(\xi_z)\right]$$

$$\mathcal{T}_6 = -\frac{2\,\alpha}{3\,\pi} \frac{\omega_{(+)}^3\, \ln\left(\frac{\omega_{(+)}}{\omega_c}\right) - \omega_{(-)}^3\, \ln\left(\frac{\omega_{(-)}}{\omega_c}\right)}{m(\omega_{(+)} - \omega_{(-)})} = \frac{\alpha}{\pi} \alpha_c^4 m \frac{\xi_z^4}{3} + \mathcal{O}(\xi_z^6)$$

State Dependence of the Self Energy

PHYSICAL REVIEW D **108,** 036004 (2023)

Quantum electrodynamic corrections to cyclotron states in a Penning trap

Ulrich D. Jentschura and Christopher Moore

Department of Physics and LAMOR, Missouri University of Science and Technology, Rolla, Missouri 65409, USA

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We analyze the leading and higher-order quantum electrodynamic corrections to the energy levels for a single electron bound in a Penning trap, including the Bethe logarithm correction due to virtual excitations of the reference quantum cyclotron state. The effective coupling parameter α_c in the Penning trap is identified as the square root of the ratio of the cyclotron frequency, converted to an energy via multiplication by the Planck constant, to the electron rest mass energy. We find a large, state-independent, logarithmic one-loop self-energy correction of order $\alpha\,\alpha_c^4 mc^2 \ln(\alpha_c^{-2})$, where m is the electron rest mass and c is the speed of light. Furthermore, we find a state-independent "trapped" Bethe logarithm. We also obtain a state-dependent higher-order logarithmic self-energy correction of order $\alpha\,\alpha_c^6 mc^2 \ln(\alpha_c^{-2})$. In the high-energy part of the bound-state self-energy, we need to consider terms with up to six magnetic interaction vertices inside the virtual photon loop.

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$$\delta E_{\mathrm{SE}} = \frac{\alpha}{\pi}\,\alpha_c^6 m\,\mathcal{A}_{61}\,\ln(\alpha_c^{-2})\,, \qquad \mathcal{A}_{61} = 2n + 1 - \frac{4s}{3}\,.$$

[higher-order $\mathcal{A}_{61}$ has a state dependence.]

Apparatus–Dependent Corrections

PHYSICAL REVIEW D **107,** 076014 (2023)

Apparatus-dependent corrections to the electron $g-2$ revisited

Ulrich D. Jentschura
Department of Physics and LAMOR, Missouri University of Science and Technology, Rolla, Missouri 65409, USA

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We revisit the derivation of the apparatus-dependent correction to the energy levels of quantum cyclotron states, as previously outlined [Boulware *et al.*, Phys. Rev. D **32**, 729 (1985)]. We evaluate the leading corrections to the axial, magnetron, cyclotron, and spin-projection-dependent energy levels due to the altered photon field quantization in the vicinity of a conducting wall. Our work significantly extends previous considerations. Quantum cyclotron states are used for the determination of the electron g factor in Penning traps. Our calculations show that the numerically largest apparatus-dependent corrections can be expected for the axial and magnetron frequencies, where they can be as large as 10^{-8} in relative units. For the cyclotron frequency, one can expect corrections on the order of 10^{-12}, which can affect the determination of the anomalous magnetic moment of the electron.

DOI: 10.1103/PhysRevD.107.076014

Apparatus–Dependent Corrections

How do the walls of the trap influence the self-energy of the electron bound in the quantum cyclotron state? At least a partial answer can be found in Phys. Rev. D **107**, 076014 (2023).

- Assumption: Idealized geometry of an electron in the vicinity of an infinitely extended conducting plane, which serves to approximate the walls of the Penning trap.
- Conclusion: The low-energy part of the self energy is modified. The calculation splits up into the modification of the photon propagator, and a matrix element of the quantum cyclotron state.
- The correction to the axial frequency is large but it can be eliminated in the evaluation of the experimental data. The correction to the spin-flip frequency is parametrically highly suppressed.

Limits for the g Factor

$$g = 2(1 + \kappa)$$

- The correction to the cyclotron frequency is the problem. Independent of the geometry of the trap, we can provide the estimate

$$\kappa + \delta\kappa = \frac{\omega_s}{\omega_c + \delta\omega_c} - 1 \approx \kappa - \frac{\delta\omega_c}{\omega_c}$$

  and so

$$\delta\kappa = -\frac{\delta\omega_c}{\omega_c} \approx -\frac{r_0}{R}\, f(x)\,, \qquad x = \frac{\omega_c\, R}{c} = \text{"retardation phase"}\,,$$

  where for an electron over a conducting plane, $f(x) = \frac{1}{2}\cos(2x)$. The function $f(x)$ can be assumed to be of order unity. Classical electron radius is $r_0 = 2.8 \times 10^{-15}$ m, and $R = 3 \times 10^{-3}$ m is the trap dimension.
- We may have a problem when $\delta\kappa \sim 10^{-13}$!
- $(\alpha/\pi)^4 \sim 10^{-11}$
- $(\alpha/\pi)^5 \sim 10^{-14}$
- What about five-loop corrections?

Further Details in the Following Papers

- A. Wienczek, C. Moore and U. D. Jentschura, "Foldy–Wouthuysen Transformation in Strong Magnetic Fields and Relativistic Corrections for Quantum Cyclotron Energy Levels", Phys. Rev. A **106**, 012816 (2022).
(Relativistic Corrections to the Energies)
- U. D. Jentschura, "Algebraic Approach to Relativistic Landau Levels in the Symmetric Gauge", Phys. Rev. D **108**, 016016 (2023).
(Relativistic Wave Functions)
- U. D. Jentschura and C. Moore, "Quantum Electrodynamic Corrections for Quantum Cyclotron States", Phys. Rev. D **108**, 036004 (2023).
(Self–Energy Corrections to the Energies)
- U. D. Jentschura, "Apparatus–Dependent Corrections to $g - 2$ Revisited", Phys. Rev. D **107**, 076014 (2023). (Ultimate Limits of Penning Traps)

Conclusions

Relativistic, quantum electrodynamic, and apparatus-dependent corrections to quantum cyclotron states in Penning traps have been analyzed in detail.

The results provide small updates of known results, provide data for a number of higher-order corrections, and clarify the level at which apparatus-dependent effects have to be considered!