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Introduction to Particle Accelerators

Lattice calculations:

- ✓ *Lattices*
- ✓ *Tune Calculations*
- ✓ *Dispersion*
- ✓ *Momentum Compaction*
- ✓ *Chromaticity*
- ✓ *Sextupoles*

Rende Steerenberg (BE/OP)

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A quick recap……

- ✓ We solved Hill's equation, which led us to the definition of transverse emittance and allowed us to describe particle motion in transverse phase space in terms of β, α, etc…
- ✓ We constructed the Transport Matrices corresponding to drift spaces and quadrupoles.
- ✓ **Now we must combine these matrices with the solution of Hill's equation to evaluate β, α, etc…**

Matrices & Hill's equation

- We can multiply the matrices of our drift spaces and quadrupoles together to form a transport matrix that describes a larger section of our accelerator.
- These matrices will move our particle from one point $(x(s_1),x'(s_1))$ on our phase space plot to another $(x(s_2),x'(s_2))$, as shown in the matrix equation below.

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- The elements of this matrix are fixed by the elements through which the particles pass from point s_1 to point s_2.
- However, we can also express (x, x') as solutions of Hill's equation.

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$ and $$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (2)

$$x = \sqrt{\varepsilon.\beta}\cos(\mu+\phi)$$

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos(\mu+\phi) - \sqrt{\varepsilon/\beta}\sin(\mu+\phi)$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

- ✓ Assume that our transport matrix describes **a complete turn** around the machine.
- ✓ Therefore : $\beta(s_2) = \beta(s_1)$
- ✓ Let μ be the change in betatron phase over one complete turn.
- ✓ Then we get for $x(s_2)$:

$$x(s_2) = \sqrt{\varepsilon.\beta}\cos(\mu+\phi) = a\sqrt{\varepsilon.\beta}\cos\phi - b\alpha\sqrt{\varepsilon/\beta}\cos\phi - b\sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (3)

- So, for the position x at s2 we have...

$$\sqrt{\varepsilon.\beta}\cos(\mu+\phi) = a\sqrt{\varepsilon.\beta}\cos\phi - b\alpha\sqrt{\varepsilon/\beta}\cos\phi - b\sqrt{\varepsilon/\beta}\sin\phi$$

$$\cos\phi\cos\mu - \sin\phi\sin\mu$$

- Equating the 'sin' terms gives: $-\sqrt{\varepsilon.\beta}\sin\mu\sin\phi = -b\sqrt{\varepsilon/\beta}\sin\phi$
- Which leads to: $b = \beta\sin\mu$
- Equating the 'cos' terms gives: $\sqrt{\varepsilon.\beta}\cos\mu\cos\phi = a\sqrt{\varepsilon.\beta}\cos\phi - \alpha\sqrt{\varepsilon.\beta}\sin\mu\cos\phi$
- Which leads to: $a = \cos u + \alpha\sin\mu$
- We can repeat this for **c** and **d**.

Matrices & Twiss parameters

- ✓ Remember previously we defined:

$$\alpha = \frac{-\beta'}{2} = -\omega\omega'$$

$$\beta = \omega^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

- ✓ These are called **TWISS parameters**

- ✓ Remember also that μ is the total betatron phase advance over one complete turn is.

$$Q = \frac{\mu}{2\pi}$$

Number of betatron oscillations per turn

- ✓ Our transport matrix becomes now:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Lattice parameters

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- ✓ This matrix describes one complete turn around our machine and will vary depending on the starting point (s).
- ✓ If we start at any point and multiply all of the matrices representing each element all around the machine we can calculate α, β, γ and μ for that specific point, which then will give us β(s) and Q
- ✓ If we repeat this many times for many different initial positions (s) we can calculate our Lattice Parameters for all points around the machine.

Lattice calculations and codes

- ✓ Obviously μ (or Q) is not dependent on the initial position 's', but we can calculate the change in betatron phase, dμ, from one element to the next.
- ✓ Computer codes like "MAD" or "Transport" vary lengths, positions and strengths of the individual elements to obtain the desired beam dimensions or envelope '$\beta(s)$' and the desired 'Q'.
- ✓ Often a machine is made of many individual and identical sections (FODO cells). In that case we only calculate a single cell and not the whole machine, as the the functions $\beta(s)$ and dμ will repeat themselves for each identical section.
- ✓ The insertion sections have to be calculated separately.

The β(s) and Q relation.

