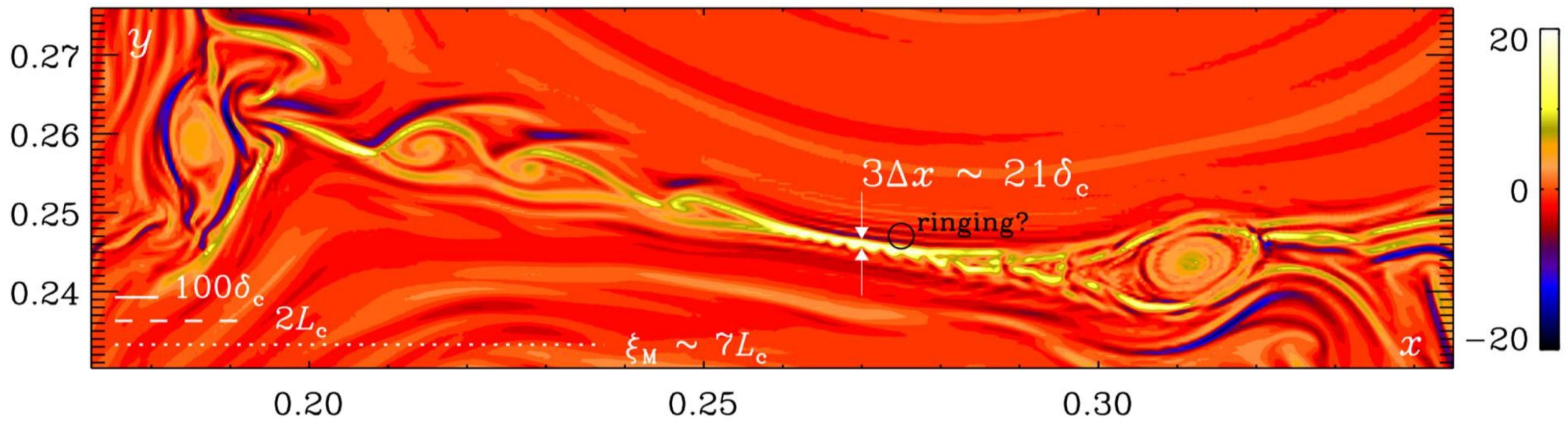
# Numerical simulations from reheating to turbulent MHD

Axel Brandenburg (Nordita)

- Possibility of magnetogenesis
  - Inflationary + reheating
    - Ratra model
    - Axion, axion-like
  - Electroweak (EW) + QCD
    - Twisted EW dumbbell decay
    - Chiral magnetic effect

- Early universe turbulence
  - From JxB
  - Other sources
  - → Gravitational waves
- Then "just" decay
  - Magnetically dominated
  - Helical
  - nonhelical

→ begin with this



# Primordial magnetic fields in the early 1990s

- Magnetic fields from phase transitions
  - Vachaspati (1991)
  - Cheng & Olinto (1994)
  - Baym, Bödeker, & McLerran (1996)
- Magnetic fields a “stable”feature of electroweak phase transitions
  - Martin & Davis (1995)
- Early universe plasma has high conductivity
  - Enqvist & Olesen (1994)
- Magnetic field would be imprinted on comoving plasma
  - Cheng, Schramm, Turan (1994)
  - Enqvist, Rez, & Semikoz (1995)
- Typical length scales: at most 3 cm
  - Too small to be of astrophysical interest

# Large-scale magnetic fields from hydromagnetic turbulence in the very early universe

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(Received 1 February 1996)

We investigate hydromagnetic turbulence of primordial magnetic fields using magnetohydrodynamics (MHD) in an expanding universe. We present the basic, covariant MHD equations, find solutions for MHD waves in the early universe, and investigate the equations numerically for random magnetic fields in two spatial dimensions. We find the formation of magnetic structures at larger and larger scales as time goes on. In three dimensions we use a cascade (shell) model that has been rather successful in the study of certain aspects of hydrodynamic turbulence. Using such a model we find that after $\sim 10^9$ times the initial time the scale of the magnetic field fluctuation (in the comoving frame) has increased by 4–5 orders of magnitude as a consequence of an inverse cascade effect (i.e., transfer of energy from smaller to larger scales). Thus *at large scales* primordial magnetic fields are considerably stronger than expected from considerations which do not take into account the effects of MHD turbulence. [S0556-2821(96)02712-9]

# Inverse cascade since the 1970s (driven turbulence)

*J. Fluid Mech.* (1975), *vol.* 68, *part* 4, *pp.* 769–778 769
*Printed in Great Britain*

**Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence**

By U. FRISCH, A. POUQUET,
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J. LÉORAT AND A. MAZURE
Université Paris VII, Observatoire de Meudon, France

*J. Fluid Mech.* (1976), *vol.* 77, *part* 2, *pp.* 321–354 321
*Printed in Great Britain*

