CERN

AXEL – 2023

Introduction to Particle Accelerators

Basic Mathematics

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Contents

- Vectors & Matrices
- Differential Equations
- Some Units we use

- **Vectors & Matrices**
- Differential Equations
- Some Units we use

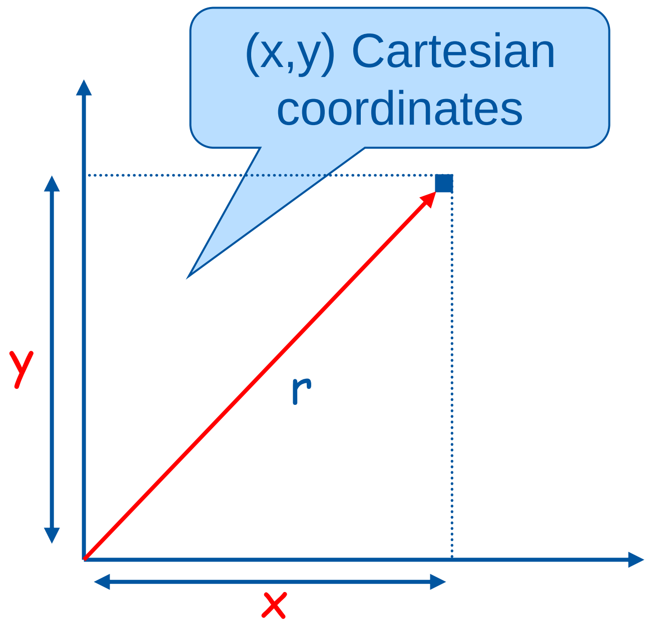
Scalars & Vectors

Scalar, a single quantity or value

Vector, (origin,) length, direction

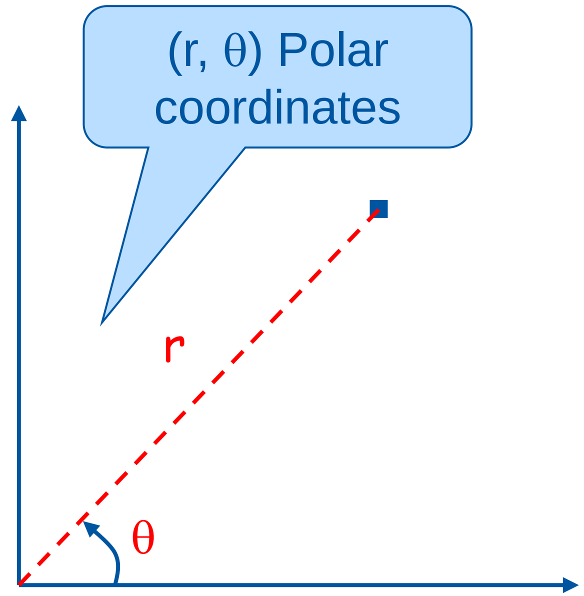
Coordinate systems

A **vector** has 2 or more quantities associated with it



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

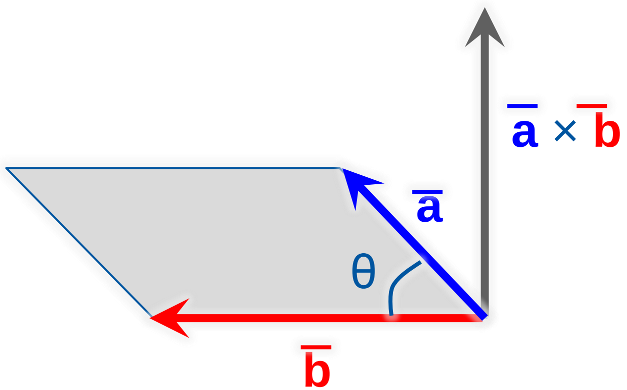


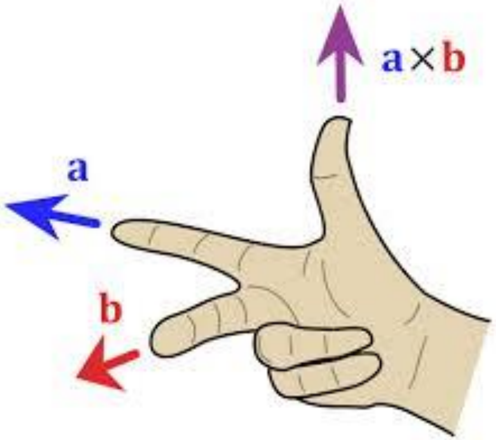
θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Vector Cross Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ





The cross product $\bar{a} \times \bar{b}$ is defined by:

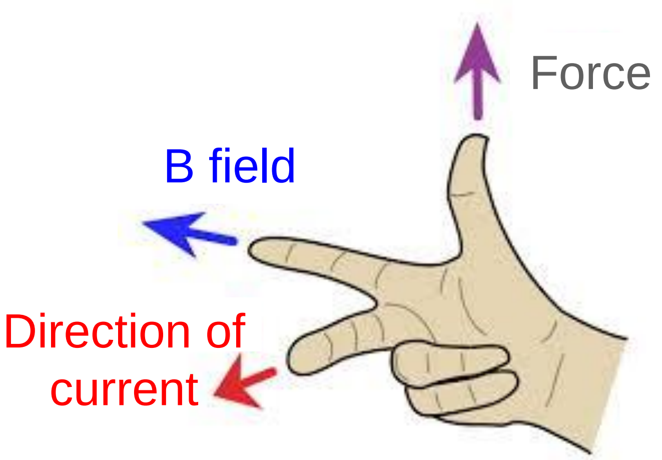
- **Direction**: $\bar{a} \times \bar{b}$ is perpendicular (normal) on the plane through $\bar{a}$ and $\bar{b}$
- The **length** of $\bar{a} \times \bar{b}$ is the surface of the parallelogram formed by $\bar{a}$ and $\bar{b}$

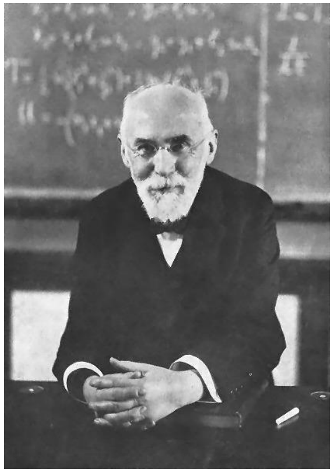
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

Cross Product & Magnetic Field

The Lorentz force in a pure magnetic field expression

$$F = e(\vec{v} \times \vec{B})$$





The reason why our particles move around our "circular" machines under the influence of the magnetic fields

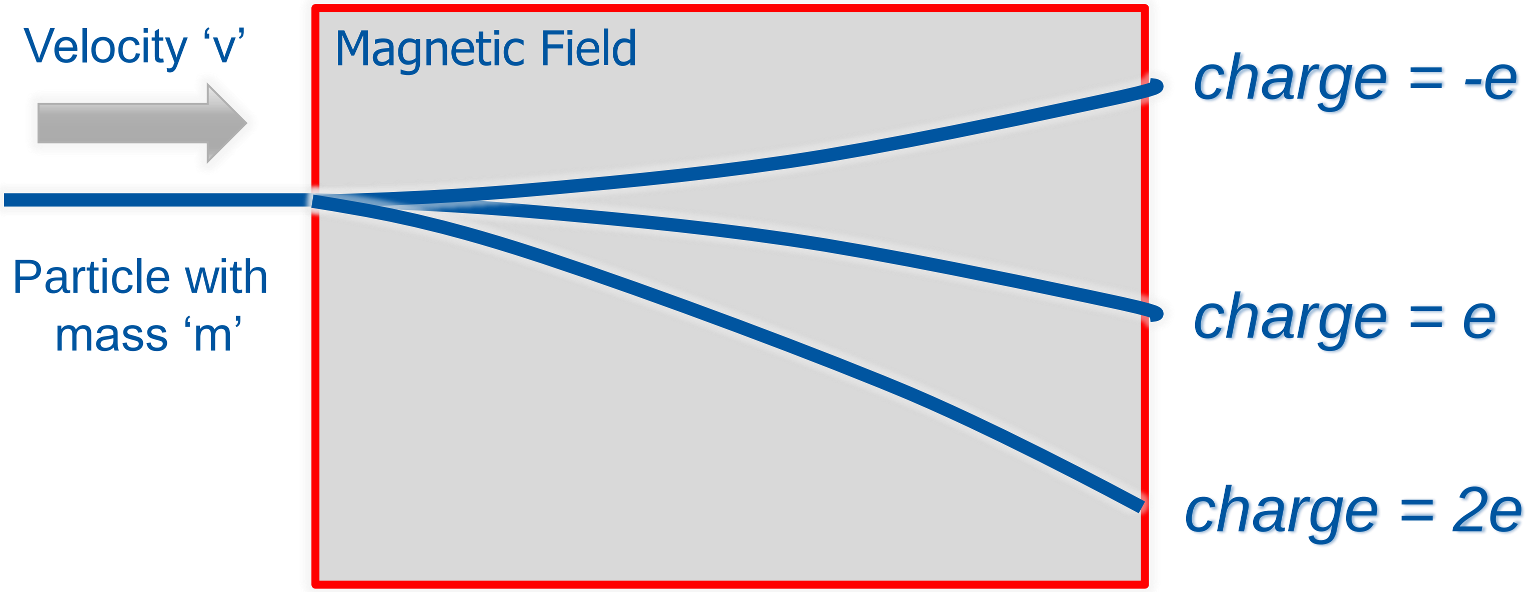
"Electro-Magnetic Elements" by Piotr Skowronski
"Transverse Beam Dynamics" by Bernhard Holzer

Today

Tue. & Wed.