- $Q = \frac{\mu}{2\pi}$,where μ = ΔΦ over a complete turn
- But we also found: $\frac{d\phi(s)}{ds} = \frac{1}{\beta(s)}$
- This leads to: $Q = \frac{1}{2\pi}\int_{o}^{s} \frac{ds}{\beta(s)}$

Over one complete turn

- Increasing the focusing strength decreases the size of the beam envelope (*β*) and increases Q and vice versa.

Tune corrections

- What happens if we change the focusing strength slightly?
- The Twiss matrix for our 'FODO' cell is given by:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- Add a small QF quadrupole, with strength dK and length ds.
- This will modify the 'FODO' lattice, and add a horizontal focusing term:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix} \qquad dk = \frac{dK}{(B\rho)} \qquad f = \frac{(B\rho)}{dKds}$$

- The new Twiss matrix representing the modified lattice is:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix}\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Tune corrections (2)

- This gives $\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dkds\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$

- This extra quadrupole will modify the phase advance μ for the FODO cell.

New phase advance — $\mu_1 = \mu + d\mu$ — Change in phase advance

- If $d\mu$ is small then we can ignore changes in β

- So the new Twiss matrix is just:

$$\begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix}$$

Tune corrections (3)

- ✓ These two matrices represent the same FODO cell therefore:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dkds\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- ✓ Which equals:

$$\begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix}$$

- ✓ Combining and compare the first and the fourth terms of these two matrices gives:

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

Only valid for change in β <<

Tune corrections (4)

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

Remember $\mu_1 = \mu + d\mu$ and $d\mu$ is small

$$2\cos\mu - 2\sin\mu\, d\mu$$

$$2\sin\mu\, d\mu = dk\, ds\,\beta\sin\mu$$

$$d\mu = \frac{1}{2} dk\, ds\,\beta$$

,but: $dQ = d\mu/2\pi$

In the **horizontal plane** this is a **QF**

$$dQh = +\frac{1}{4\pi} dk.ds.\beta h$$

If we follow the same reasoning for both transverse planes for both **QF** and **QD** quadrupoles

QD | **QF**

$$dQv = +\frac{1}{4\pi}\beta v.dk_D.ds_D - \frac{1}{4\pi}\beta v.dk_F.ds_F$$

$$dQh = -\frac{1}{4\pi}\beta h.dk_D.ds_D + \frac{1}{4\pi}\beta h.dk_F.ds_F$$

Tune corrections (5)

Let $\mathbf{dk_F} = \mathbf{dk}$ for **QF** and $\mathbf{dk_D} = \mathbf{dk}$ for **QD**

β_{hF}, β_{vF} = β at **QF** and β_{hD}, β_{vD} = β at **QD**

Then:

$$\begin{pmatrix} dQv \\ dQh \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi}\beta_{vD} & \frac{-1}{4\pi}\beta_{vF} \\ \frac{-1}{4\pi}\beta_{hD} & \frac{1}{4\pi}\beta_{hF} \end{pmatrix} \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

This matrix relates the change in the tune to the change in strength of the quadrupoles.
We can invert this matrix to calculate change in quadrupole field needed for a given change in tune

Dispersion (1)

- Until now we have assumed that our beam has no energy or momentum spread:

$$\frac{\Delta E}{E}=0 \quad \text{and} \quad \frac{\Delta p}{p}=0$$

- Different energy or momentum particles have different radii of curvature (ρ) in the main dipoles.
- These particles no longer pass through the quadrupoles at the same radial position.
- Quadrupoles act as dipoles for different momentum particles.
- Closed orbits for different momentum particles are different.
- This horizontal displacement is expressed as the dispersion function D(s)
- D(s) is a function of 's' exactly as β(s) is a function of 's'

Dispersion (2)

- ✓ The displacement due to the change in momentum at any position (s) is given by:

$$\Delta x(s) = D(s).\frac{\Delta p}{p}$$

Local radial displacement due to momentum spread

Dispersion function

- ✓ D(s) the dispersion function, is calculated from the lattice, and has the unit of meters.
- ✓ The beam will have a finite horizontal size due to it's momentum spread.
- ✓ In the majority of the cases we have no vertical dipoles, and so D(s)=0 in the vertical plane.

Momentum compaction factor

- ✓ The **change in orbit** with the **changing momentum** means that the average length of the orbit will also depend on the beam momentum.
- ✓ This is expressed as the **momentum compaction factor**, α_p, where:

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

- ✓ α_p tells us about the change in the length of radius of the closed orbit for a change in momentum.