**Strong MHD helical turbulence and the nonlinear dynamo effect**

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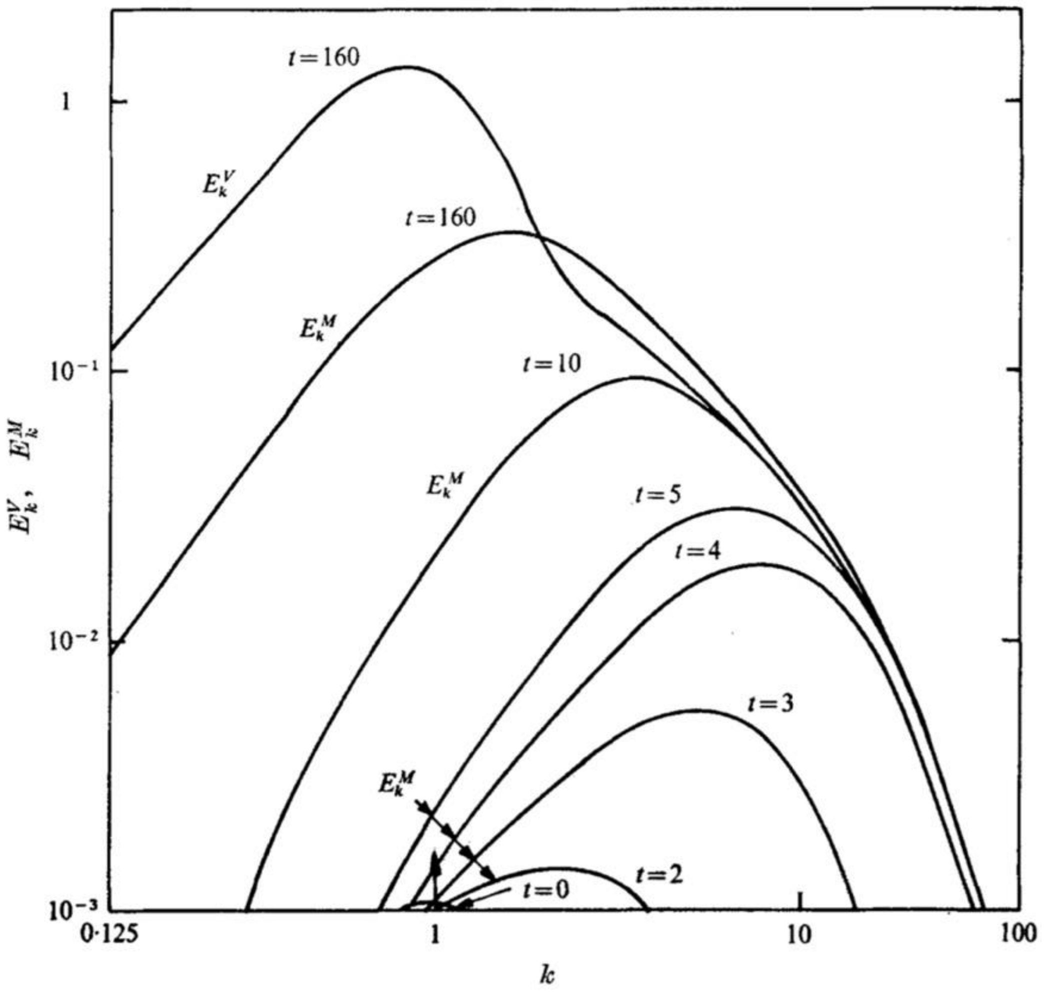
AND J. LÉORAT
Université Paris VII, Observatoire de Meudon, France



# Relativistic equations in expanding Universe

Energy momentum tensor

Brandenburg, Enqvist, Olesen
Phys Rev D 54, 1291 (1996)

$$T^{\mu\nu} = (p+\rho)U^{\mu}U^{\nu} + pg^{\mu\nu} + \frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right),$$

Conformal time, rescaled equations $\mathbf{S} = (p+\rho)\gamma^2\mathbf{v}$

$$\tilde{t} = \int dt/R. \qquad \tilde{\mathbf{S}} = R^4\mathbf{S}, \quad \tilde{p} = R^4 p, \quad \tilde{\rho} = R^4\rho, \quad \tilde{\mathbf{B}} = R^2\mathbf{B},$$

$$\tilde{\mathbf{J}} = R^3\mathbf{J}, \quad \text{and} \quad \tilde{\mathbf{E}} = R^2\mathbf{E}.$$

Equivalent to usual magneto-hydrodynamics

$$\frac{\partial\tilde{\mathbf{S}}}{\partial\tilde{t}} = -(\boldsymbol{\nabla}\cdot\mathbf{v})\tilde{\mathbf{S}} - (\mathbf{v}\cdot\boldsymbol{\nabla})\tilde{\mathbf{S}} - \boldsymbol{\nabla}\tilde{p} + \tilde{\mathbf{J}}\times\tilde{\mathbf{B}}.$$

$$\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}} = -\nabla\times\tilde{\mathbf{E}}, \qquad \boldsymbol{\nabla}\cdot\tilde{\mathbf{B}} = 0,$$

# Small Lorentz factors, γ~1

$$\frac{\partial \ln\widetilde{\rho}}{\partial\widetilde{t}} = -\frac{4}{3}(\mathbf{v}\cdot\boldsymbol{\nabla}\ln\widetilde{\rho}+\boldsymbol{\nabla}\cdot\mathbf{v}) - \frac{\widetilde{\mathbf{J}}\cdot\widetilde{\mathbf{E}}}{\widetilde{\rho}},$$