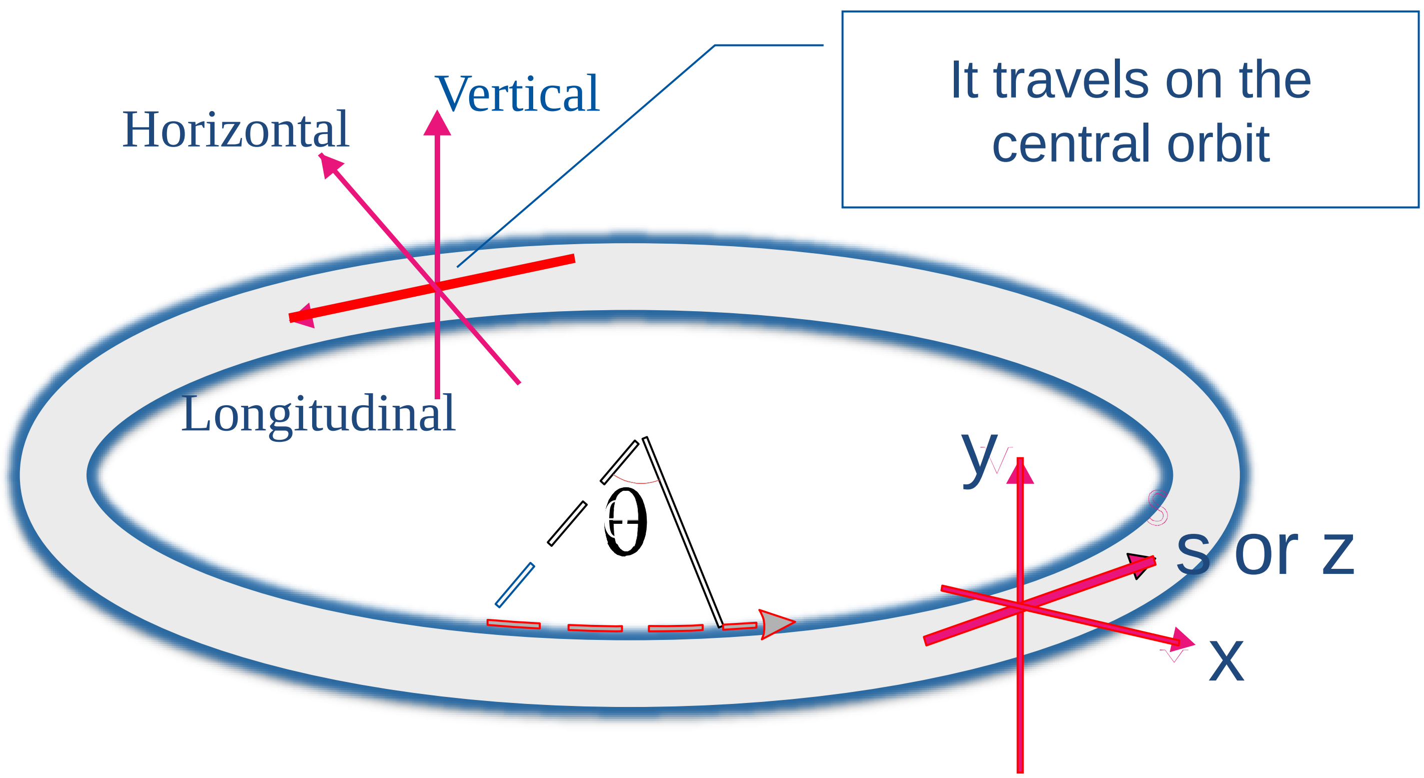
Lorentz Force in Action

$$F = e(\vec{v} \times \vec{B})$$



The larger the energy of the beam the larger the radius of curvature

A Rotating Coordinate System



Magnetic Rigidity

$$F = e(\vec{v} \times \vec{B})$$

- As a formula this is:

$$F = evB = \frac{mv^2}{\rho}$$

Radius of curvature

Like for a stone attached to a rotating rope

- Which can be written as:

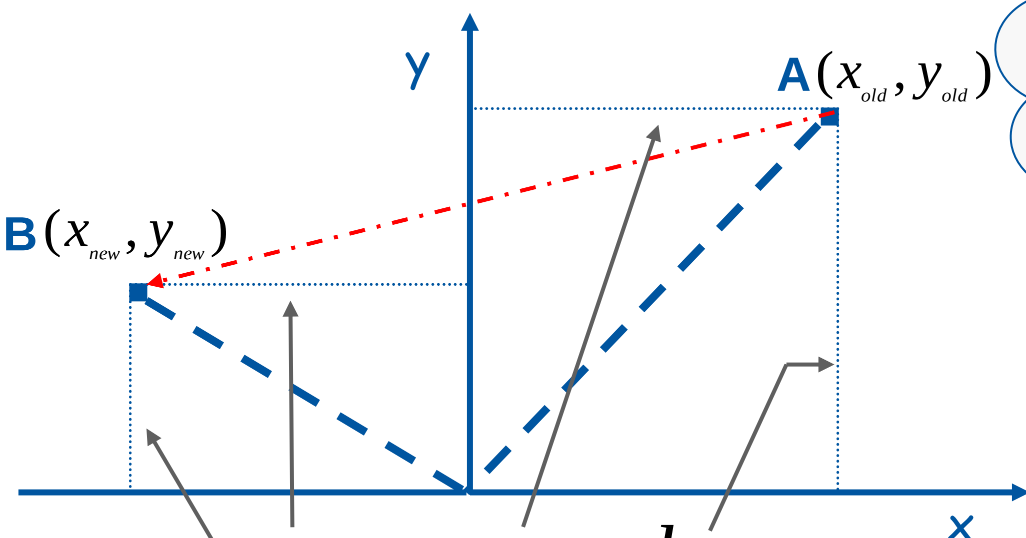
$$B\rho = \frac{mv}{e} = \frac{p}{e}$$

Momentum p=mv

- Bρ is called the magnetic rigidity, and if we put in all the correct units we get:
- Bρ = 33.356·p [kG·m] = 3.3356·p [T·m] (if p is in [GeV/c])

Moving a Point in a Coordinate System

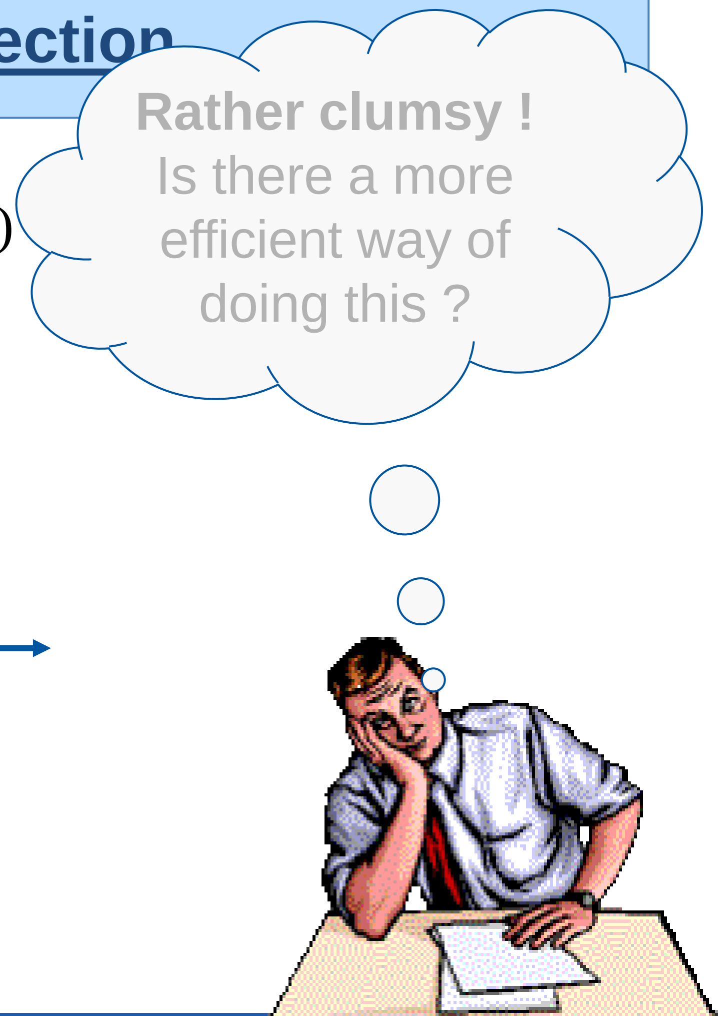
To move from one point (A) to any other point (B) one needs control of both **Length** and **Direction**



$$x_{new} = ax_{old} + by_{old}$$

$$y_{new} = cx_{old} + dy_{old}$$

2 equations needed !!!



Matrices & Vectors

So, we have:
$$\begin{cases} x_{new} = a x_{old} + b y_{old} \\ y_{new} = c x_{old} + d y_{old} \end{cases}$$

Let's write this as **<u>one</u>** equation:

$$\overline{B} = M\overline{A}$$

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Rows

Columns

- $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are **<u>Vectors</u>** or **<u>Matrices</u>**
- $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ have 2 rows and 1 column
- **M** is a **<u>Matrix</u>** and has 2 rows and 2 columns

Matrix Multiplications

This implies:

$$\left.\begin{aligned} x_{new} &= ax_{old} + by_{old} \\ y_{new} &= cx_{old} + dy_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

