AXEL-2023

Introduction to Particle Accelerators

Transverse optics 2:

- ✓*Hill's equation*
- ✓*Phase Space*
- ✓*Emittance & Acceptance*
- ✓*Matrix formalism*

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Hill's equation

- ✓ The **betatron oscillations** exist in both horizontal and vertical planes.
- ✓ The number of betatron oscillations per turn is called the **betatron tune** and is defined as **Qx** and **Qy**.
- ✓ Hill's equation describes this motion mathematically

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- ✓ If the restoring force, K is constant in 's' then this is just a **Simple Harmonic Motion**.
- ✓ 's' is the longitudinal displacement around the accelerator.

Hill's equation (2)

- In a real accelerator **K varies strongly with 's'**.
- Therefore, we need to solve Hill's equation for K varying as a function of 's'

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- Remember what we concluded on the mechanical equivalent concerning the shape of the gutter……
  - The **phase advance** and the **amplitude modulation** of the oscillation are determined by the shape of the gutter.
  - The overall **oscillation amplitude** will depend on the **initial conditions**, i.e. how the motion of the ball started.

Solution of Hill's equation (1)

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- ✓ Remember, this is a 2nd order differential equation.
- ✓ In order to solve it lets try to guess a solution:

$$x = \sqrt{\varepsilon.\beta(s)}\cos\left(\phi(s) + \phi_0\right)$$

- ✓ **ε** and **Φ_0** are constants, which depend on the **initial conditions**.
- ✓ **β(s)** = the **amplitude modulation** due to the changing focusing strength.
- ✓ **Φ(s)** = the **phase advance**, which also depends on focusing strength.

Solution of Hill's equation (2)

- ✓ Define some parameters

$$\alpha = \frac{-\beta'}{2}$$

$$\beta = \omega^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

- ✓ …and let $\phi = (\phi(s) + \phi_0)$

Remember Φ is still a function of s

$$x = \sqrt{\varepsilon}.\omega(s)\cos\phi$$

- ✓ In order to solve Hill's equation we differentiate our guess, which results in:

$$x' = \sqrt{\varepsilon}\frac{d\omega}{ds}\cos\phi - \sqrt{\varepsilon}\omega\phi'\sin\phi$$

- ✓ ……and differentiating a second time gives:

$$x'' = \sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$

- ✓ Now we need to substitute these results in the original equation.

Solution of Hill's equation (3)

- ✓ So we need to substitute $x = \sqrt{\varepsilon.\beta(s)}\cos(\phi(s) + \phi_0)$

  and its second derivative

$$x'' = \sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$

  into our initial differential equation $\frac{d^2x}{ds^2} + K(s)x = 0$

- ✓ This gives:

$$\sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$
$$+ K(s)\sqrt{\varepsilon}\omega\cos\phi = 0$$

Sine and Cosine are orthogonal and will never be 0 at the same time

The sum of the coefficients must vanish separately to make our guess valid for all phases

Solution of Hill's equation (4)

$$\sqrt{\varepsilon}\omega''\cos\phi - \underline{\sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi} - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$
$$+K(s)\sqrt{\varepsilon}\omega\cos\phi = 0$$

- ✓ Using the 'Sin' terms $\longrightarrow$ $2\omega'\phi' + \omega\phi'' = 0$ $\longrightarrow$ $2\omega\omega'\phi' + \omega^2\phi'' = 0$
- ✓ We defined $\beta = \omega^2$, which after differentiating gives $\beta' = 2\omega\omega'$
- ✓ Combining $2\omega\omega'\phi' + \omega^2\phi'' = 0$ and $\beta' = 2\omega\omega'$ gives: $\beta'\phi' + \beta\phi'' = (\beta\phi')' = 0$

$$\frac{d\beta}{ds} = \frac{d\beta}{d\omega}\frac{d\omega}{ds}$$

- ✓ As condition for our guessed solution to be valid we get:

$$\beta\phi' = const. = 1 \quad \text{hence} \quad \phi' = \frac{d\phi}{ds} = \frac{1}{\beta}$$

- ✓ So our **guess** seems to be **correct**

Solution of Hill's equation (5)

- Since our solution was correct, we have the following for x:

