MWAPP meeting

Unbinned quasi-model-independent approach to γ measurement

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Paper

These slides are based on the paper I co-authored with Dr. Evelina Gersabeck (Manchester) and Prof Jonas Rademacker (Bristol), on arXiv in May (published in JHEP in September)

https://arxiv.org/abs/2305.10787

$D^0\bar{D}^0$ interference in γ measurements

Decay amplitude for $D^0(\bar{D}^0) \to f(\mathbf{p}) = A_D^f(\bar{A}_D^f)(\mathbf{p})$, $\mathbf{p}$ is the phase-space for the final state f, decay rate for $B^- \to D(\to f(\mathbf{p}))K^-$, $D = (D^0, \bar{D}^0)$:

$$\Gamma^-(\mathbf{p}) \equiv \Gamma(B^- \to D(\to f(\mathbf{p}))K^-)$$

$$\begin{aligned}\Gamma^-(\mathbf{p}) &= \left|A_D^f(\mathbf{p}) + r_B e^{i(\delta_B - \gamma)}\bar{A}_D^f(\mathbf{p})\right|^2 \\ &= \left|A_D^f(\mathbf{p})\right|^2 + r_B^2\left|\bar{A}_D^f(\mathbf{p})\right|^2 \\ &\quad + 2r_B\left|A_D^f(\mathbf{p})\bar{A}_D^f(\mathbf{p})\right|\cos\left(\delta_D^f(\mathbf{p}) + \delta_B - \gamma\right)\end{aligned}$$

$$\delta_D^f(\mathbf{p}) \equiv \mathrm{Arg}\left(A_D^f(\mathbf{p})\right) - \mathrm{Arg}\left(\bar{A}_D^f(\mathbf{p})\right)$$

Relative phase/notation

We are interested in $f(\mathbf{p}) = K_S^0\pi^+\pi^-(s_+, s_-)$

$$A_D \equiv A_D^{K_S^0\pi^+\pi^-}(s_+, s_-)$$

$$\bar{A}_D^{K_S^0\pi^+\pi^-}(s_+, s_-) = A_D(s_-, s_+) \equiv \bar{A}_D$$

$$\delta_D \equiv \delta_D^{K_S^0\pi^+\pi^-}(s_+, s_-)$$

With CP conservation, $\delta_D =$ relative strong phase:

$$\bar{\delta}_D \equiv \delta_D^{K_S^0\pi^+\pi^-}(s_-, s_+) = -\delta_D$$

$$s_\pm = \left(E_{K_S^0} + E_{\pi^\pm}\right)^2 - \left|\mathbf{p}_{K_S^0} + \mathbf{p}_{\pi^\pm}\right|^2$$

Back to $B^\pm \to DK^\pm$

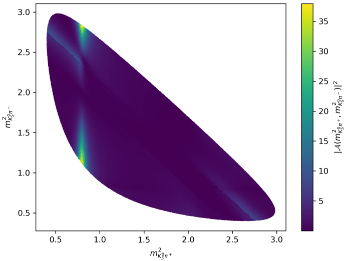
$$\Gamma^\pm(s_+, s_-) = |A_D(s_\pm, s_\mp)|^2 + (x_\pm^2 + y_\pm^2)|A_D(s_\mp, s_\pm)|^2 + 2|A_D(s_\pm, s_\mp)||A_D(s_\mp, s_\pm)| (x_\pm \cos\delta_D + y_\pm \sin\delta_D)$$

$$x_\pm + iy_\pm \equiv r_B \cos(\delta_B \pm \gamma) + ir_B \sin(\delta_B \pm \gamma)$$

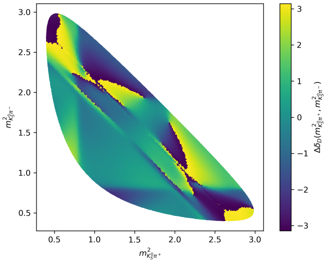
Can obtain $x_\pm, y_\pm$ by maximising the associated likelihood for $\Gamma^\pm(s_+, s_-)$ from $B^\pm \to D(\to K_S^0\pi^+\pi^-)K^\pm$, this is the Model Dependent (MD) method.

Models for $D^0 \rightarrow K^0_S \pi^+ \pi^-$

Most recent published model was from a combined Belle and BaBar analysis https://arxiv.org/abs/1804.06153



(a) $|A_D|$



(b) δ_D

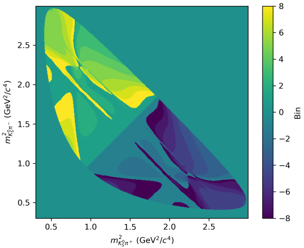
Pros/cons of models

Pros	Cons
$\lvert A_D \rvert$ very well constrained	Models do not produce reliable results for δ_D
Optimal statistical precision in γ	Mismodelling δ_D leads to large systematic uncertainties in γ

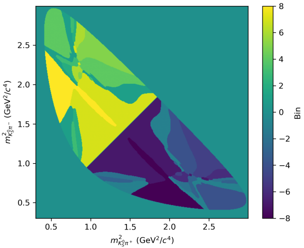
Can avoid model systematics by moving to a model independent measurement of δ_D and γ