This defines the rules for **matrix multiplicatio**

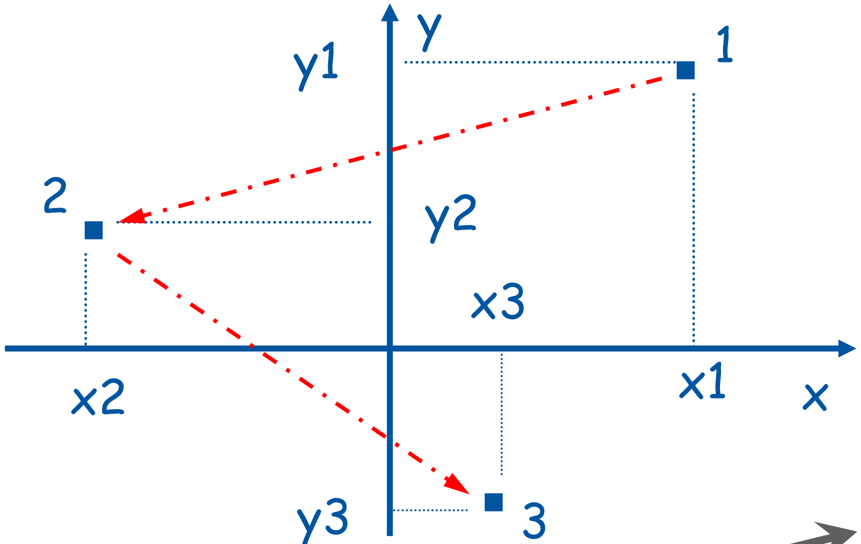
$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

This matrix multiplication results in:

$$i = ae + bg, j = af + bh, k = ce + dg, l = cf + dh$$

Moving a Point & Matrices

Lets apply what we just learned and move a point around:



- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2.M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Matrices & Accelerators

- We use matrices to describe the various magnetic elements in our accelerator.
  - The **x** and **y** co-ordinates are the **position** and **angle** of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we **multiply all the matrices** describing the magnetic elements between the two points to give a single matrix

- Now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

The Unit Matrix

There is a special matrix that when multiplied with an initial point will result in the same final point.

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

The **Unit matrix** has **no effect** on x and y

Going Backwards

What about **going back** from a **final** point to the corresponding **initial** point ?

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

or

$$\overline{B} = M\overline{A}$$

For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B}$$

such that

$$\overline{B} = MM^{-1}\overline{B}$$

The combination of M and M^{-1} does have no effect

$$MM^{-1} = Unit\ Matrix$$

M^{-1} is the "**inverse**" or "**reciprocal**" matrix of **M.**

Calculating the Inverse Matrix

If we have a 2 x 2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

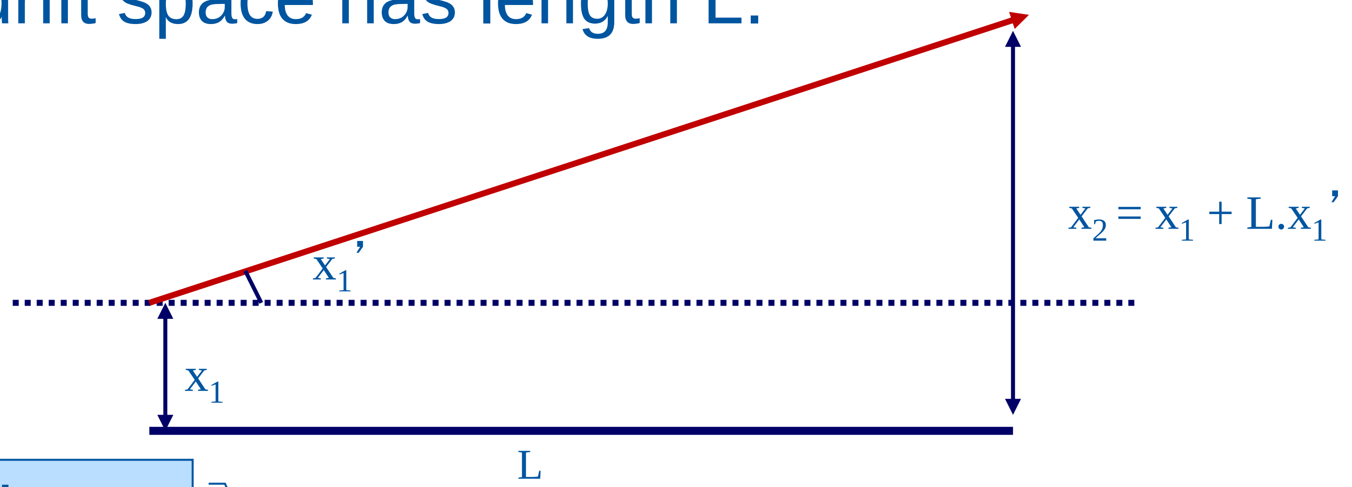
Then the **inverse matrix** is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term **(ad – bc)** is called the **determinate**, which is just a **number** (scalar).

