p-adic reconstruction of rational functions in multi-loop amplitude calculations

based on arXiv:2312.03672

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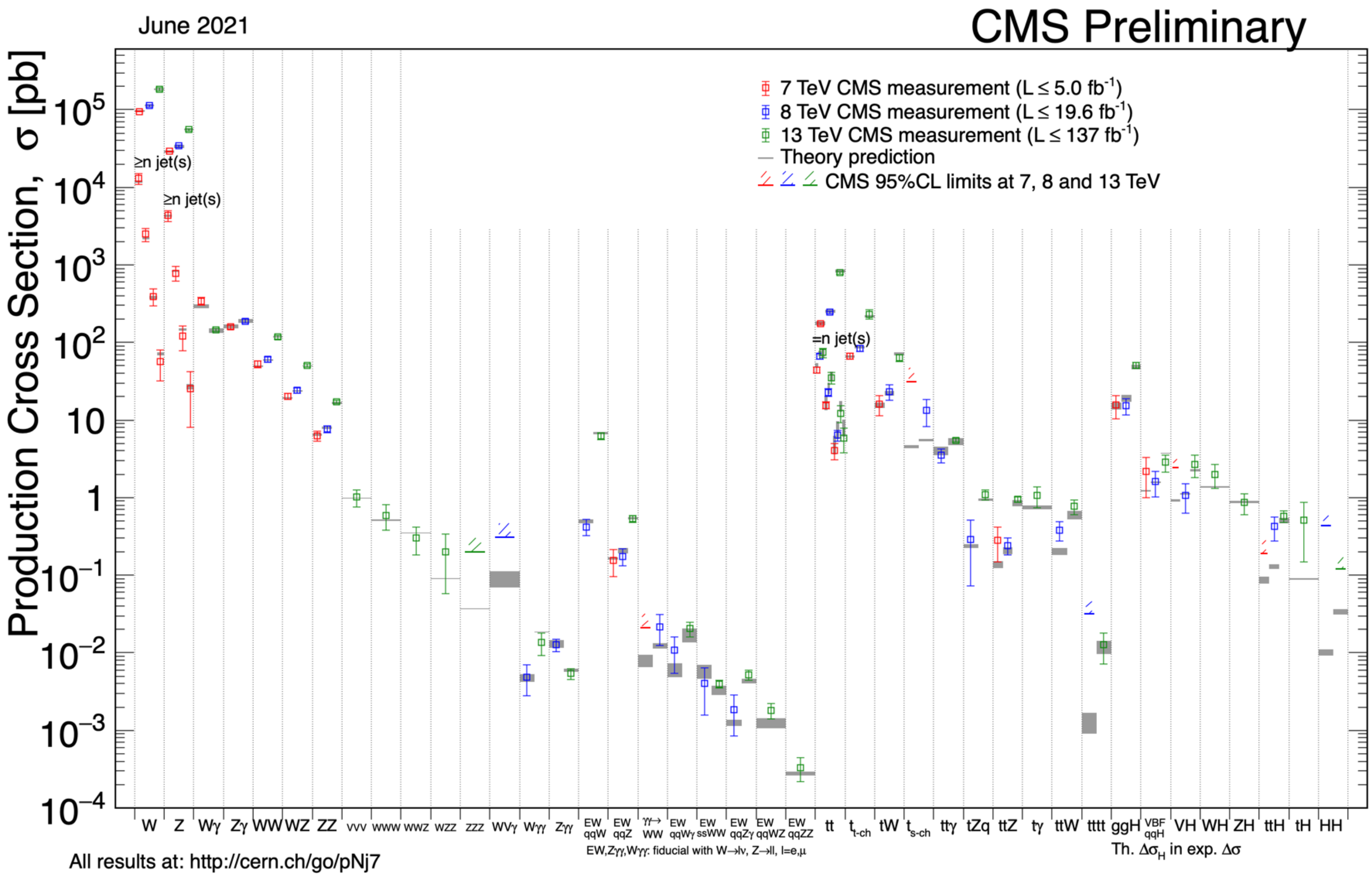
ACAT conference 2024 at Stony Brook (NY), USA

Wednesday 13th March 2024

Context: precise predictions for high-energy collider processes



Executive summary (for experts)

- Performing elementary arithmetic on large rational functions is a central bottleneck in high-precision predictions for high-energy collider processes
  - Specifically: in multi-loop amplitude calculations
- In recent years, finite-field interpolation methods have widely been employed to improve speed+reach of these calculations
- In parallel, it has been observed that symbolic expressions can be greatly simplified by partial fractioning
- This talk: can we interpolate directly in partial-fractioned form and hence benefit from the best of both worlds?