Model independent measurement of δ_D

We split the $K_S^0\pi^+\pi^-$ phase-space into bins (see chapter 6 : https://inspirehep.net/literature/1669241)



(a) Equal binning scheme



(b) Optimal binning scheme

Model independent measurement of δ_D

Define the associated binned parameters

$$F_i = \int\int_i ds_+ ds_- |A_D|^2, F_{-i} = \int\int_i ds_+ ds_- |\bar{A}_D|^2$$

$$c_i = \frac{1}{\sqrt{F_i F_{-i}}} \int\int_i ds_+ ds_- |A_D||\bar{A}_D| \cos\delta_D$$

$$s_i = \frac{1}{\sqrt{F_i F_{-i}}} \int\int_i ds_+ ds_- |A_D|\bar{A}_D| \sin\delta_D$$

Charm threshold experiments

Colliding e^+e^- at $\sqrt{s} = 3.773\ \mathrm{GeV}$ (the $\psi(3770)$ $c\bar{c}$ resonance) just above the $D^0\bar{D}^0$ threshold ($2m_D = 3.730\ \mathrm{GeV}$), one produces quantum correlated $D^0\bar{D}^0$ pairs from $\psi(3770) \to D^0\bar{D}^0 \to f(\mathbf{p})g(\mathbf{q})$ decays.

$$\Gamma_{\psi(3770)}(\mathbf{p},\mathbf{q}) \propto \frac{1}{2}\left|A_D^f(\mathbf{p})\bar{A}_D^g(\mathbf{q}) - \bar{A}_D^f(\mathbf{p})A_D^g(\mathbf{q})\right|^2$$

BESIII has the largest sample of charm threshold data (2011 : $\int \mathcal{L}dt = 2.93\ \mathrm{fb}^{-1}$) https://arxiv.org/abs/2002.12791 , now has (2023 : $12\ \mathrm{fb}^{-1}$ aims to get $20\ \mathrm{fb}^{-1}$ by 2024)

Tagged $D \to K_S^0\pi^+\pi^-$ decays

In our case, we already have $f(\mathbf{p}) = K_S^0\pi^+\pi^-(s_+, s_-)$, we then choose different 'tag' decays for $g(\mathbf{q})$.

Tag type	Decay	$(A_D^g(\mathbf{q}), \bar{A}_D^g(\mathbf{q}))$
CP even	$D \to K^+K^-$	$(1, 1)$
CP odd	$D \to K_S^0\pi^0$	$(1, -1)$
D^0 flavour	$D \to K^-\pi^+$	$(1, 0)$
$\bar{D}^0$ flavour	$D \to K^+\pi^-$	$(0, 1)$
Double tag	$D \to K_S^0\pi^+\pi^-$	$(A_D(s_+, s_-), A_D(s_-, s_+))$

MI measurement of δ_D at charm threshold

Tag	$\langle N_i \rangle \propto$
CP Even	$F_i + F_{-i} - 2\sqrt{F_i F_{-i}} c_i$
CP Odd	$F_i + F_{-i} + 2\sqrt{F_i F_{-i}} c_i$
D^0 flavour	F_i
$\bar{D}^0$ flavour	F_{-i}
Double tag (i, j)	$F_i F_{-j} + F_{-i} F_j - 2\sqrt{F_i F_j F_{-i} F_{-j}}(c_i c_j + s_i s_j)$

Extract c_i and s_i from fitting a Poisson distribution of $f(N_i | \langle N_i \rangle)$

Back (again) to $B^{\pm} \to DK^{\pm}$

Replace the decay probabiltiy $\Gamma^{\pm}(s_+, s_-)$ with binned version, expected yield per bin i from $B^{\pm} \to D(\to K_S^0\pi^+\pi^-)K^{\pm}$ decays

$$N_i^{\pm} \propto F_{\pm i} + (x_{\pm}^2 + y_{\pm}^2)F_{\mp i} \\ + 2\sqrt{F_{\pm i}F_{\mp i}}\,(c_i x_{\pm} + s_i y_{\pm})$$

Again fit with a Poisson distribution - we call this the binned Model Independent (MI) method

Pros/cons of binning

Pros	Cons
No model dependence - zero systematic uncertainty from model	Acquires a new systematic from inputting c_i, s_i - dependent on statistical uncertainty of c_i, s_i
Can optimise the binning scheme to minimise statistical uncertainty on γ	Loss of precision is inherent to binning $K_S^0\pi^+\pi^-$ - will show that σ_γ increases by approximately 20%

Motivation for QMI method

We propose a different approach to both MD and binned method, an unbinned (quasi-) model independent method aka the QMI method. Basic assumptions:

- Models for A_D constrain $|A_D|$ very well but not δ_D - treat $|A_D|$ as the 'true' magnitude
- No CP violation in $D \to K_S^0\pi^+\pi^-$ decays
- The difference between the model version of δ_D ($\delta_D^{\mathrm{model}}$) and the 'true' δ_D can be closed with a polynomial