Example: Drift Space Matrix

- A drift space contains no magnetic field.
- A drift space has length L.



x_1' small

$$\left.\begin{aligned} x_2 &= x_1 + L x_1' \\ x_2' &= 0 + x_1' \end{aligned}\right\} \rightarrow \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

A Practical Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M\,\overline{\Delta I}$$

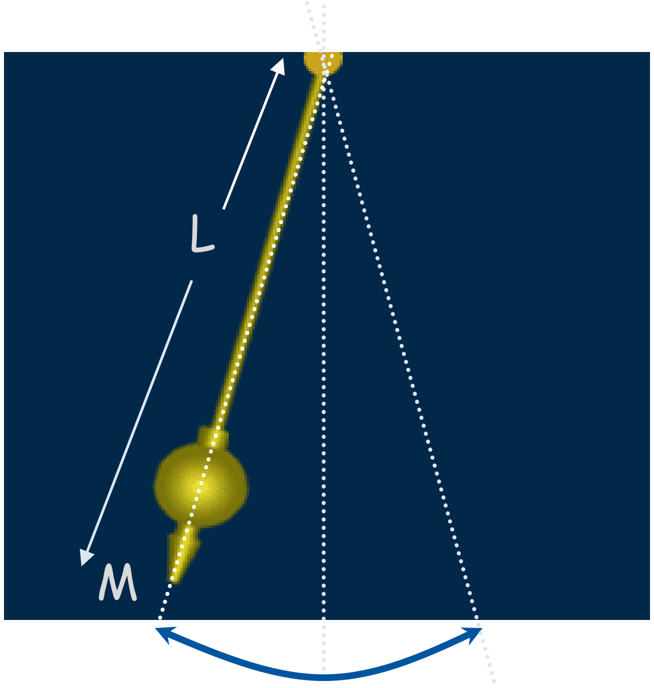
- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1}\,\overline{\Delta Q}$$

- Vectors & Matrices
- **Differential Equations**
- Some Units we use

The Pendulum

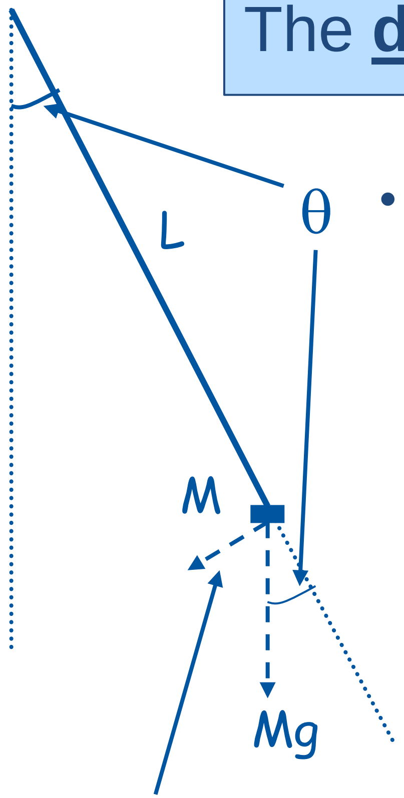
- Lets use a pendulum as example
- The **length** of the pendulum is **L**
- It has a **mass m** attached to it
- It moves back and forth under the **influence of gravity**



- Lets try to find an **equation** that **describes** the **motion** of the mass **m** makes.
- This will result in a **Differential Equation**

Differential Equation

The **distance** from the centre = **Lθ** (since θ is small)



- The **velocity** of mass M is: $v = \frac{d(L\theta)}{dt}$
- The **acceleration** of mass M is: $a = \frac{d^2(L\theta)}{dt^2}$
- Newton: **F**orce = **m**ass x **a**cceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity is

-M g sinθ

(force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

θ is small

L is constant

Solving a Differential Equation

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

Differential equation describing the motion of a pendulum at small amplitudes.

Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t)$$ and $$\frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving a Differential Equation

Now we have to find the solution for the following equation:

$$-\omega^2 \cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving this equation gives: $\omega = \sqrt{\frac{g}{L}}$

The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}\,t$$

Oscillation amplitude

Oscillation frequency

Position & Velocity

The differential equation that describes the transverse motion of the particles as they move around our accelerator.

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

The solution of this second order describes **oscillatory motion**

For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

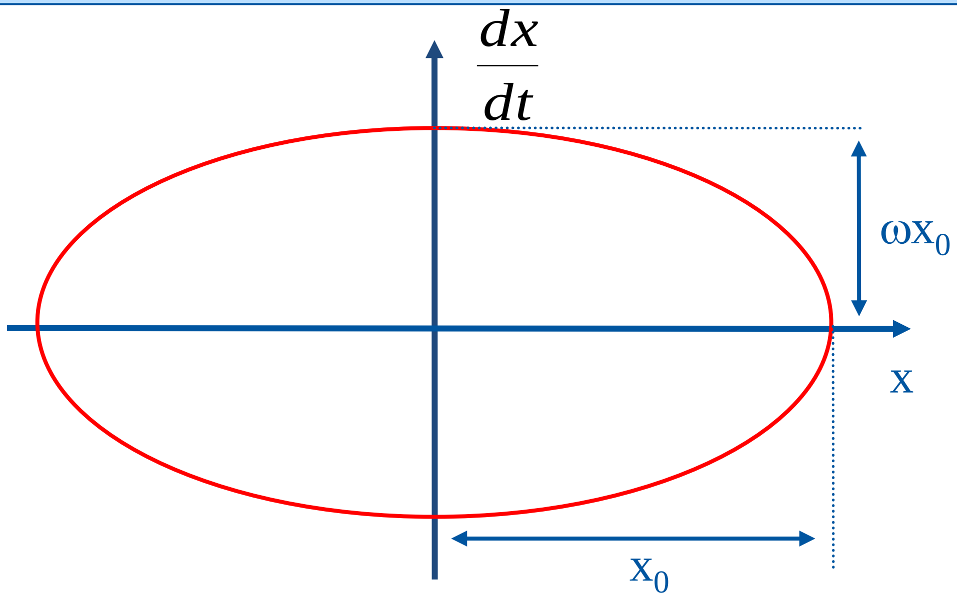
$$x = x_0 \cos(\omega t) \qquad \frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Phase Space Plot