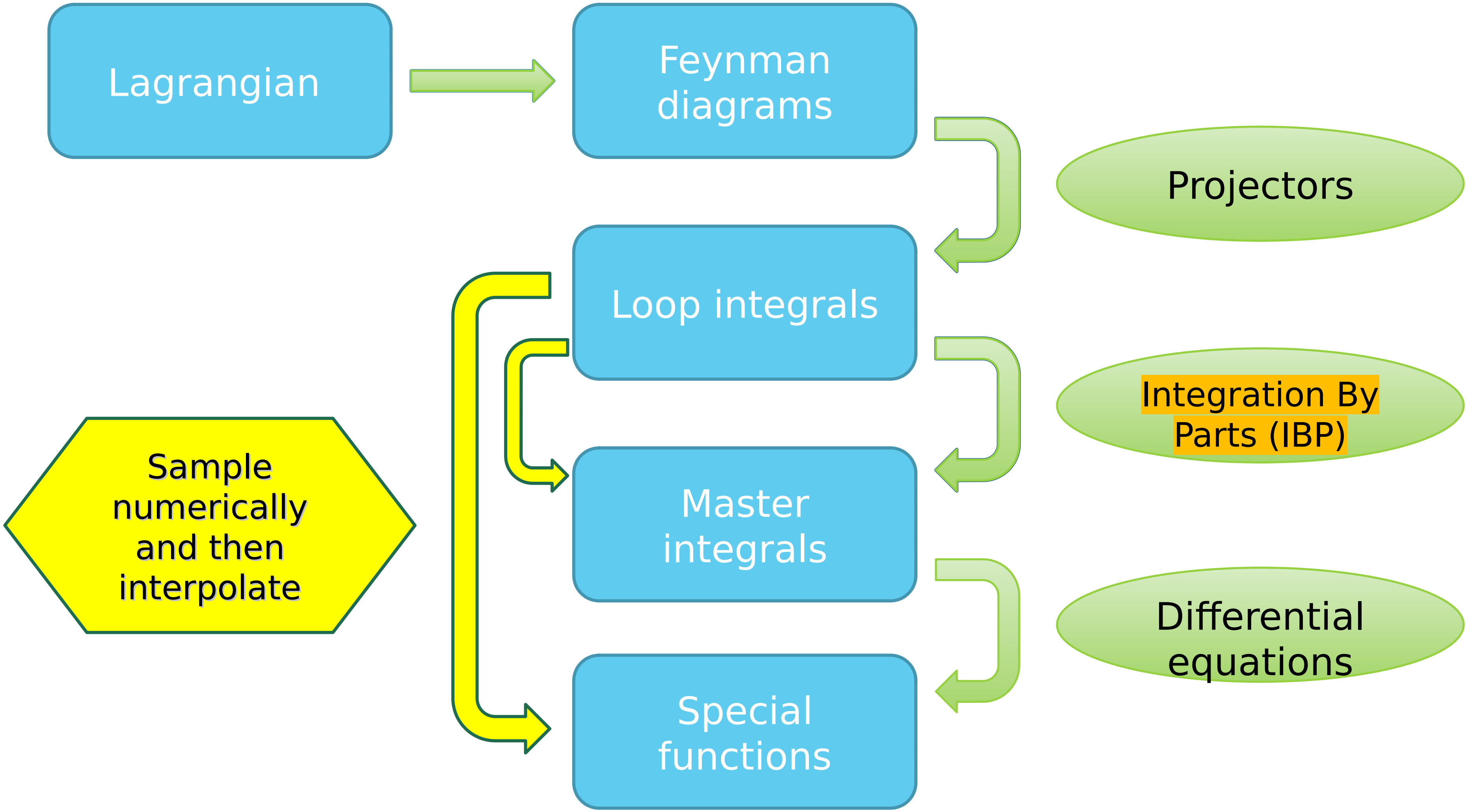
Outline

1. Introduction
   1. Why numerical reconstruction?
   2. Why partial-fractioned form?
   3. *p*-adic numbers
2. Details of interpolation strategy
3. Results
4. Conclusion

Why numerical (finite-field) interpolation methods?