Chromaticity

- ✓ The focusing strength of our quadrupoles depends on the beam momentum, 'p'

$$k = \frac{dBy}{dx} \times \frac{1}{B\rho}$$

$$3.3356\,p$$

- ✓ Therefore a spread in momentum causes a spread in focusing strength

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p}$$

- ✓ But Q depends on the 'k' of the quadrupoles

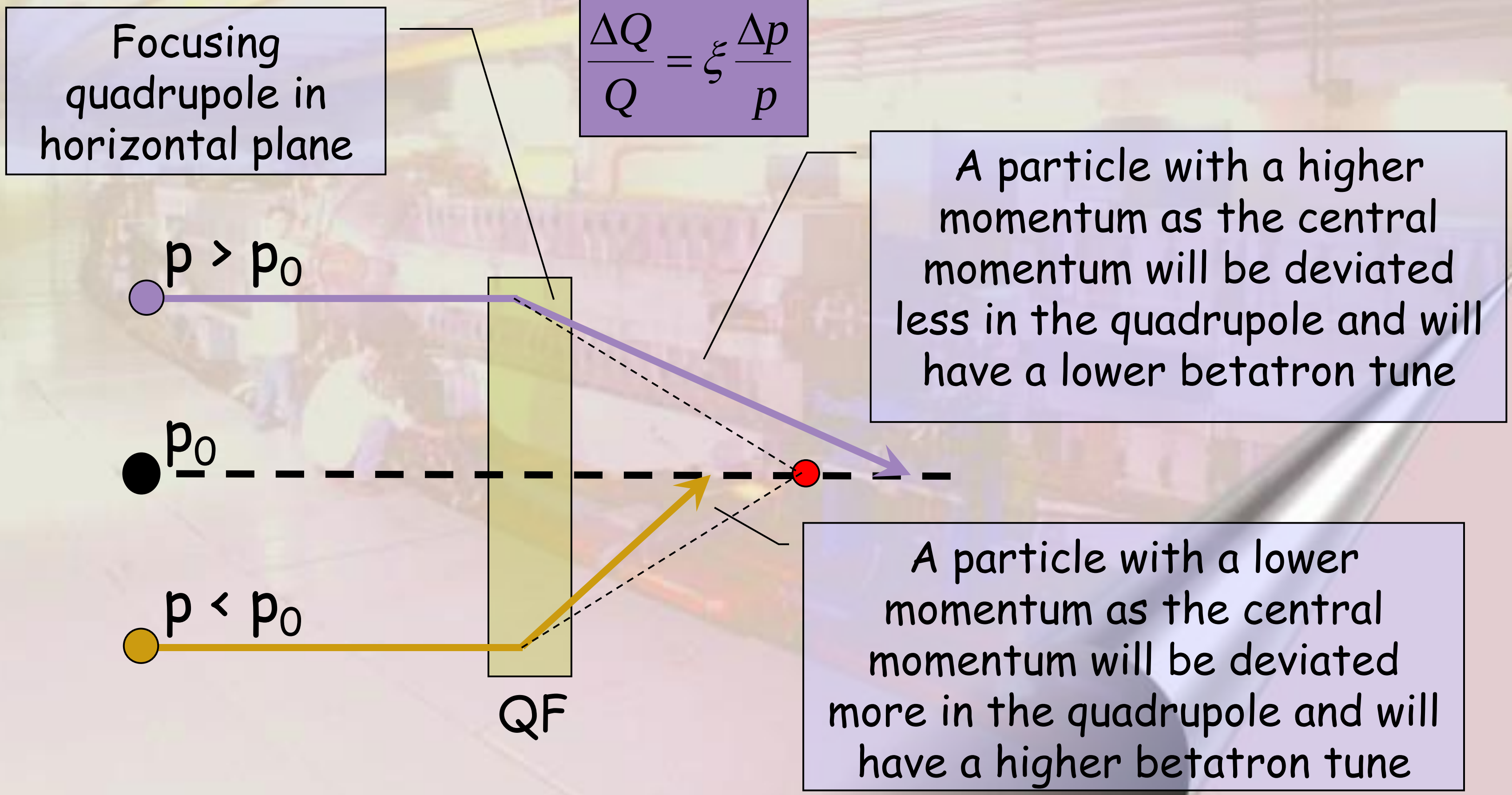
$$\frac{\Delta Q}{Q} \alpha \frac{\Delta p}{p} \longrightarrow \frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

- ✓ The constant here is called : **Chromaticity**

Chromaticity visualized

- ✓ The chromaticity relates the tune spread of the transverse motion with the momentum spread in the beam.

$$\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$



Chromaticity calculated

- ✓ Remember $\Delta Q = \frac{1}{4\pi}(\beta dk ds)$ and $\frac{\Delta k}{k} = -\frac{\Delta p}{p}$ ⇒ $\Delta k = -k\frac{\Delta p}{p}$