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$

$$\frac{d\omega}{ds} = \frac{\beta'}{2\omega} = -\frac{\alpha}{\sqrt{\beta}}$$

- For x' we have now:

$$x' = \sqrt{\varepsilon}\frac{d\omega}{ds}\cos\phi - \sqrt{\varepsilon}\omega\phi'\sin\phi$$

$$\omega = \sqrt{\beta}$$

- Thus, the expression for x' finally becomes:

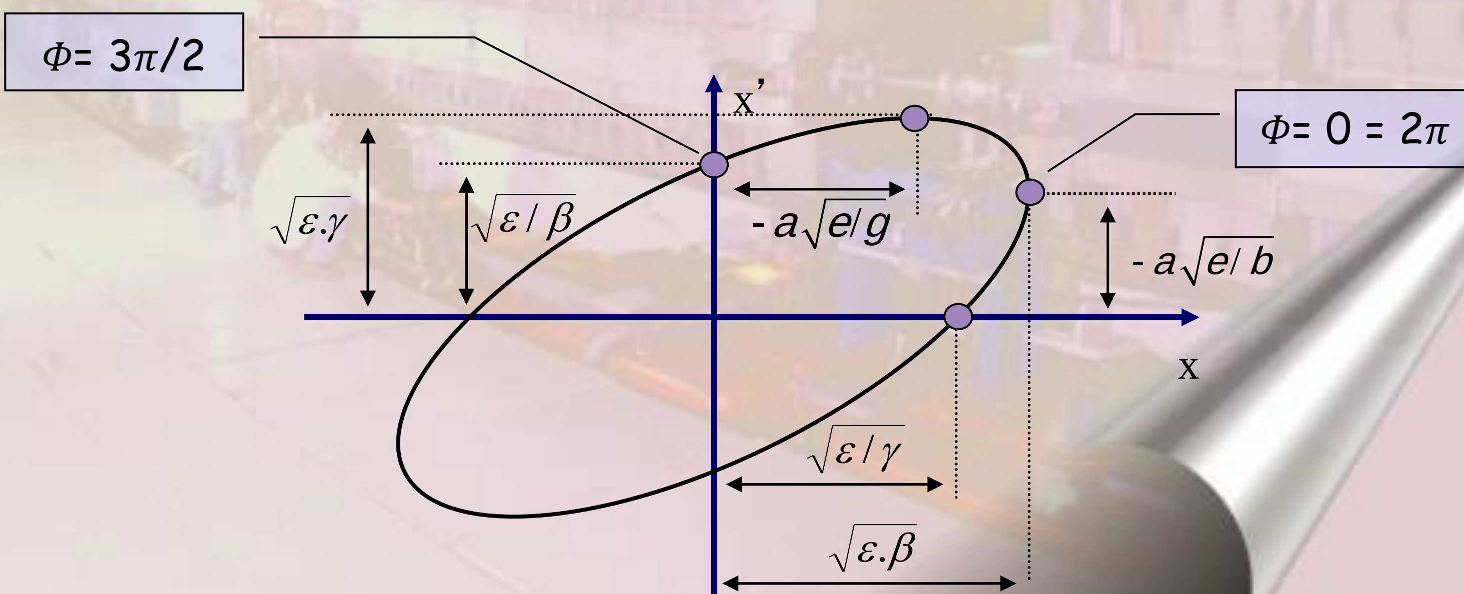
$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

Phase Space Ellipse

- ✓ So now we have an expression for x and x'

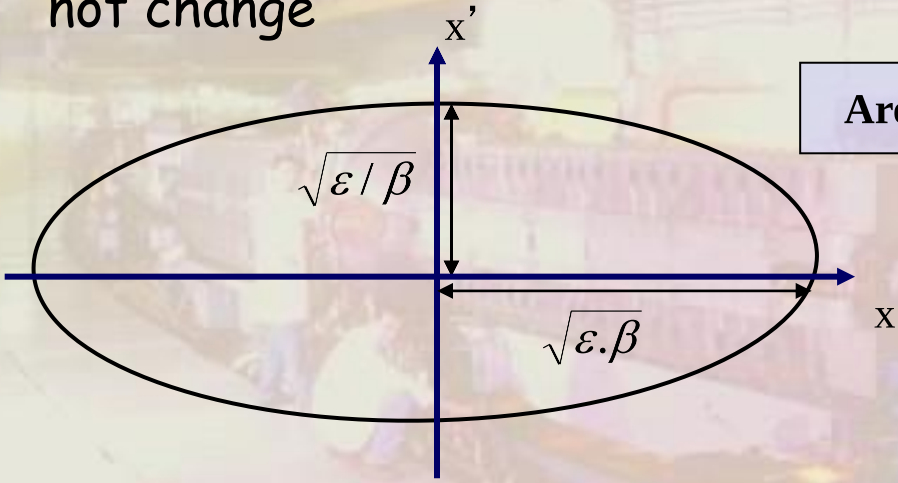
$$x = \sqrt{\varepsilon.\beta}\cos\phi \quad \text{and} \quad x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