- Core idea: perform repeated numerical calculations and then interpolate result
  - Bypasses large intermediate expressions
    - Generic feature of symbolic calculations (not specific to physics)
  - Use finite-field numbers instead of real numbers. (advantage: exact results)
- Long-used in computer algebra (e.g. Mathematica), now also used in physics
  - e.g. [1406.4513 – Manteuffel, Schabinger], [1608.01902 – Peraro]
  - Various libraries, e.g. FiniteFlow, FireFly, ...
  - Has enabled calculation of many new multi-loop multi-scale amplitudes
- Most computing time is spent evaluating the numerical probes
  - Number of probes is determined by the polynomial degrees of the expressions in the final result

A typical multi-loop toolbox



Previous work on optimising numerical reconstruction (in HEP)

- Guess denominators → reduces probes by factor of 2
  - e.g. [1812.04586], [2101.08283], and many more
- Single-variable partial fractioning
  - e.g. [Wang '81], [1908.04301], [2106.08664], [2102.02516], and more
- Reconstructing in partial-fractioned form using very high precision floating-point evaluations [1904.04067]
- using algebraic-geometry techniques + p-adic numbers to study the singular limits of rational functions [2203.04269]
- Combining several of the above, e.g. [2010.14525], [2203.17170]
- Evaluate at roots of unity, exploiting fast Fourier transform
- ...

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Partial fractioning

- Widely used in recent years to simplify final (and intermediate) results of heavy calculations
  - Popular libraries: Singular, MultivariateApart
- Example throughout this talk: the largest rational function in the largest IBP expression needed for the 2-loop full-colour QCD amplitudes for $pp \to \gamma\gamma j$
  - Analytic expression courtesy of authors (Agarwal, Buccioni, von Manteuffel, Tancredi) of [2105.04585]
  - Partial-fractioned form is O(100) times smaller than common-denominator form:

Form of expression	Size	Parameters to fit
Common-denominator	605 MB	1,369,559
Partial-fractioned	4 MB	14,558

- This talk: from numeric evaluations, reconstruct such expressions directly in partial-fractioned form
  - Aim to improve speed (and hence reach) of loop calculations
  - Reconstruct piece-by-piece
    - Capability to better understand and exploit the structure of these functions

Why does partial fractioning simplify our rational functions?

- Rational functions in multi-loop calculations are special
  - In common-denominator form, the denominator factorises: $\Delta(x) = \prod_a [f_a(x)]^{\nu_a}$
    - But this doesn't explain the simplification
  - Consider 2 rational functions with identical common denominators.
    - Now partial-fraction them both

Taken from a multi-loop calculation → $R = \sum_i \frac{n_i}{d_i}$

"Random" rational function → $\tilde{R} = \sum_j \frac{\tilde{n}_j}{d_j}$

- Surprise: only R simplifies under partial fractioning
  - So there is some interesting physics hiding here!
- Furthermore, $\{d_i\} \subset \{d_j\}$
  - Observation: just knowing which d_i to keep would give a 25x simplification.

Recall:

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A brief history of *p*-adic numbers

- Described/explored by Kurt Hensel in 1897
- Widely used in computer algebra for several decades
  - Finding rational solutions to various types of equations
    - Reconstruct a rational number from its *p*-adic expansion
- Appearance in particle physics too!
  - *p*-adic / adelic quantum mechanics / string theory [since '80s/'90s]
  - Ansatze for amplitudes [De Laurentis & Page, 2022]
    - Constrained ansatze in common-denominator form, to then be fitted with standard finite-field methods
- This talk: interpolate rational functions directly in partial-fractioned form, from p-adic evaluations.
  - In practice, will perform evaluations at special integer points that are “p-adically small” (but not small according to usual absolute value)

Brief intro to *p*-adic numbers

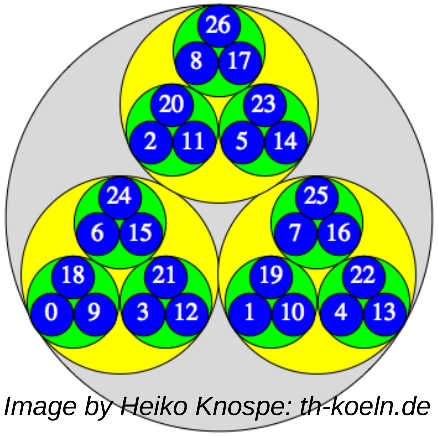

Image by Heiko Knospe: th-koeln.de

- *p*-adic numbers Q_p are an alternative completion of the rationals Q
  - Alternative absolute value: $|(a * p^n / b)|_p = 1/p^n$, where p is prime and a,b,p are coprime
  - For each prime *p*, a separate field Q_p
    - Nice results, e.g. Hasse's local-global principle: certain equations have solutions in Q iff they have solutions in R and in each Q_p
- Can expand any rational number *x* as a power series in *p*
    - e.g. 80 = $3 + 4*7 + 1*7^2$
    - e.g. -1 = $6 + 6*7 + 6*7^2 + 6*7^3 + O(7^4)$
    - e.g. ½ = $4 + 3*7 + 3*7^2 + 3*7^3 + O(7^4)$
    - e.g. (2 / 21) = $3*7^{-1} + 2 + 2*7 + 2*7^2 + 2*7^3 + O(7^4)$
  - If *x* is integer then the coefficient of p^0 is (*x* mod *p*)
  - Expansion operation commutes with all arithmetical operations + * - /