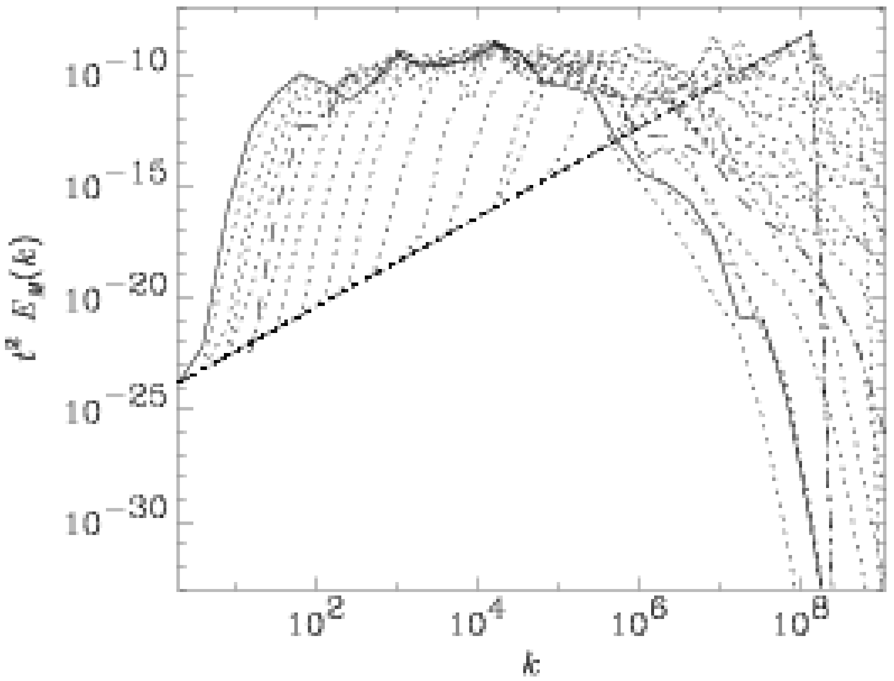
$$\frac{D\mathbf{v}}{D\widetilde{t}} = -\mathbf{v}\left(\frac{D\ln\widetilde{\rho}}{D\widetilde{t}}+\boldsymbol{\nabla}\cdot\mathbf{v}\right) - \frac{1}{4}\boldsymbol{\nabla}\ln\widetilde{\rho} + \frac{\widetilde{\mathbf{J}}\times\widetilde{\mathbf{B}}}{\frac{4}{3}\widetilde{\rho}},$$

where $D/D\widetilde{t} = \partial/\partial\widetilde{t} + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the total derivative, and

$$\frac{\partial\widetilde{\mathbf{B}}}{\partial\widetilde{t}} = \nabla\times(\mathbf{v}\times\widetilde{\mathbf{B}}), \quad \widetilde{\mathbf{J}} = \boldsymbol{\nabla}\times\widetilde{\mathbf{B}}.$$

# Magnetohydrodynamic turbulence

The conclusion from the above expressions is thus that *the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled "tilde" variables, and provided conformal time $\tilde{t}$ is used.* The effect of this is, as usual, that




shell models $$\frac{db_n}{d\tilde{t}} = M_n(v,b)$$

$$M_n(v,b) = ik_n(A-C)(v_{n+1}^* b_{n+2}^* - b_{n+1}^* v_{n+2}^*) + ik_n(B+\tfrac{1}{2}C)(v_{n-1}^* b_{n+1}^* - b_{n-1}^* v_{n+1}^*) - ik_n(\tfrac{1}{2}B - \tfrac{1}{4}A)(v_{n-2}^* b_{n-1}^* - b_{n-2}^* v_{n-1}^*),$$

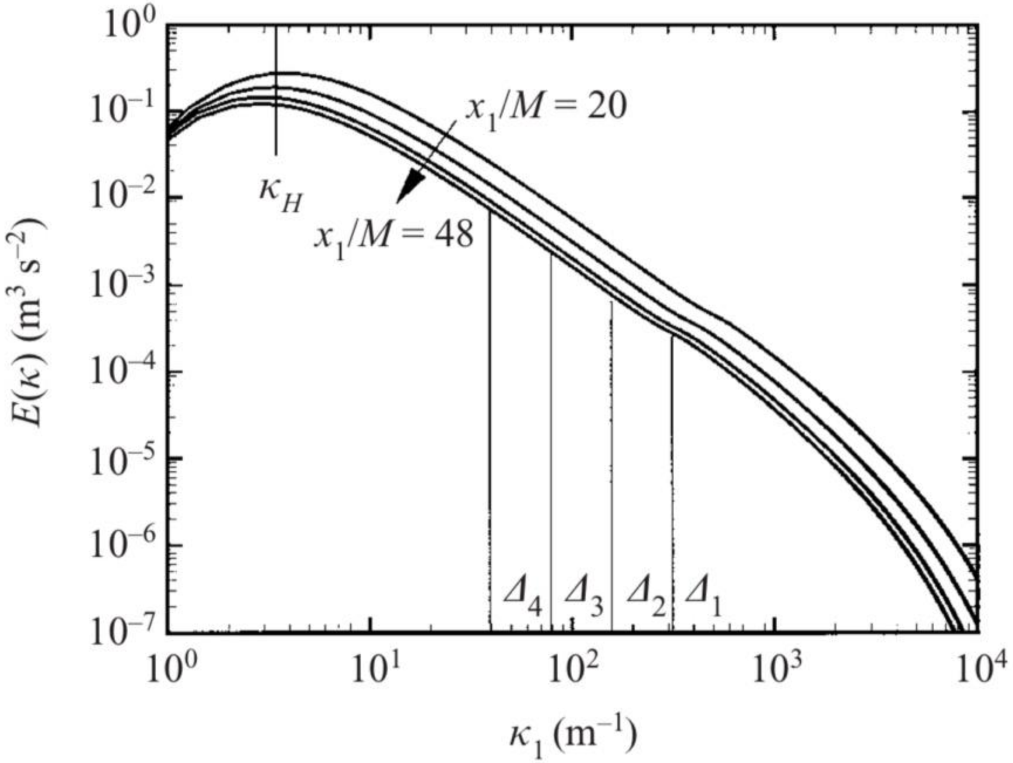
$$\frac{4}{3}\rho_0 \frac{dv_n}{d\tilde{t}} = N_n(v,b)$$