The QMI method

Core of the method - define a ‘correction’ term for δ_D from a model:

$$\delta_D = \delta_D^{\mathrm{model}} + \delta_D^{\mathrm{corr}}$$

Different $\delta_D^{\mathrm{model}}$ (choice of model) leads to different δ_D^{corr} (correction to the model), hence ‘quasi model independent’ - δ_D is unaffected by choice of model (as long as the correcting term is accurate)

Setting up δ_D^{corr}

Use a two-dimensional polynomial in (s_+, s_-), CP conservation implies

$$\delta_D^{\mathrm{corr}}(s_+, s_-) = -\delta_D^{\mathrm{corr}}(s_-, s_+)$$

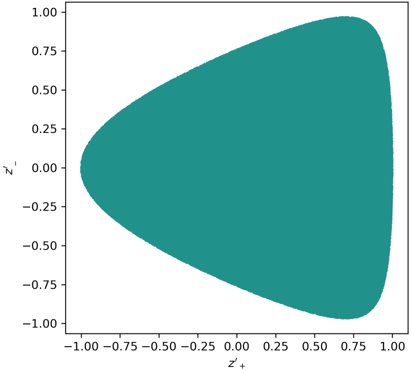
so we define $z_\pm$

$$z_\pm = s_+ \pm s_-$$

Setting up δ_D^{corr}

We rotate s_+, s_- into z'_+, z'_- such that $|z'_\pm| \leq 1$

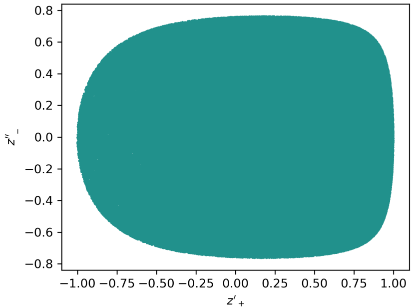
$$z'_\pm = \frac{2z_\pm - (z_\pm^{\max} + z_\pm^{\min})}{z_\pm^{\max} - z_\pm^{\min}}$$



Setting up δ_D^{corr}

We then stretch z'_- into z''_- to avoid varying δ_D^{corr} in regions where there is no data

$$z''_- = \frac{2z'_-}{z'_+ + 2}$$



The form of the polynomial

The correcting polynomial of order O is formed out of the free parameters, $C_{i,2j+1}$, defined by the sum

$$\delta_D^{\mathrm{corr}} = \sum_{i=0}^{i \leq O} \sum_{j=0}^{j \leq \frac{O-i-1}{2}} C_{i,2j+1} p_i(z'_+) p_{2j+1}(z''_-)$$

where $p_n(x)$ is a one-dimensional Legendre polynomial of order n. The odd ordered polynomials in z''_- ensure that $\delta_D^{\mathrm{corr}}(z'_+, -z''_-) = -\delta_D^{\mathrm{corr}}(z'_+, z''_-)$.

Simulation studies

We use a modified version of `AmpGen` to generate and fit to simulated (signal only) decays:

A_D, δ_D come from the Belle-BaBar 2018 model: https://arxiv.org/abs/1804.06153

BESIII (https://arxiv.org/abs/2002.12791)

$\psi(3770) \to D^0\bar{D}^0 \to K_S^0\pi^+\pi^-(s_+, s_-), g(\mathbf{q})$	
g	Sample size
K^+K^-	2546
$K_S^0\pi^0$	1725
$K^-\pi^+$	23457
$K^+\pi^-$	23457
$K_S^0\pi^+\pi^-$	1833

LHCb (https://arxiv.org/abs/2010.08483)

$B^\pm \to D(\to K_S^0\pi^+\pi^-)K^\pm$	
Sample size (each sign)	6267
r_B, δ_B	$0.093, 119.5^\circ$
γ	69.5°
$x_+ + iy_+$	$-0.092 - 0.015i$
$x_- + iy_-$	$0.060 + 0.071i$

Fitting with the QMI method

In our studies, we replace δ_D with $\delta_D^{\mathrm{model}} + \delta_D^{\mathrm{corr}}$. We then combine both simulation of $\psi(3770)$ and $B^\pm \to D(\to K_S^0\pi^+\pi^-)K^\pm$ decays into a single simulatenous fit, constraining $C_{i,2j+1}$ and $x_\pm, y_\pm$ simultaneously. Following results are from our paper: https://arxiv.org/abs/2305.10787

Precision comparison between methods

	σ_{r_B} $\times 10^2$	σ_{δ_B}	σ_γ
binned fit (fixed c_i, s_i)	0.879	5.33°	5.09°
unbinned QMI	0.664	4.24°	4.21°
unbinned MD	0.660	4.19°	4.23°

i.e. $\sigma^{\mathrm{stat}}_{\mathrm{MD}} \approx \sigma^{\mathrm{stat}}_{\mathrm{QMI}} < \sigma^{\mathrm{stat}}_{\mathrm{MI}}$