- ✓ If we plot **x' versus x as Φ goes from 0 to 2π** we get an ellipse, which is called the **phase space ellipse**.

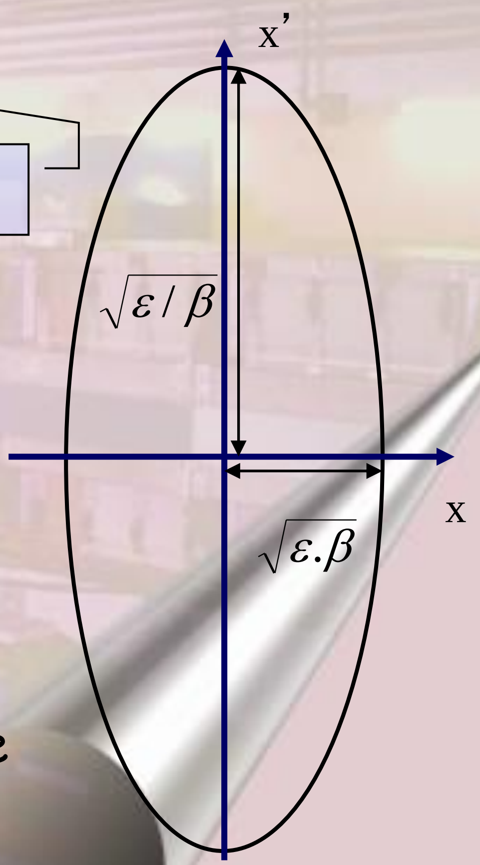


Phase Space Ellipse (2)

- ✓ As we move around the machine the shape of the ellipse will change as β changes under the influence of the quadrupoles
- ✓ However, the area of the ellipse ($\pi\varepsilon$ does not change