# Turbulent decay: early results & expectations

*J. Fluid Mech.* (2003), *vol.* 480, *pp.* 129–160. © 2003 Cambridge University Press 129
DOI: 10.1017/S0022112002003579 Printed in the United Kingdom

**Decaying turbulence in an active-grid-generated flow and comparisons with large-eddy simulation**

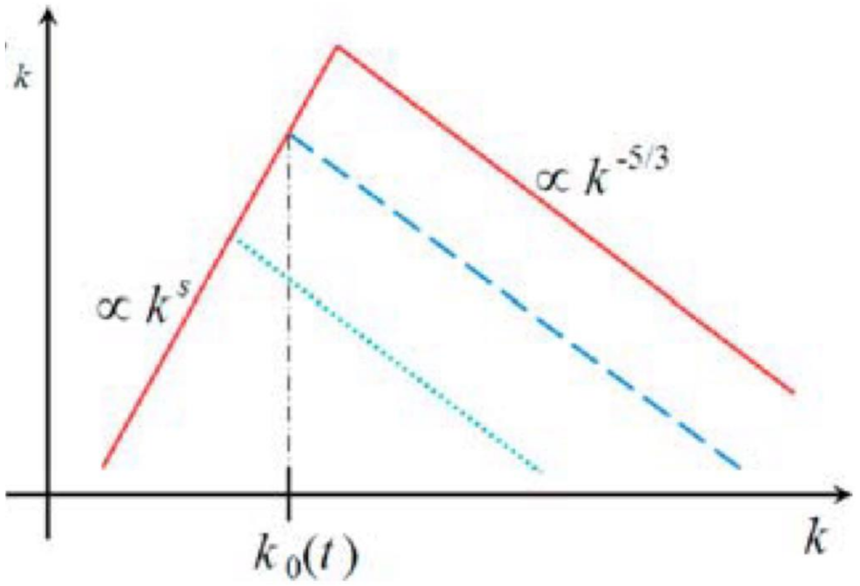
By HYUNG SUK KANG[1], STUART CHESTER[1] AND CHARLES MENEVEAU[1,2]




Mon. Not. R. Astron. Soc. **366,** 1437–1454 (2006) doi:10.1111/j.1365-2966.2006.09918.x

**Evolving turbulence and magnetic fields in galaxy clusters**

Kandaswamy Subramanian,[1,4★] Anvar Shukurov[1,2,4★] and Nils Erland L. Haugen[3,5★]



# Increase at small wavenumbers already in 2000

**THE DYNAMO EFFECT IN STARS**

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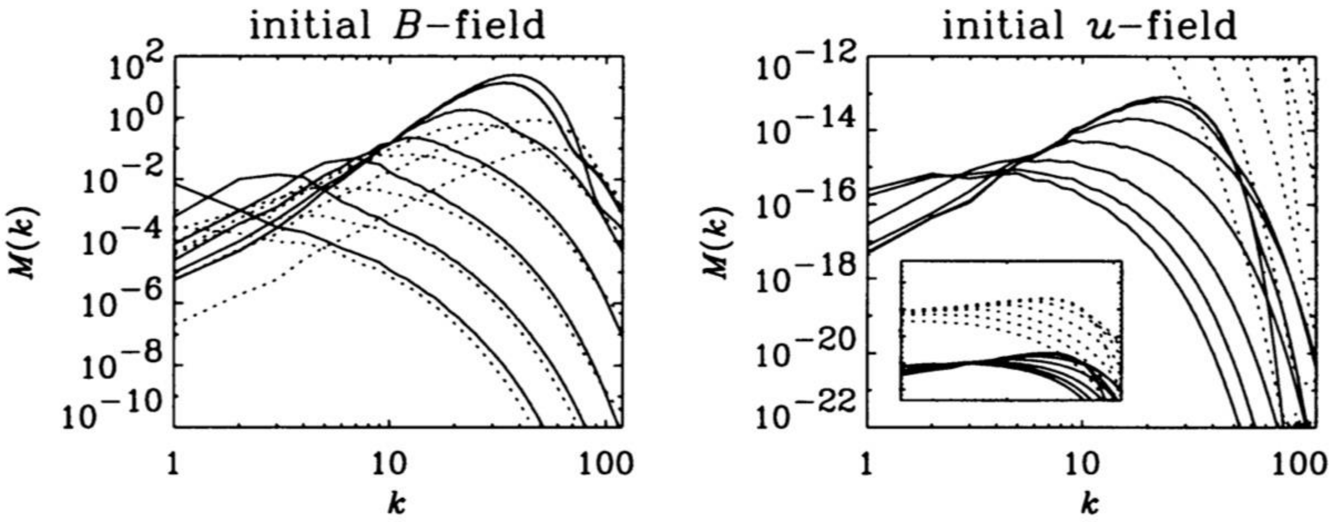



*K.S. Cheng et al. (eds.), Stellar Astrophysics,* 1–8.