Plot the **velocity** as a function of **displacement**:

$$x = x_0 \cos(\omega t)$$

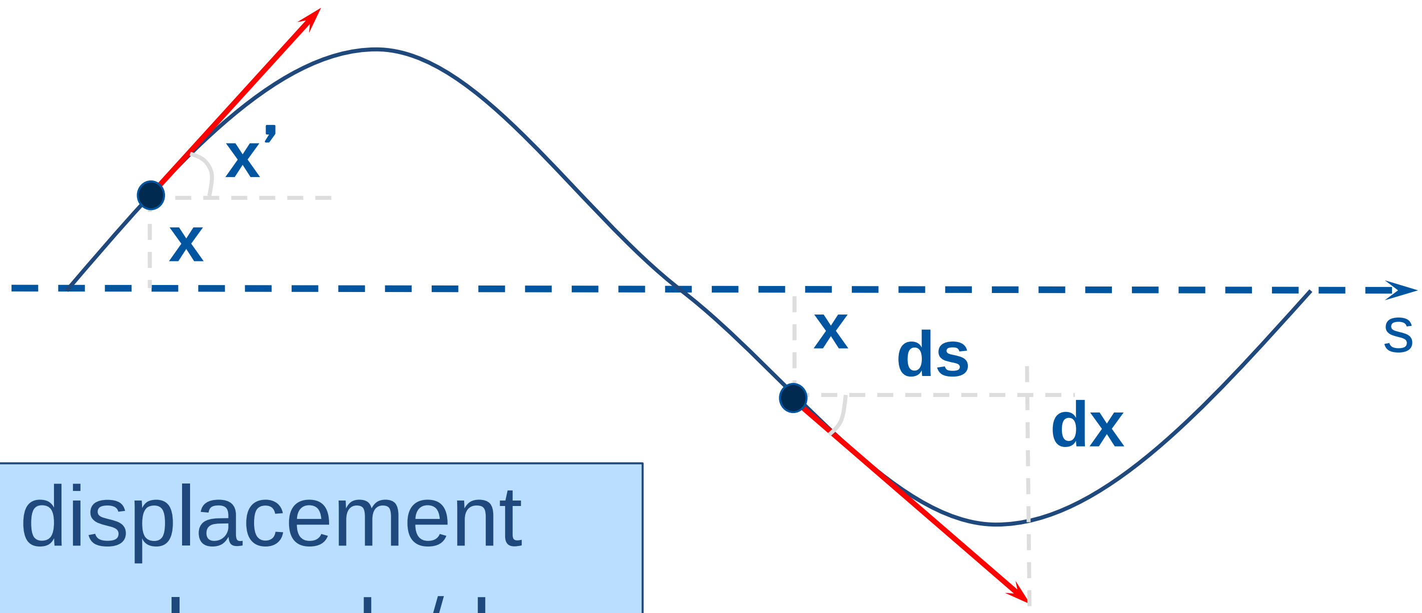
$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$



- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

Oscillations in Accelerators

Under the influence of the **magnetic fields** the **particle oscillate**



x = displacement

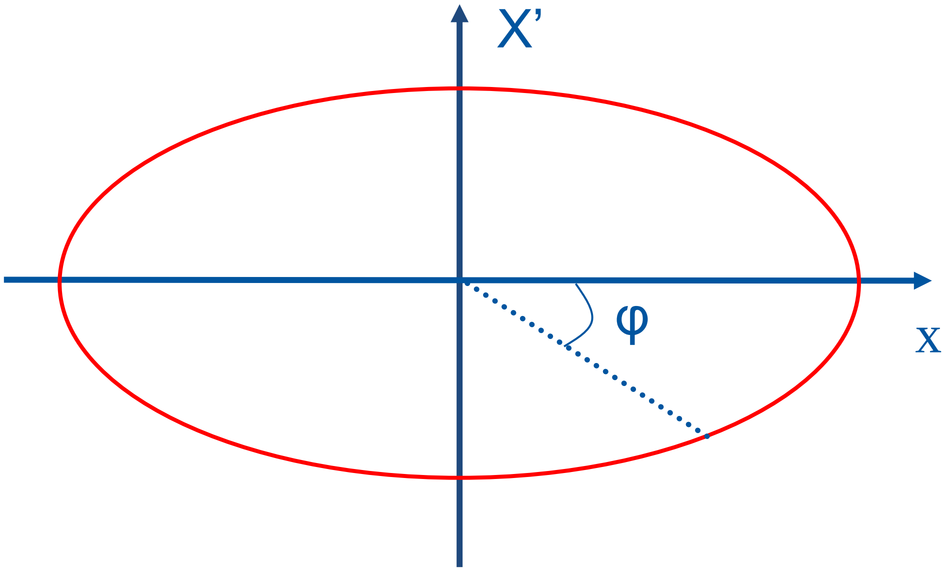
x' = angle = dx/ds

Transverse Phase Space Plot

This changes slightly the Phase Space plot

Position x

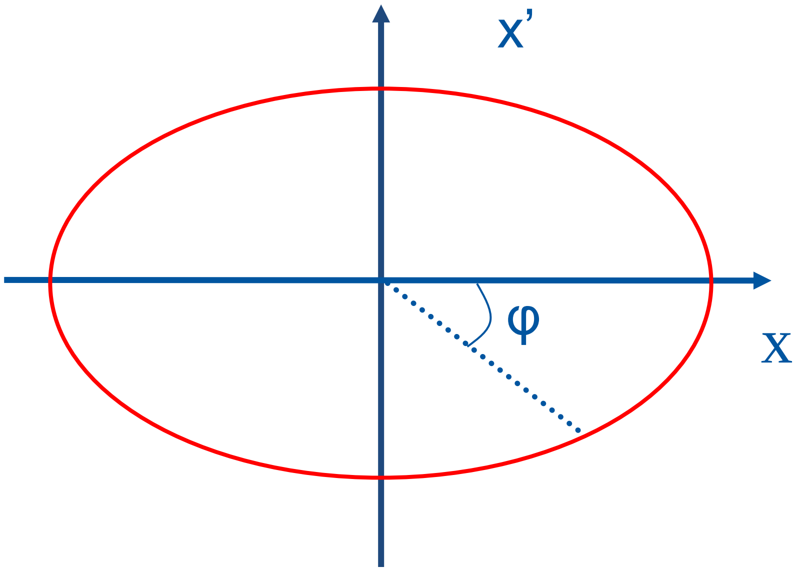
Angle $x' = \frac{dx}{ds}$



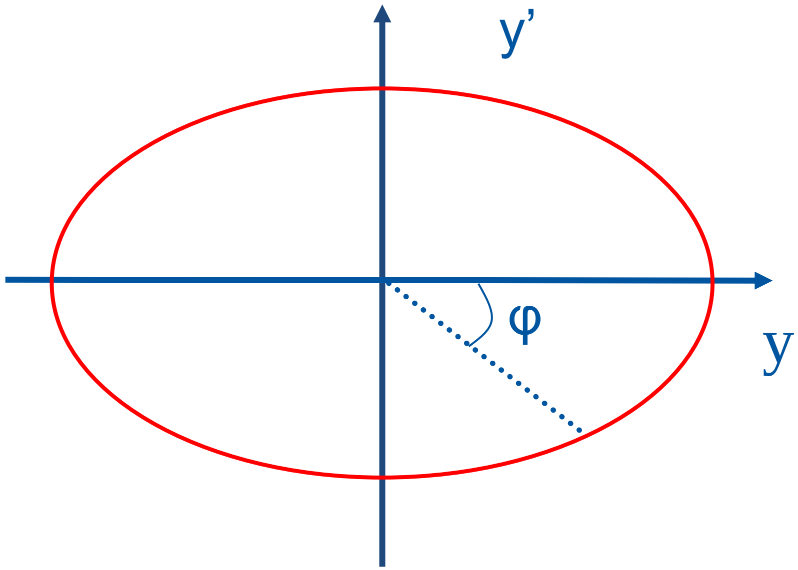
- $\varphi = \omega t$ is called the **phase angle**
- X-axis is the horizontal or vertical position (or time).
- Y-axis is the horizontal or vertical phase angle (or energy).

Transverse Phase Space Plot

We distinguish motion in the Horizontal & Vertical Plane

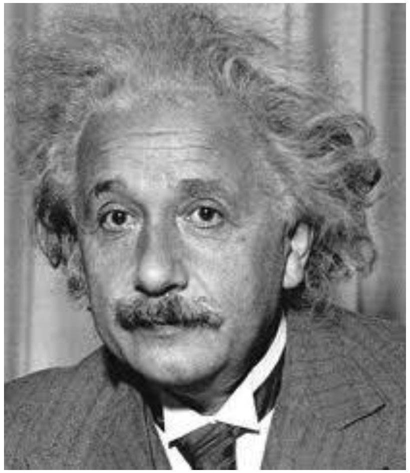


Horizontal Phase Space



Vertical Phase Space

"Pure mathematics is, in its way, the poetry of logical ideas."



Albert Einstein