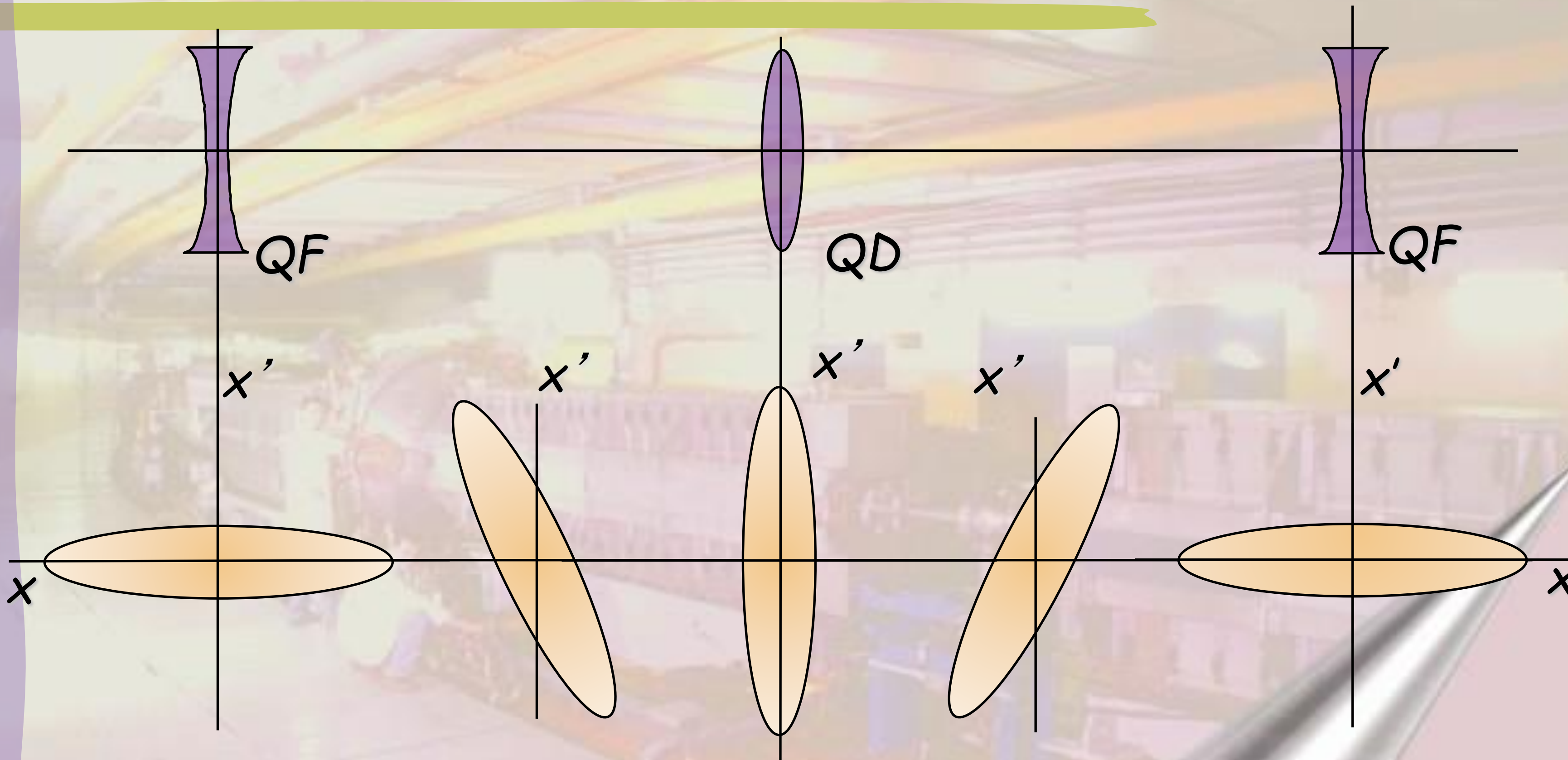


Area = $\pi \cdot r_1 \cdot r_2$



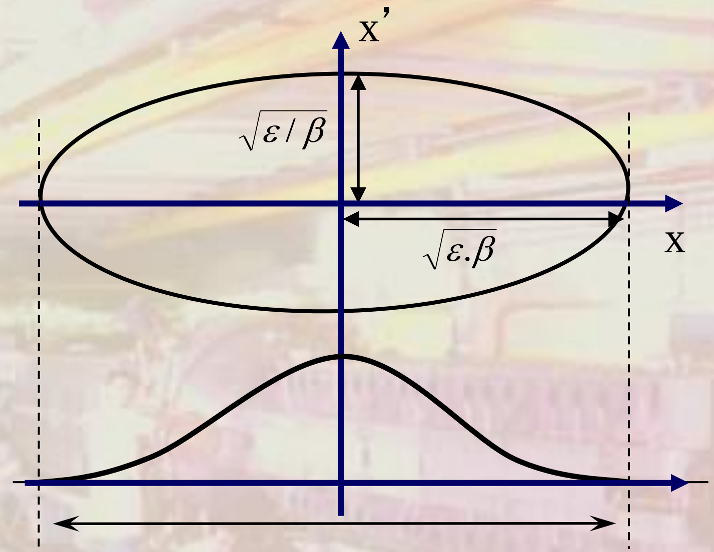
- ✓ ε is called the **transverse emittance** and is determined by the initial beam conditions.
- ✓ The units are meter·radians, but in practice we use more often **mm·mrad**.

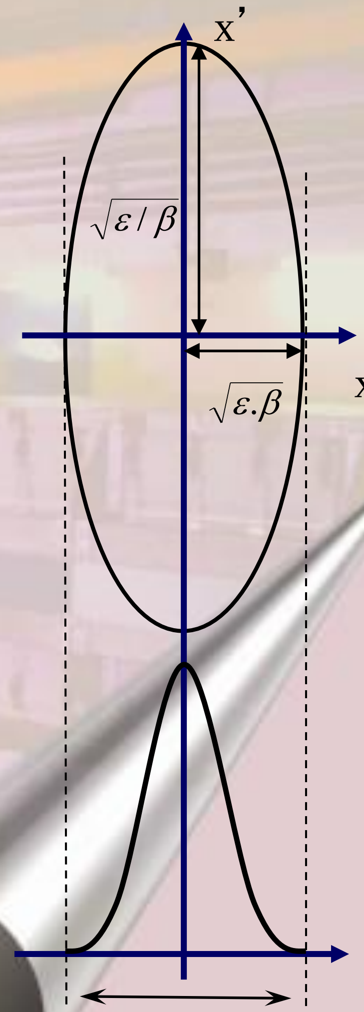
Phase Space Ellipse (3)