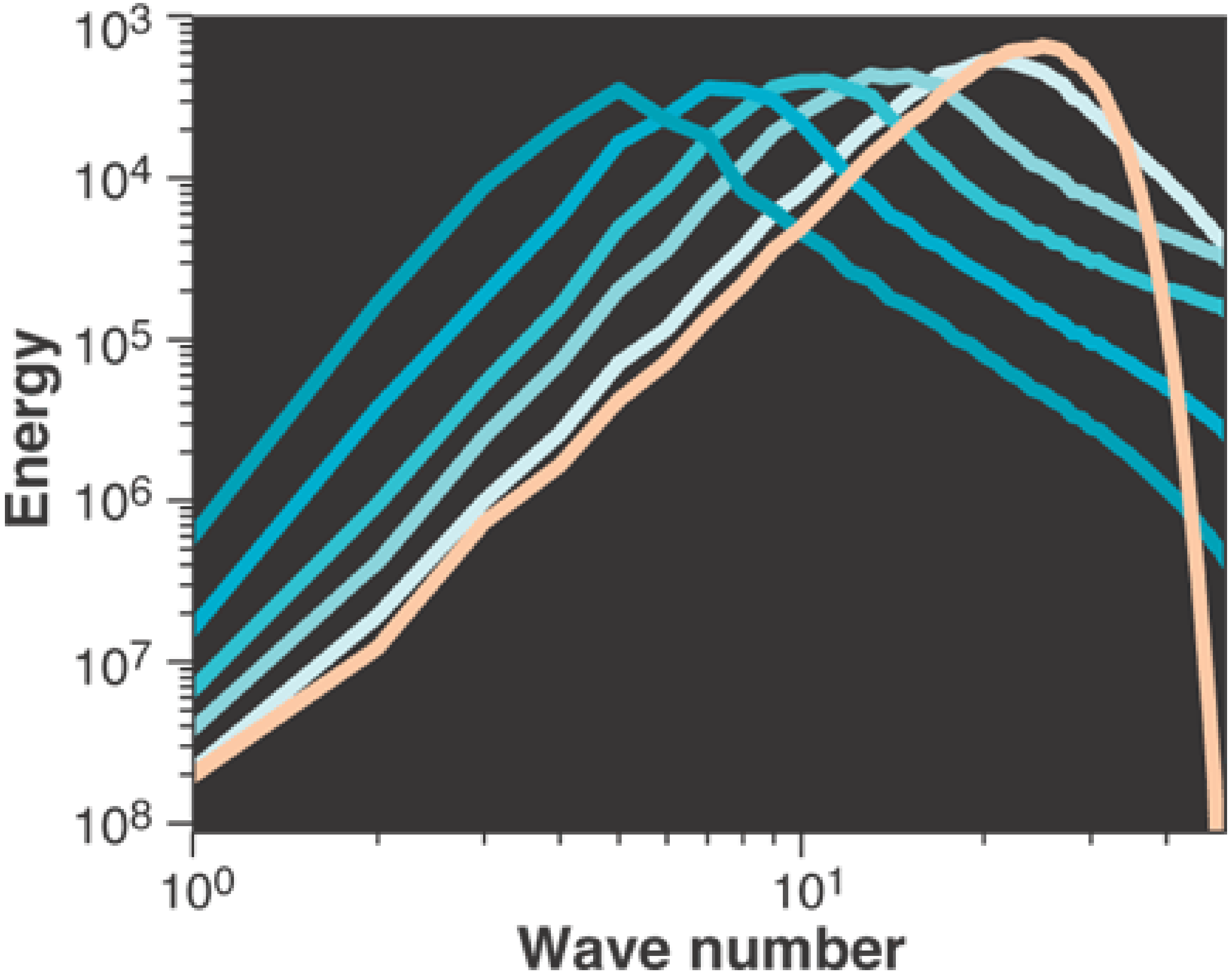

- Magnetically dominated
  - Started from random vector potential
  - K4 spectrum for magnetic energy
  - Kinetic energy (dotted) similar, but without the peak
- Kinetically dominated
  - Very similar inverse transfer
  - But kinetic energy much larger

# 3-D decay simulations with & without helicity

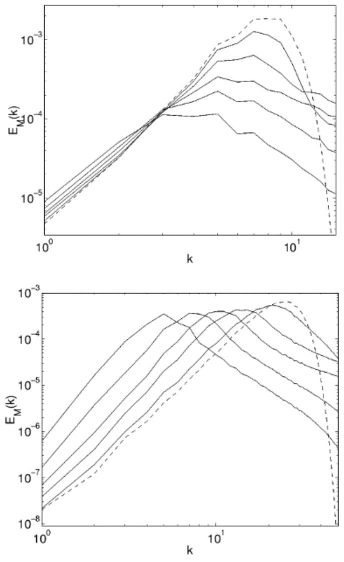
Initial slope

$E\sim k^4$

Causality (Durrer & Caprini 2003)
shell-integrated spectra
δ-correlated vector potential




helical vs
nonhelical



Christensson et al.
(2001, PRE 64, 056405)

$$E_{\rm M}(k) = \frac{1}{2}\sum_{k_-<|\boldsymbol{k}|\leqslant k_+} |\tilde{\boldsymbol{B}}(\boldsymbol{k})|^2,$$

$$H_{\rm M}(k) = \frac{1}{2}\sum_{k_-<|\boldsymbol{k}|\leqslant k_+} (\tilde{\boldsymbol{A}}\cdot\tilde{\boldsymbol{B}}^* + \tilde{\boldsymbol{A}}^*\cdot\tilde{\boldsymbol{B}}),$$