The gradient seen by the particle depends on its momentum

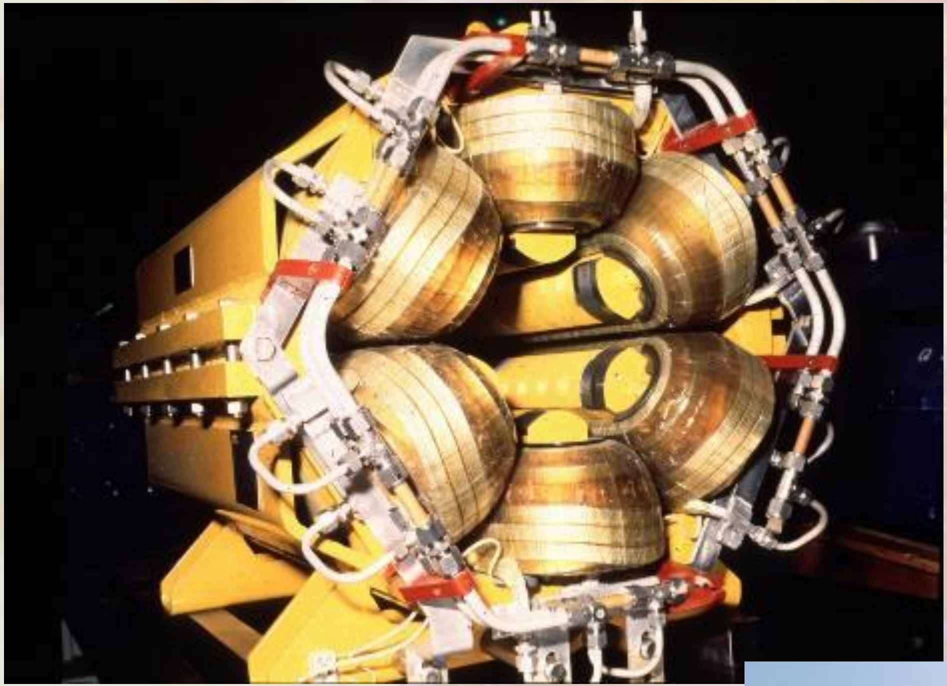
- ✓ Therefore $$\frac{\Delta Q}{Q} = -\frac{1}{4\pi}\left(\beta\frac{k}{Q}ds\right)\frac{\Delta p}{p}$$

- ✓ This term is the **Chromaticity ξ**

- ✓ To correct this tune spread we need to increase the quadrupole focusing strength for higher momentum particles, and decrease it for lower momentum particles.