Implementing p-adic arithmetic on a computer

- There are many ways to implement p-adic arithmetic on a computer.
  - This work focusses on reducing number of probes (regardless of how they're performed)
- One option: work directly with power-series in *p* up to some chosen *p*-adic order.
  - Possible loss of precision during probe (albeit better controlled than in floating-point real numbers)
- Another option (employed here): perform probes at special integer points

e.g.

$$4 + 3*7 + 3*7^2 + 3*7^3 + \mathcal{O}(7^4)$$

$$4 + 3*7 + 3*7^2 + 3*7^3 = 1201$$

  - No loss of precision at intermediate stages of calculation

Important caveat:

  - It can be beneficial to use small primes e.g. Q_{101} (in contrast to finite-field practice)
    - <u>But for now, still prudent to assume p-adic probes to be slower than finite-field probes.</u>
  - Can use finite fields to perform the integer probes (here, typically at 10-digit points)
    - The size of the finite field does not need to match the *p* of the *p*-adic field
      - e.g. use 64-bit finite fields to evaluate at an integer point that is special when viewed in Q_{101}

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Partial-fractioned reconstruction: key idea

- Suppose we can find a p-adic point that makes one of the (candidate) denominators become p-adically smaller than all of the others
  - i.e. $\exists k : \forall i \neq k, \quad |d_k(\bar{x})|_p < |d_i(\bar{x})|_p$
- Then if we evaluate R at that point, we get a p-adic series

$$R(\bar{x}) = \frac{n_k(\bar{x})}{d_k(\bar{x})} + \mathcal{O}\left(\frac{1}{p^{m-1}}\right)$$

  - If the candidate term is absent, we will notice and can skip reconstruction
  - If the candidate term is present, we can perform several evaluations and interpolate n_k
- Then repeat for other candidate terms

A complication: bases/relations between partial-fractioned terms

- There are relations between partial-fractioned candidate terms
  e.g. $\frac{1}{x^2 y} - \frac{1}{x^2 (x+y)} - \frac{1}{x y (x+y)} == 0$
- Choice of basis:
  - Basis in MultivariateApart / Leinartas's decomposition
    - Chosen depending on a specified variable ordering
    - Avoids introducing new spurious factors, but can still introduce spurious higher powers of existing factors
    - Unique basis -> allows vectorised addition in symbolic calculations
  - Basis in this work: prioritises avoiding introducing spurious higher powers
    - Basis customised to given rational function, so that no partial-fractioned term has stronger divergence than the overall function
    - Further study needed to see which basis choices are "best"

Choice of probe weights

- A first attempt: exponential weights
  - Naively, might expect to need large powers to uniquely pick out one partial-fractioned term.
    - e.g. if singularity degrees are known to be bounded to be below 10, we can set $(s_{12}, s_{23}, s_{34}) \sim (p^{100}, p^{10}, p)$. Then if the rational function diverges there like $1/p^{273}$, we know we have picked out the term $1/(s_{12}^2\, s_{23}^7\, s_{34})$
    - But this strategy would require evaluating to very high p-adic precision.
- Smart choice: low weights
  - Choose a limited set of small weights
    - e.g. $(s_{12}, s_{23}, s_{34}) \sim (p, p^2, p)$
    - Repeatedly cycle through this set, trying to find a probe point that picks out a single candidate partial-fractioned term.
      - In this work, used 6k distinct combinations of weights
    - Heuristically, seems to work

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Results

- Reconstructed a very large rational function at the edge of current computational techniques
  - Specifically: largest rational function in one of the rank 5 IBP expressions needed for calculating non-planar 2-loop 5-point massless QCD amplitudes.

Expression	Size	Parameters to fit
Original	605 MB	1,369,559
Reconstructed	4.5 MB	52,527 (*of which 15,403 non-zero*)

- Required ~60k p-adic evaluations (per prime), whereas conventional approach would require 1.4M finite-field evaluations (per prime)
- Reconstructed result is 130 times smaller than in conventional approach
- Signs of further patterns and simplifications (see next slide)

Hints of additional pattern and structure

- Of the 52k fitted parameters, only 15k are non-zero
  - Can we predict these in advance?