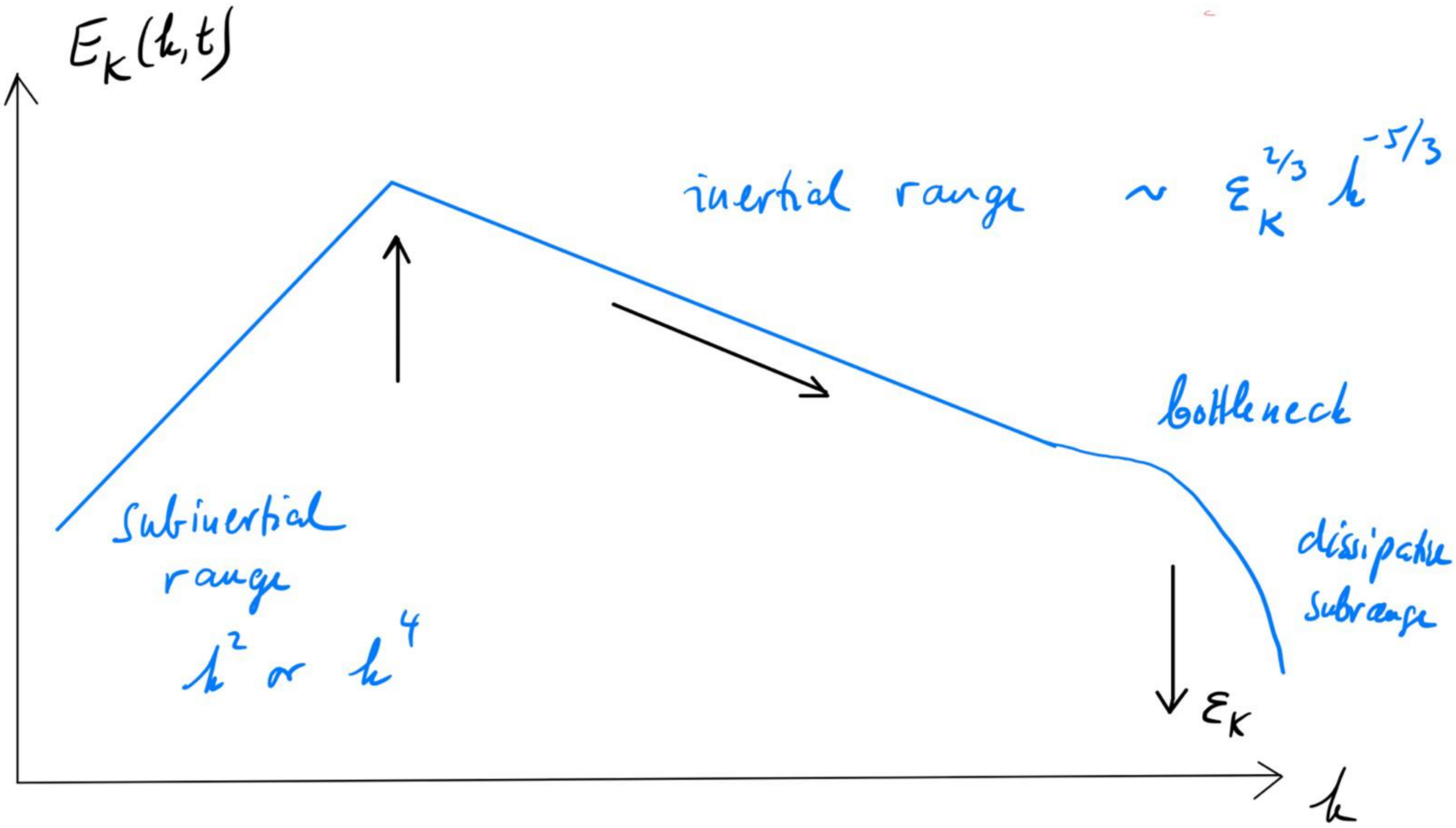
$k_\pm = k \pm \delta k/2$ and $\delta k = 2\pi/L$ is the

# Considerations

- Difficulties in seeing (nonhelical) inverse cascade
  - Must have: $k_{\mathrm{peak}} >> k_{\mathrm{min}}$ (enough $k$-range to the left of the peak)
  - Causal spectrum $E_{\mathrm{M}}(k)$ ~$k^4$ (must be steep enough)
- Not seen for velocity spectrum
  - Even if incompressible
  - → long-range interactions immediately driven by ***B***-field
- Tools
  - *pq* diagram
  - conservation laws
  - study resistive effects