- ✓ This we will obtain using a **Sextupole** magnet

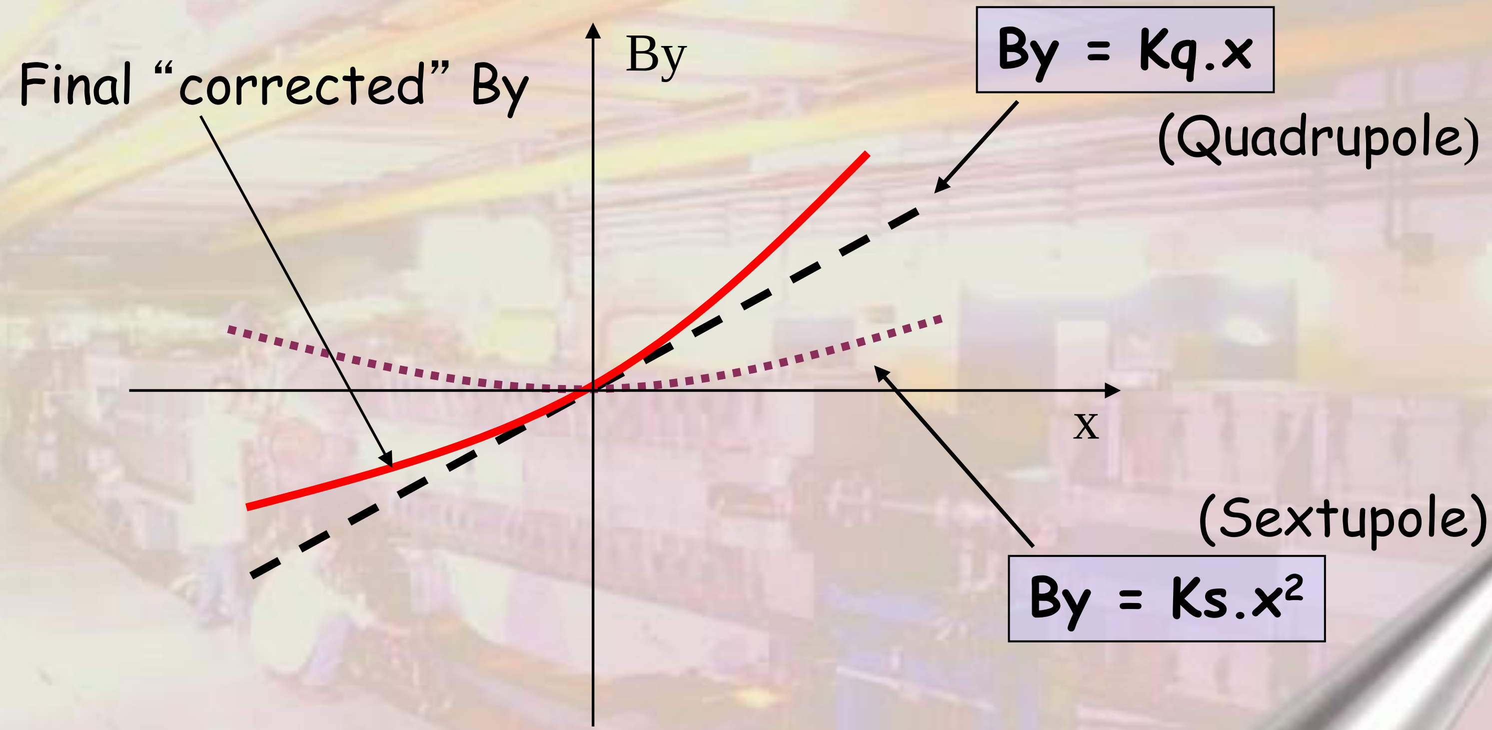
Sextupole Magnets



- ✓ Conventional Sextupole from LEP, but looks similar for other 'warm' machines.
- ✓ ~ 1 meter long and a few hundreds of kg.

- ✓ Correction Sextupole of the LHC
- ✓ 11cm, 10 kg, 500A at 2K for a field of 1630 T/m^2

Chromaticity correction



✓ Vertical magnetic field versus horizontal displacement in a quadrupole and a sextupole.

Chromaticity correction (2)

- The effect of the sextupole field is to increase the magnetic field of the quadrupoles for the positive 'x' particles and decrease the field for the negative 'x' particles.
- However, the dispersion function, D(s), describes how the radial position of the particles change with momentum.
- Therefore the sextupoles will alter the focusing field seen by the particles as a function of their momentum.
- This we can use to compensate the natural chromaticity of the machine.

Sextupole & Chromaticity

- In a sextupole for **y = 0** we have a field **By = C.x²**
- Now calculate 'k' the focusing gradient as we did for a quadrupole:

$$k = \frac{1}{(B\rho)}\frac{dB_y}{dx}$$

- Using $B_y = Cx^2$ which after differentiating gives $\frac{dB_y}{dx} = 2Cx$
- For k we now write $k = \frac{1}{(B\rho)}2Cx$
- We conclude that 'k' is no longer constant, as it depends on 'x'
- So for a Δx we get $\Delta k = \frac{2C}{(B\rho)}\Delta x$ and we know that $\Delta x = D(s)\frac{\Delta p}{p}$
- Therefore

$$\Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$$

Sextupole & Chromaticity

- We know that the tune changes with : $\Delta Q = \frac{1}{4\pi}\beta(s)dkds$
- Where: $ds = \text{sextupole length}$ and $dk = \Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$
- Remember $B = C \cdot x^2$ with $C = \frac{1}{2}\frac{d^2By}{dx^2}$
- The effect of a sextupole with length l on the particle tune Q as a function of Δp/p is given by:

$$\frac{\Delta Q}{Q} = \frac{1}{4\pi}\ell\beta(s)\frac{d^2By}{dx^2}\frac{D(s)}{(B\rho)Q}\frac{\Delta p}{p}$$

- If we can make this term exactly balance the natural chromaticity then we will have solved our problem.

Sextupole & Chromaticity (2)

- There are two chromaticities:
  - horizontal → ξ_h
  - vertical → ξ_v
- However, the effect of a sextupole depends on $\beta(s)$, which varies around the machine
- Two types of sextupoles are used to correct the chromaticity.
  - One (SF) is placed near QF quadrupoles where β_h is large and β_v is small, this will have a large effect on ξ_h
  - Another (SD) placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v
- Also sextupoles should be placed where D(s) is large, in order to increase their effect, since Δk is proportional to D(s)

Questions….,Remarks…?