Pull studies

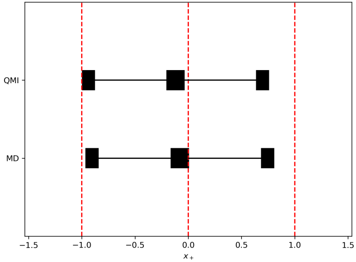
We generate $N = 100$ independent samples of both $\psi(3770) \to D^0\bar{D}^0$ and $B^\pm \to DK^\pm$ decays, we obtain $x_\pm, y_\pm$ using the MD, binned MI and QMI methods (show just MD and QMI). The pull of a fitted parameter $\pm$ its calculated uncertainty from the fit $x \pm \sigma_x$ relative to the value of x used to generate the sample, x_0 is

$$\mathrm{pull}(x) \equiv \frac{x - x_{\mathrm{gen}}}{\sigma_x}$$

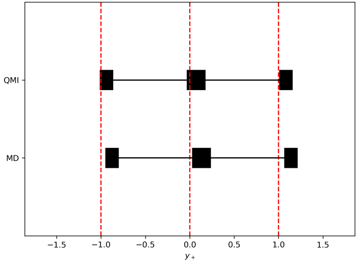
will ideally follow $\mathrm{pull}(x) \sim N(0, 1)$, a standard normal distribution.

Pull results (x_+, y_+) : self-consistency

The black bars centre $= \langle \text{pull} \rangle \pm \frac{s(\text{pull})}{\sqrt{N}}$ and edge $= \langle \text{pull} \pm s(\text{pull}) \rangle \pm \frac{s(\text{pull})}{\sqrt{2N}}$, red : ideal unbiased pull

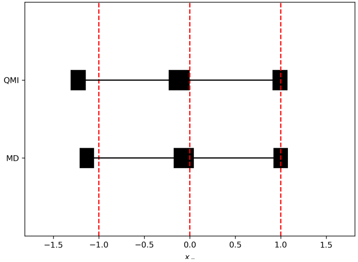


(a) x_+ pulls

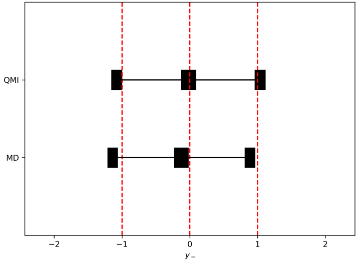


(b) y_+ pulls

Pull results (x_-, y_-) : self-consistency



(a) x_- pulls



(b) y_- pulls

Both unbiased and correct widths

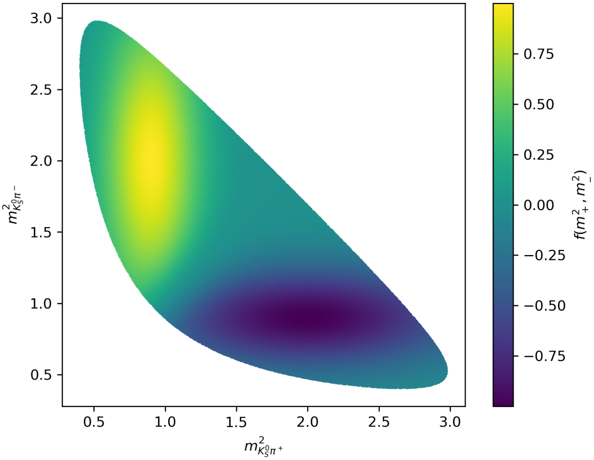
Adding a bias to δ_D

To test if our method can actually account for mismodelling δ_D, define a bias, $f_{\text{bias}}(s_+, s_-)$ and replace the δ_D at the generation stage with $\delta_D^{\text{model}} + f_{\text{bias}}(s_+, s_-)$ then repeat the pull studies

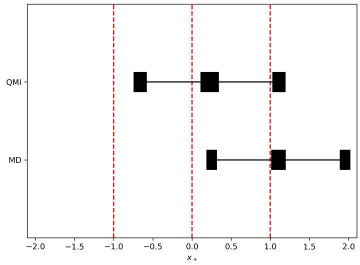
$$f_{\text{bias}} = A\,\text{erf}\left(\frac{s_- - s_+}{\varepsilon}\right) G(s_+, s_-)$$

where $\text{erf}(x) \equiv \int_\infty^x e^{-u^2} du$, $G(s_+, s_-)$ is a the product of two Gaussians with means μ_+, μ_- and widths σ_+, σ_-, $(s_+ < s_- : \mu_\pm \to \mu_\mp, \sigma_\pm \to \sigma_\mp)$