# Turbulent magnetic fields: cascades & dissipation

- Turbulence spectrum important diagnostics
  - Spectral slope (inertial range)
  - Dimensional arguments
  - Subinertial range
  - Random (δ-correlated): $k^2$, because integrated over shells in $k$-space
- Length of inertial range
  - → Reynolds number
  - Very large in astrophysics
  - Not in simulations
  - Some effects sensitive to this
- Bottleneck effect
  - Is a real effect
  - Less pronounced in 1-D spectra
  - Important for some small-scale dynamos



# Different approaches to decays laws

- Initial slope matters
  - "selective decay"
- Olesen (1997)
  - Initial slope $k^{\alpha}$
  - Invariance under rescaling: $x \rightarrow x\ell$, $t \rightarrow t\,\ell^{1/q}$
  - $\rightarrow q=2/(3+\alpha)$
- Inverse cascade criterion
  - $q>0$, so $\alpha > -3$

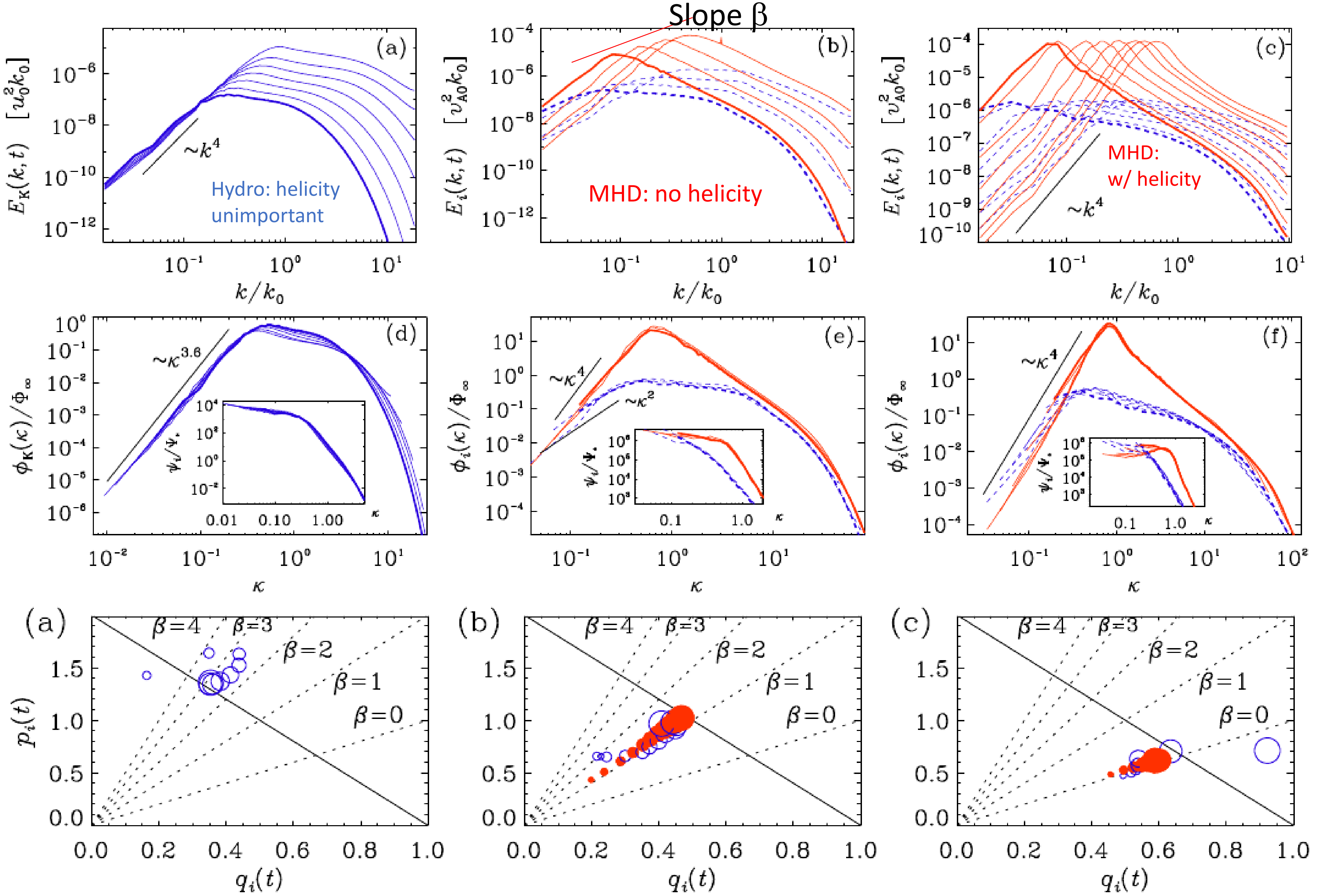
- Self-similarity matters
  - Measure empirically $\beta$
  - $\rightarrow q=2/(3+\beta)$
- Inverse cascade criterion
  - $\alpha - \beta > 0$, so
  - $\alpha > \beta$ cc
- Hosking integral:
  - $\beta = 3/2$

- Conservation law matters
  - Just dimensional arguments
  - Get nondim. prefactors from simulations

# Collapsed spectra and *pq* diagrams

$-p_i(t) = d \ln \mathcal{E}_i / d \ln t, \quad q_i(t) = d \ln \xi_i / d \ln t,$

Brandenburg & Kahniashvili (2017)