- ✓ For each point along the machine the ellipse has a particular orientation, but the area remains the same

Phase Space Ellipse (4)

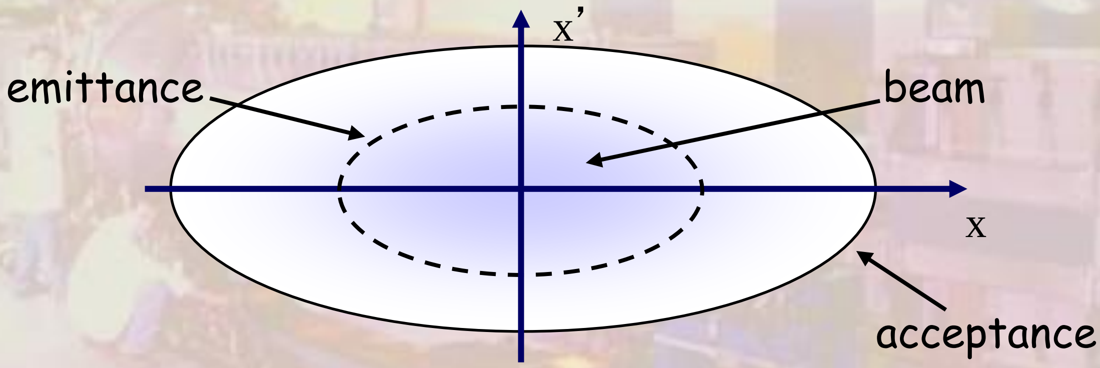




- ✓ The projection of the ellipse on the x-axis gives the **Physical transverse beam size.**
- ✓ Therefore the variation of $\beta(s)$ around the machine will tell us how the transverse beam size will vary.

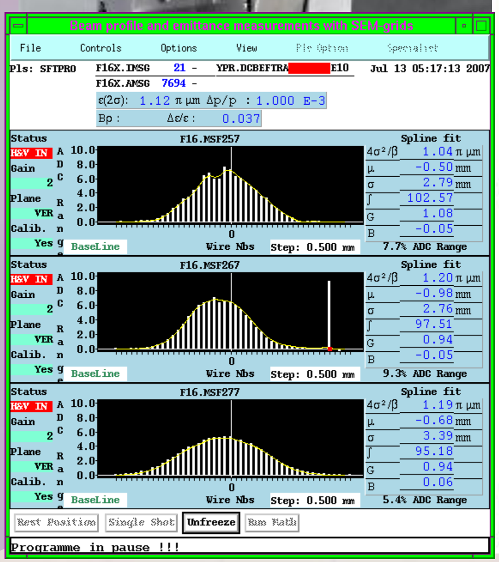
Emittance & Acceptance

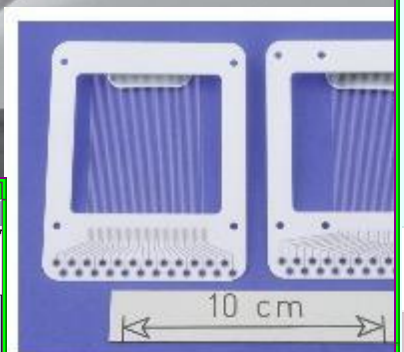
- To be rigorous we should define the emittance slightly differently.
  - Observe all the particles at a single position on one turn and measure both their position and angle.
  - This will give a large number of points in our phase space plot, each point representing a particle with its co-ordinates x, x'.