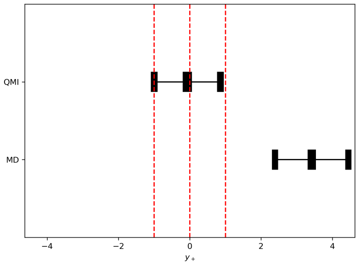
Bias to δ_D



Pull study results (x_+, y_+) : $\delta_D + \delta_D^{\text{bias}}$

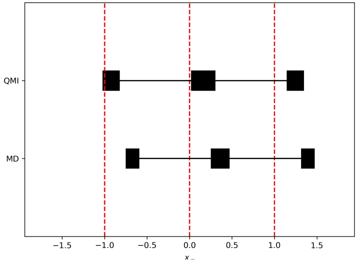


(a) x_+ pulls

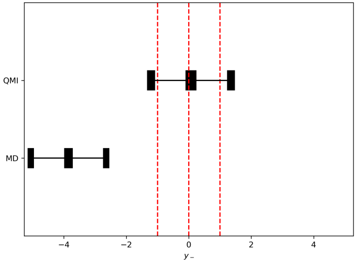


(b) y_+ pulls

Pull study results (x_-, y_-) : $\delta_D + \delta_D^{\text{bias}}$



(a) x_- pulls



(b) y_- pulls

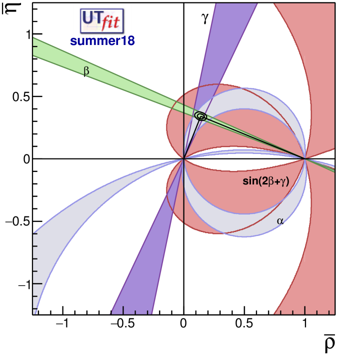
Conclusion

- The QMI method is self-consistent (from pull studies without a bias to δ_D)
- The QMI method has similar statistical uncertainty to the MD method (optimal statistical precision)
- The QMI method is able to recover a bias imposed to δ_D, avoiding mismodelling δ_D which biases CKM measurements massively

Backup slides

Unitarity triangle

From https://arxiv.org/abs/2212.03894

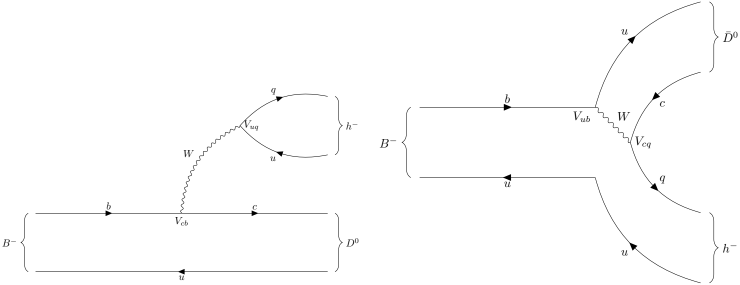


(a) Constraints on the Unitarity triangle from UTFit

Unitarity triangle

- Quark mixing matrix (CKM matrix) must be unitary in the SM : $V^\dagger V = 1$
- Construct 'unitarity triangle' in the complex plane
- $e^{i\alpha} = \frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}$, $e^{i\beta} = \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}$, $e^{i\gamma} = \frac{V_{cd}V_{cb}^*}{V_{ud}V_{ub}^*}$
- $\alpha + \beta + \gamma = 180^\circ$ for unitarity
- World average: $\left(\left(85.2^{+4.8}_{-4.3}\right)_\alpha + (22.2 \pm 0.7)_\beta + \left(65.9^{+3.3}_{-3.5}\right)_\gamma\right)^\circ = (173 \pm 6)^\circ$
- Want sub-degree precision of each angle

Tree level determination of γ



(a) $B^- \to D^0 h^-$ (b) $B^- \to \bar{D}^0 h^-$

$$\Gamma(B^- \to Dh^-) \propto \left| D^0 + r_B e^{i(\delta_B - \gamma)} \bar{D}^0 \right|^2$$

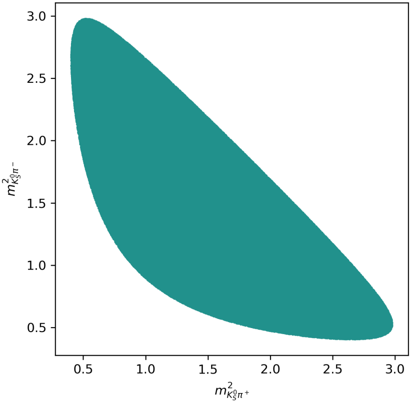
$$\Gamma(B^+ \to Dh^+) \propto \left| \bar{D}^0 + r_B e^{i(\delta_B + \gamma)} D^0 \right|^2$$

$D^0 \to K_S^0 \pi^+ \pi^-$ decays

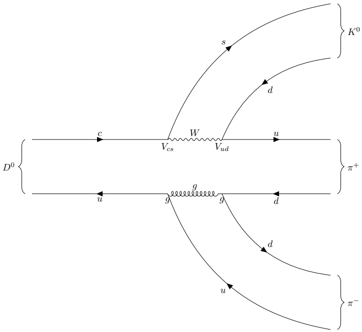
- Self conjugate decay mode, both D^0 and $\bar{D}^0$ decay to the final state, relatively large BF $= 2.80\%$
- Three body decay = two-dimensional phase-space ($N_{\mathrm{dim}} = 3N_{\mathrm{body}} - 7 = 3 \times 3 - 7 = 2$)

$$s_\pm = \left(E_{K_S^0} + E_{\pi^\pm}\right)^2 - \left|\mathbf{p}_{K_S^0} + \mathbf{p}_{\pi^\pm}\right|^2$$
$$\in ((m_{K_S^0} + m_{\pi^\pm})^2, (m_{D^0} - m_{\pi^\pm})^2)$$
$$\in (0.406, 2.977)\ \mathrm{GeV}^2$$