Expression	Size	Parameters to fit
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A few of the
reconstructed terms:

$$\frac{\frac{45}{1024}s_{45}^6 s_{12}^3}{(D-3)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3} + \frac{\frac{9}{5120}s_{45}^6 s_{12}^3}{(D-1)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3} - \frac{\frac{693}{5120}s_{45}^6 s_{12}^3}{(2D-7)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3} - \frac{\frac{3}{1024}s_{45}^6 s_{12}^3}{s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3}$$
$$+ \frac{-\frac{45 s_{45}^6 s_{51}^2}{1024} - \frac{135 s_{45}^6 s_{51} s_{12}}{1024} - \frac{135 s_{45}^6 s_{12}^2}{1024}}{(D-3)s_{34}^4(s_{23}-s_{45}-s_{51})^3} + \frac{-\frac{9 s_{45}^6 s_{51}^2}{5120} - \frac{27 s_{45}^6 s_{51} s_{12}}{5120} - \frac{27 s_{45}^6 s_{12}^2}{5120}}{(D-1)s_{34}^4(s_{23}-s_{45}-s_{51})^3}$$
$$+ \frac{\frac{693 s_{45}^6 s_{51}^2}{5120} + \frac{2079 s_{45}^6 s_{51} s_{12}}{5120} + \frac{2079 s_{45}^6 s_{12}^2}{5120}}{(2D-7)s_{34}^4(s_{23}-s_{45}-s_{51})^3} + \frac{-\frac{3 s_{45}^6 s_{51}^2}{1024} - \frac{9 s_{45}^6 s_{51} s_{12}}{1024} - \frac{9 s_{45}^6 s_{12}^2}{1024}}{s_{34}^4(-s_{23}+s_{45}+s_{51})^3}$$

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A few of the reconstructed terms:

$$
\begin{aligned}
&\frac{\frac{45}{1024}s_{45}^6 s_{12}^3}{(D-3)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3}
+\frac{\frac{9}{5120}s_{45}^6 s_{12}^3}{(D-1)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3}
-\frac{\frac{693}{5120}s_{45}^6 s_{12}^3}{(2D-7)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3}
-\frac{\frac{3}{1024}s_{45}^6 s_{12}^3}{s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3} \\
&+\frac{-\frac{45 s_{45}^6 s_{51}^2}{1024}-\frac{135 s_{45}^6 s_{51} s_{12}}{1024}-\frac{135 s_{45}^6 s_{12}^2}{1024}}{(D-3)s_{34}^4(s_{23}-s_{45}-s_{51})^3}
+\frac{-\frac{9 s_{45}^6 s_{51}^2}{5120}-\frac{27 s_{45}^6 s_{51} s_{12}}{5120}-\frac{27 s_{45}^6 s_{12}^2}{5120}}{(D-1)s_{34}^4(s_{23}-s_{45}-s_{51})^3} \\
&+\frac{\frac{693 s_{45}^6 s_{51}^2}{5120}+\frac{2079 s_{45}^6 s_{51} s_{12}}{5120}+\frac{2079 s_{45}^6 s_{12}^2}{5120}}{(2D-7)s_{34}^4(s_{23}-s_{45}-s_{51})^3}
+\frac{-\frac{3 s_{45}^6 s_{51}^2}{1024}-\frac{9 s_{45}^6 s_{51} s_{12}}{1024}-\frac{9 s_{45}^6 s_{12}^2}{1024}}{s_{34}^4(-s_{23}+s_{45}+s_{51})^3}
\end{aligned}
$$

$$
= -\frac{\frac{3}{512}D\left(D^2-4\right)s_{45}^6(s_{51}+s_{12})^3}{(D-3)(D-1)(2D-7)s_{34}^4 s_{51}(-s_{23}+s_{45}+s_{51})^3}
$$

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Summary and Outlook

- Method to reconstruct rational functions directly in partial-fractioned form
  - Uses p-adic numbers to harness the major (100 times!) simplification of rational functions under partial fractioning
  - Simplification is specific to loop calculations, and is surprising from computer algebra viewpoint – physical origin?
- Demonstrated by reconstructing the largest rational function in one of largest IBP coefficients needed for non-planar 2-loop 5-point massless QCD amplitudes.

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- Promising technique for:
  - Calculating rational functions (in amplitudes and IBPs)
  - Studying the reasons for these simplifications (analytically?)
  - Exploring the further simplification hinted at in these results

Find out more: arXiv:2312.03672