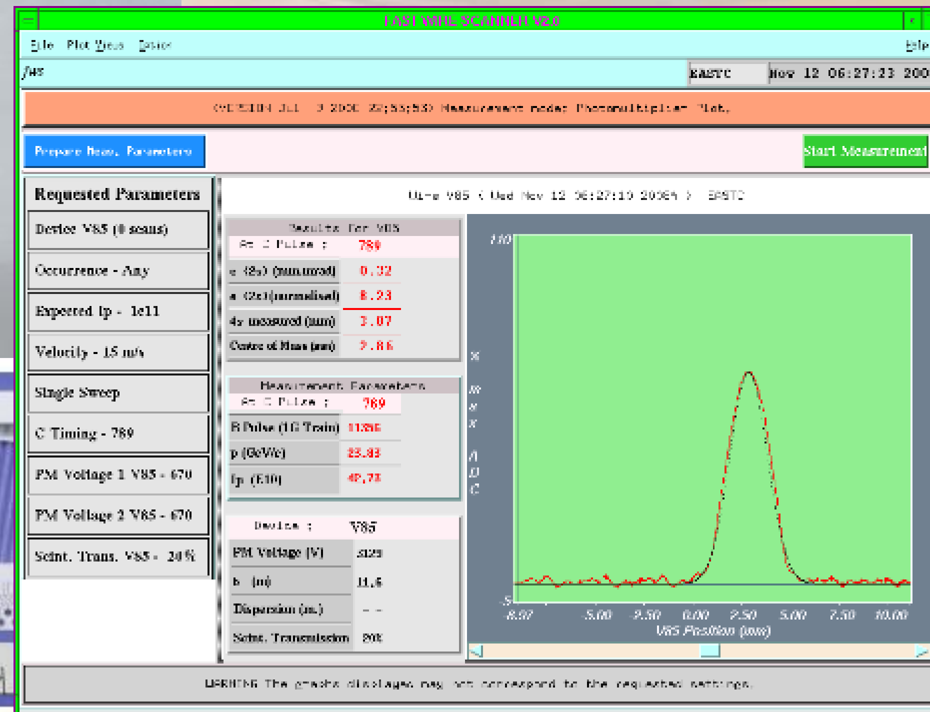


- The **emittance** is the **area** of the ellipse, which contains all, or a defined percentage, of the particles.
- The **acceptance** is the maximum **area** of the ellipse, which the emittance can attain without losing particles.

Emittance measurement







Matrix Formalism

- ✓ Lets represent the particles transverse position and angle by a column matrix.

$$\begin{pmatrix} x \\ x' \end{pmatrix}$$

- ✓ As the particle moves around the machine the values for x and x' will vary under influence of the dipoles, quadrupoles and drift spaces.
- ✓ These modifications due to the different types of magnets can be expressed by a **Transport Matrix M**
- ✓ If we know x_1 and x_1' at some point s_1 then we can calculate its position and angle after the next magnet at position s_2 using:

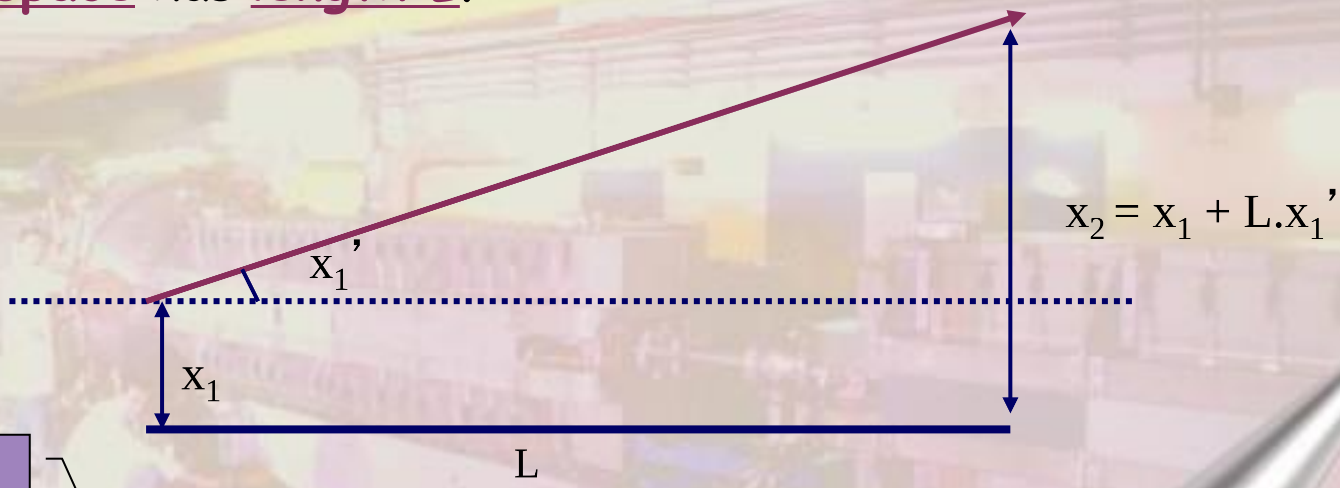
$$\begin{pmatrix} x(s_2) \\ x(s_2)' \end{pmatrix} = M \begin{pmatrix} x(s_1) \\ x(s_1)' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(s_1) \\ x(s_1)' \end{pmatrix}$$