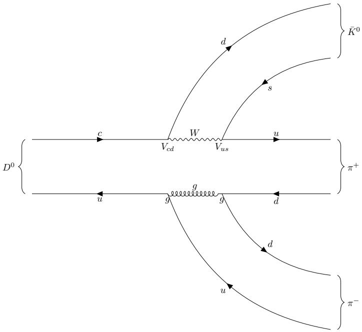
$D^0 \to K_S^0 \pi^+ \pi^-$ decays



$D^0 \to K_S^0\pi^+\pi^-$



(a) $D^0 \to K^0\pi^+\pi^-$



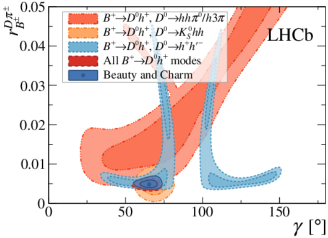
(b) $D^0 \to \bar{K}^0\pi^+\pi^-$

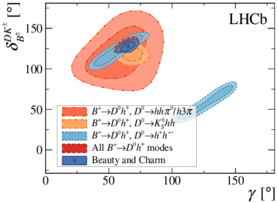
- $CP|K_S^0\pi^+\pi^-\rangle = |K_S^0\pi^-\pi^+\rangle$

Current precision on γ from $B^{\pm} \to D(\to K_S^0\pi^+\pi^-)K^{\pm}$

From https://cds.cern.ch/record/2743058?ln=en

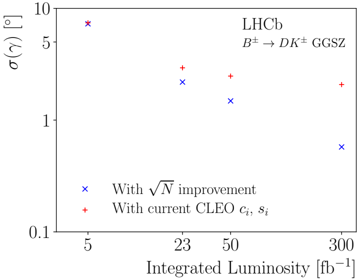
- Most precise measurement of γ in a single decay comes from $B^{\pm} \to D(\to K_S^0\pi^+\pi^-)K^{\pm}$ (GGSZ method)



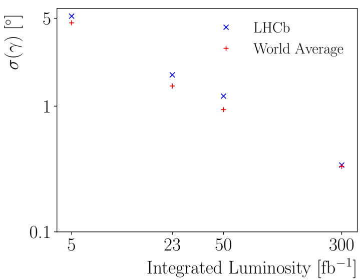


Projected precision on γ

From https://arxiv.org/abs/1808.08865



(a) Projection of γ with GGSZ method



(b) Projection of γ from all D decay modes

Measuring δ_D at charm threshold

For $f(\mathbf{p}) = K_S^0\pi^+\pi^-(s_+, s_-)$, we have the correlated probability

$$\begin{aligned}\Gamma_{\psi(3770)}(s_+, s_-, \mathbf{q}) = {} & |A_D|^2|\bar{A}_g(\mathbf{q})|^2 + |\bar{A}_D|^2|A_g(\mathbf{q})|^2 \\ & - 2|A_D||\bar{A}_D||A_g(\mathbf{q})||\bar{A}_g(\mathbf{q})| \\ & (\cos\delta_D \cos\delta_g(\mathbf{q}) + \sin\delta_D \sin\delta_g(\mathbf{q}))\end{aligned} \tag{1}$$

Measuring δ_D at charm threshold

Tag	$\Gamma_{\psi(3770)}(s_+, s_-, \mathbf{q})$
CP Even	$\lvert A_D\rvert^2 + \lvert\bar{A}_D\rvert^2 - 2\lvert A_D\rvert\lvert\bar{A}_D\rvert \cos\delta_D$
CP Odd	$\lvert A_D\rvert^2 + \lvert\bar{A}_D\rvert^2 + 2\lvert A_D\rvert\lvert\bar{A}_D\rvert \cos\delta_D$
D^0 flavour	$\lvert A_D\rvert^2$
$\bar{D}^0$ flavour	$\lvert\bar{A}_D\rvert^2$
Double tag	$\lvert A_D^1\rvert^2\lvert\bar{A}_D^2\rvert^2 + \lvert\bar{A}_D^1\rvert^2\lvert A_D^2\rvert^2 - 2\lvert A_D^1\rvert\lvert\bar{A}_D^1\rvert\lvert A_D^2\rvert\lvert\bar{A}_D^2\rvert \left(\cos\delta_D^1\cos\delta_D^2 + \sin\delta_D^1\sin\delta_D^2\right)$

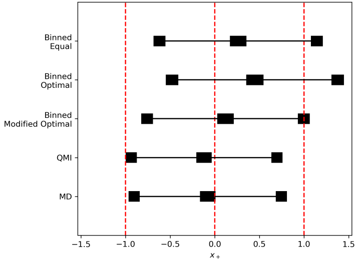
^{1,2} indicies refer to opposite $K_S^0\pi^+\pi^-$ states

Precision comparison between methods

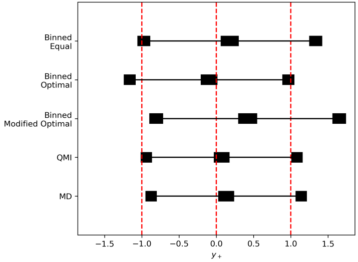
	σ_{x_+} $\times 10^2$	σ_{y_+} $\times 10^2$	σ_{x_-} $\times 10^2$	σ_{y_-} $\times 10^2$
binned fit (fixed c_i, s_i)	0.886	1.482	1.189	1.328
unbinned QMI	0.780	1.091	0.877	0.945
unbinned MD	0.784	1.081	0.878	0.939

Pull results (x_+, y_+) : self-consistency

The black bars centre $= \langle \mathrm{pull} \rangle \pm \frac{s(\mathrm{pull})}{\sqrt{N}}$ and edge $= \langle \mathrm{pull} \pm s(\mathrm{pull}) \rangle \pm \frac{s(\mathrm{pull})}{\sqrt{2N}}$, red : ideal unbiased pull

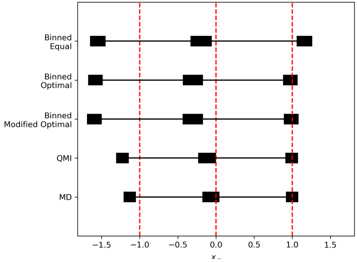


(a) x_+ pulls

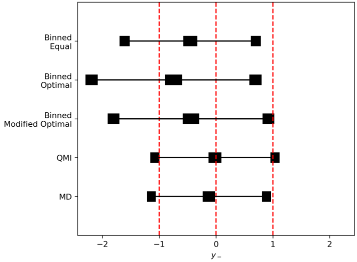


(b) y_+ pulls

Pull results (x_-, y_-) : self-consistency



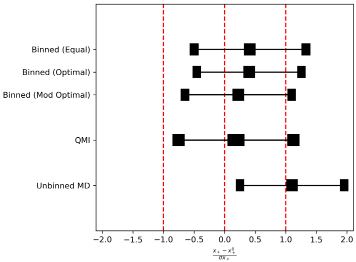
(a) x_- pulls



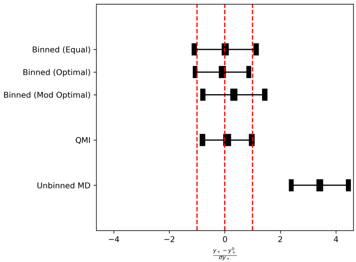
(b) y_- pulls

Both unbiased and correct widths

Pull study results (x_+, y_+) : $\delta_D + \delta_D^{\mathrm{bias}}$

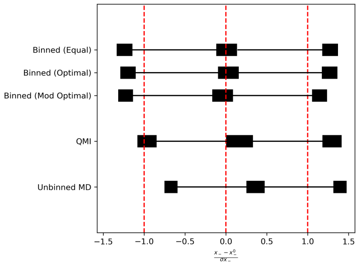


(a) x_+ pulls

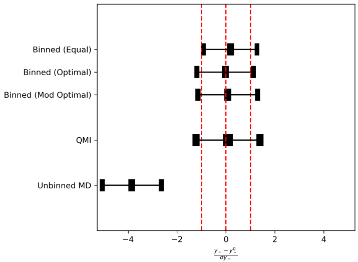


(b) y_+ pulls

Pull study results (x_-, y_-) : $\delta_D + \delta_D^{\mathrm{bias}}$



(a) x_- pulls



(b) y_- pulls

Other considerations

- What impact does the order of the correcting polynomial have?
- How will this method work in the presence of background?
- What is the optimal order for a correcting polynomial?
- How will increasing $\psi(3770)$ and $B^{\pm} \to DK^{\pm}$ sample sizes change the outcome of these tests? (BESIII and LHCb will have $O(10)$ and $O(100)$ times more data repsectively (at different time scales)

Order by order - single fit

Order	$\Delta x_+ \cdot 100$	$\Delta y_+ \cdot 100$	$\Delta x_- \cdot 100$	$\Delta y_- \cdot 100$
MD	-0.9 ± 0.8	-1.1 ± 1.1	-1.5 ± 0.9	$+1.0 \pm 0.9$
1	-0.9 ± 0.8	-1.0 ± 1.1	-1.5 ± 0.9	$+0.9 \pm 0.9$
2	-0.9 ± 0.8	-1.0 ± 1.1	-1.5 ± 0.9	$+1.0 \pm 0.9$
3	-0.9 ± 0.8	-1.2 ± 1.1	-1.5 ± 0.9	$+1.1 \pm 0.9$
4	-0.8 ± 0.8	-1.1 ± 1.1	-1.6 ± 0.9	$+1.2 \pm 0.9$
5	-0.9 ± 0.8	-1.1 ± 1.1	-1.6 ± 0.9	$+1.2 \pm 0.9$
6	-0.9 ± 0.8	-1.1 ± 1.2	-1.5 ± 0.9	$+1.1 \pm 0.9$
7	-0.9 ± 0.8	-1.1 ± 1.2	-1.5 ± 0.9	$+1.1 \pm 0.9$
8	-0.8 ± 0.8	-1.3 ± 1.2	-1.6 ± 0.9	$+1.3 \pm 0.9$
9	-0.8 ± 0.8	-1.4 ± 1.2	-1.6 ± 0.9	$+1.3 \pm 0.9$