How to apply the formalism

- If we want to know how a particle behaves in our machine as it moves around using the matrix formalism, we need to:
  - **Split** our machine **into separate elements** as dipoles, focusing and defocusing quadrupoles, and drift spaces.
  - **Find the matrices** for all of these components
  - **Multiply them** all together
  - **Calculate** what happens to an individual particle as it makes **one or more turns** around the machine

Matrix for a drift space

- ✓ A **drift space** contains **no magnetic field**.
- ✓ A **drift space** has **length L**.

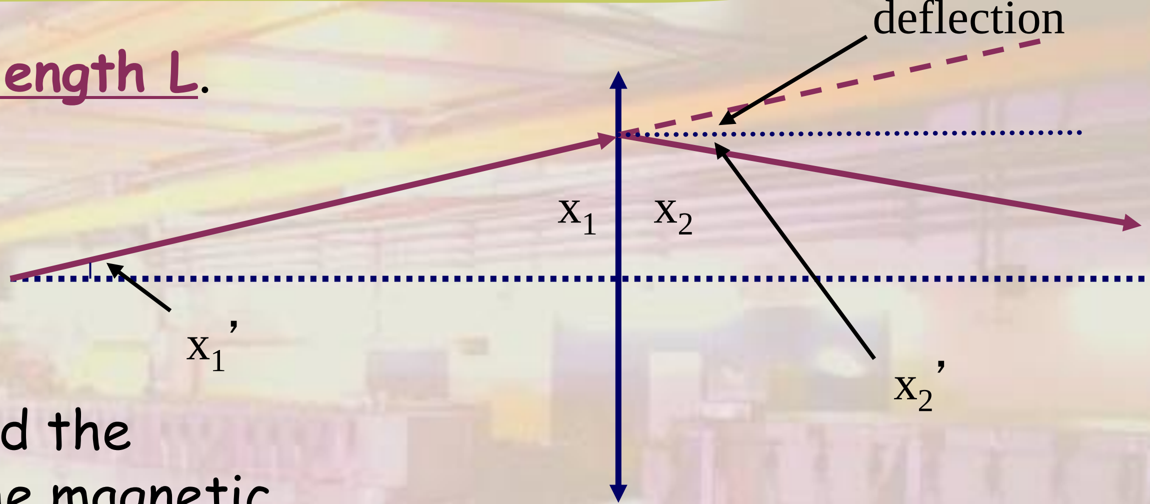


x_1' small

$$\left.\begin{aligned} x_2 &= x_1 + Lx_1' \\ x_2' &= 0 + x_1' \end{aligned}\right\} \longrightarrow \boxed{\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}}$$

Matrix for a quadrupole

- A quadrupole of length L.



Remember $\mathbf{B_y} \propto \mathbf{x}$ and the deflection due to the magnetic field is: $\frac{LB_y}{(B\rho)} = -\frac{LK}{(B\rho)} \cdot x$

Provided L is small

$$\left.\begin{aligned} x_2 &= x_1 + 0 \\ x_2' &= -\frac{LK}{(B\rho)} x_1 + x_1' \end{aligned}\right\} \longrightarrow \boxed{\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}}$$

Matrix for a quadrupole (2)

- We found :

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

- Define the focal length of the quadrupole as f= $\dfrac{(B\rho)}{KL}$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

How now further ?

- ✓ For our purpose we will treat dipoles as simple drift spaces as they bend all the particles by the same amount.
- ✓ We have **Transport Matrices** corresponding to **drift spaces** and **quadrupoles**.
- ✓ These matrices describe the real discrete focusing of our quadrupoles.
- ✓ Now we must **combine these matrices with** our solution to **Hill's equation**, since they describe the same motion……

Questions….,Remarks…?

Hill's equation

Emittance & Acceptance

Phase space

Matrix formalism