Ideal order - unbiased δ_D

Order	$\frac{\langle\chi^2_{\psi(3770)}\rangle}{n^{\mathrm{dof}}_{\psi(3770)}}$	$\frac{\langle\chi^2_{B\pm}\rangle}{n^{\mathrm{dof}}_{B\pm}}$	$(x_+ \pm \sigma_{x_+})$ $\times 100$	$(y_+ \pm \sigma_{y_+})$ $\times 100$	$(x_- \pm \sigma_{x_-})$ $\times 100$	$(y_- \pm \sigma_{y_-})$ $\times 100$
MD	212.9/241	498.7/502	-10.3 ± 0.8	-2.5 ± 1.1	4.4 ± 0.9	7.9 ± 0.9
1	213.2/238	498.8/500	-10.3 ± 0.8	-2.4 ± 1.1	4.4 ± 0.9	7.8 ± 0.9
2	212.7/235	499.0/498	-10.3 ± 0.8	-2.4 ± 1.1	4.4 ± 0.9	7.9 ± 0.9
3	212.2/229	498.7/494	-10.3 ± 0.8	-2.6 ± 1.1	4.4 ± 0.9	8.0 ± 0.9
4	211.7/223	499.8/490	-10.2 ± 0.8	-2.5 ± 1.1	4.3 ± 0.9	8.1 ± 0.9
5	212.0/214	499.9/484	-10.3 ± 0.8	-2.5 ± 1.1	4.3 ± 0.9	8.1 ± 0.9
6	212.2/205	499.4/478	-10.3 ± 0.8	-2.5 ± 1.2	4.4 ± 0.9	8.0 ± 0.9
7	210.3/193	498.9/470	-10.3 ± 0.8	-2.5 ± 1.2	4.4 ± 0.9	8.0 ± 0.9
8	210.4/181	498.5/462	-10.2 ± 0.8	-2.7 ± 1.2	4.3 ± 0.9	8.2 ± 0.9
9	210.0/166	498.8/452	-10.2 ± 0.8	-2.8 ± 1.2	4.3 ± 0.9	8.2 ± 0.9

Ideal order - biased δ_D

Order	$\frac{\langle \chi^2_{\psi(3770)} \rangle}{n^{\mathrm{dof}}_{\psi(3770)}}$	$\frac{\langle \chi^2_{B^\pm} \rangle}{n^{\mathrm{dof}}_{B^\pm}}$	$(x_+ \pm \sigma_{x_+})$ $\times 100$	$(y_+ \pm \sigma_{y_+})$ $\times 100$	$(x_- \pm \sigma_{x_-})$ $\times 100$	$(y_- \pm \sigma_{y_-})$ $\times 100$
MD	394.6/241	562.1/502	-7.6 ± 0.8	2.6 ± 1.1	6.9 ± 0.9	3.2 ± 0.9
1	279.5/238	528.8/500	-8.8 ± 0.8	-0.4 ± 1.0	7.3 ± 0.9	6.1 ± 0.9
2	276.2/235	529.6/498	-8.7 ± 0.8	-0.5 ± 1.0	7.3 ± 0.9	6.2 ± 0.9
3	246.8/229	512.4/494	-9.1 ± 0.8	-1.1 ± 1.1	7.0 ± 0.9	7.4 ± 1.0
4	242.0/223	510.8/490	-9.1 ± 0.8	-1.2 ± 1.0	7.0 ± 0.9	7.6 ± 1.0
5	237.9/214	508.9/484	-9.2 ± 0.8	-1.3 ± 1.1	7.0 ± 0.9	7.8 ± 1.0
6	236.0/205	510.7/478	-9.2 ± 0.8	-1.5 ± 1.0	6.9 ± 0.9	7.8 ± 1.0
7	238.9/193	509.4/470	-9.2 ± 0.8	-1.6 ± 1.0	6.9 ± 0.9	7.9 ± 1.0
8	237.7/181	508.4/462	-9.2 ± 0.8	-1.7 ± 1.0	7.0 ± 0.9	7.8 ± 1.0
9	239.0/166	509.5/452	-9.2 ± 0.8	-1.6 ± 1.0	7.0 ± 0.9	7.8 ± 1.0

Optimal order seems to be $O = 6$

Alternate sample sizes - MD and MI

LHCb	$\sigma_{x_+} \cdot 10^2$		$\sigma_{y_+} \cdot 10^2$		$\sigma_{x_-} \cdot 10^2$		$\sigma_{y_-} \cdot 10^2$		σ_γ ($^\circ$)	
Lumi	MD	bin	MD	bin	MD	bin	MD	bin	MD	bin
$\times 1$	0.780	0.886	1.081	1.482	0.878	1.189	0.939	1.328	4.23	5.09
$\times 100$	0.078	0.089	0.108	0.149	0.088	0.118	0.093	0.134	0.42	0.52

Alternative sample sizes - QMI

Lumi scenario: LHCb	BESIII	$\sigma_{x_+} \cdot 10^2$	$\sigma_{y_+} \cdot 10^2$	$\sigma_{x_-} \cdot 10^2$	$\sigma_{y_-} \cdot 10^2$	σ_γ (°)
$\times 1$	$\times 1$	0.780	1.091	0.877	0.945	4.21
$\times 1$	$\times 10$	0.773	1.062	0.866	0.924	4.18
$\times 100$	$\times 1$	0.079	0.122	0.090	0.104	0.45
$\times 100$	$\times 10$	0.078	0.115	0.089	0.099	0.43