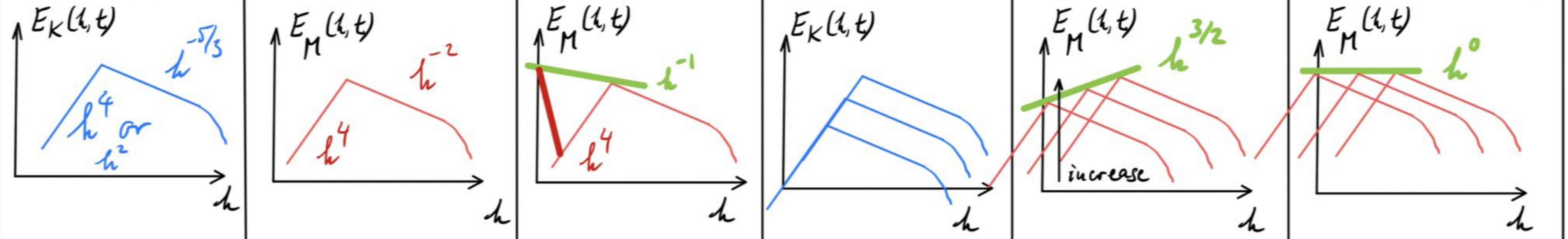

Explanations for slope β

Exponents *p,q*

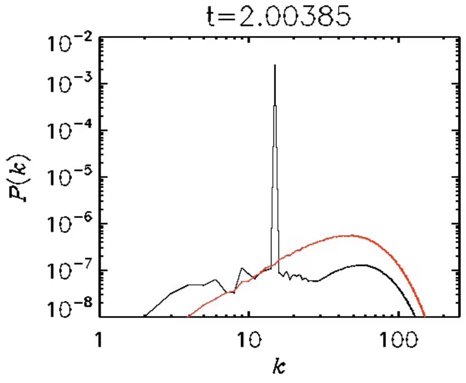
(Hosking & Schekochihin 2021+2023)

# Cascades (periodic box)

| forced turbulence | | | decaying turbulence | | |
|---|---|---|---|---|---|
| | MHD nonhel | MHD hel | | MHD nonhel | MHD hel |




| | | | | | |
|---|---|---|---|---|---|
| | | | just decay<br>no increase at small $k$ | decay & growth<br>shift and decay<br>$E_M(k_1,t) \sim t^{(\alpha-\beta)q}$ | selfsimilar shift to the left |



# Self-similar turbulent decay

$$E_\mathrm{M}(k\xi_\mathrm{M}(t),t) \approx \xi_\mathrm{M}^{-\beta_\mathrm{M}} \phi(k\xi_\mathrm{M}).$$

$$\int E_i(k,t)\,\mathrm{d}k = \mathcal{E}_i \qquad \xi_i(t) = \int_0^\infty k^{-1} E_i(k,t)\,\mathrm{d}k/\mathcal{E}_i(t)$$

## instantaneous scaling exponents

$$p_i(t) = d\ln\mathcal{E}_i/d\ln t, \quad q_i(t) = d\ln\xi_i/d\ln t,$$

| $\beta$ | $p$ | $q$ | inv. | dim. |
|---|---|---|---|---|
| 4 | $10/7 \approx 1.43$ | $2/7 \approx 0.286$ | $\mathcal{L}$ | $[x]^7[t]^{-2}$ |
| 3 | $8/6 \approx 1.33$ | $2/6 \approx 0.333$ | | |
| 2 | $6/5 = 1.20$ | $2/5 = 0.400$ | | |
| 1 | $4/4 = 1.00$ | $2/4 = 0.500$ | $\langle \boldsymbol{A}_\mathrm{2D}^2 \rangle$ | $[x]^4[t]^{-2}$ |
| 0 | $2/3 \approx 0.67$ | $2/3 \approx 0.667$ | $\langle \boldsymbol{A}\cdot\boldsymbol{B} \rangle$ | $[x]^3[t]^{-2}$ |
| $-1$ | $0/2 = 0.00$ | $2/1 = 1.000$ | | |

3/2

$$1+\beta_\mathrm{M} = p_\mathrm{M}/q_\mathrm{M}$$

## growth at small $k$

$$E_\mathrm{M}(k,t) = \xi_\mathrm{M}^{\alpha-\beta}\, k^\alpha \quad (k\xi_\mathrm{M} \ll 1)$$

| $\alpha$ | $\beta$ | $q$ | $(\alpha-\beta)\,q$ | comment, property |
|---|---|---|---|---|
| 1.7 | 3/2 | 4/9 | 4/45 | possible scaling, assuming Hosking integral conserved |
| 1.7 | 2 | 2/5 | $-3/25$ or 0 | alternative scaling if Saffman integral conserved |
| 2 | 2 | 2/5 | 0 | Saffman scaling, assuming Saffman integral conserved |
| 2 | 3/2 | 4/9 | $2/9 \approx 0.22$ | Saffman scaling, assuming Hosking integral conserved |
| 3 | 2 | 2/5 | $2/5 = 0.4$ | cubic scaling, assuming Saffman integral conserved |
| 3 | 3/2 | 4/9 | $2/3 \approx 0.67$ | cubic scaling, assuming Hosking integral conserved |
| 4 | 3/2 | 4/9 | $10/9 \approx 1.11$ | Batchelor scaling, assuming Hosking integral conserved |
| 4 | 0 | 2/3 | $8/3 \simeq 2.7$ | fully helical |

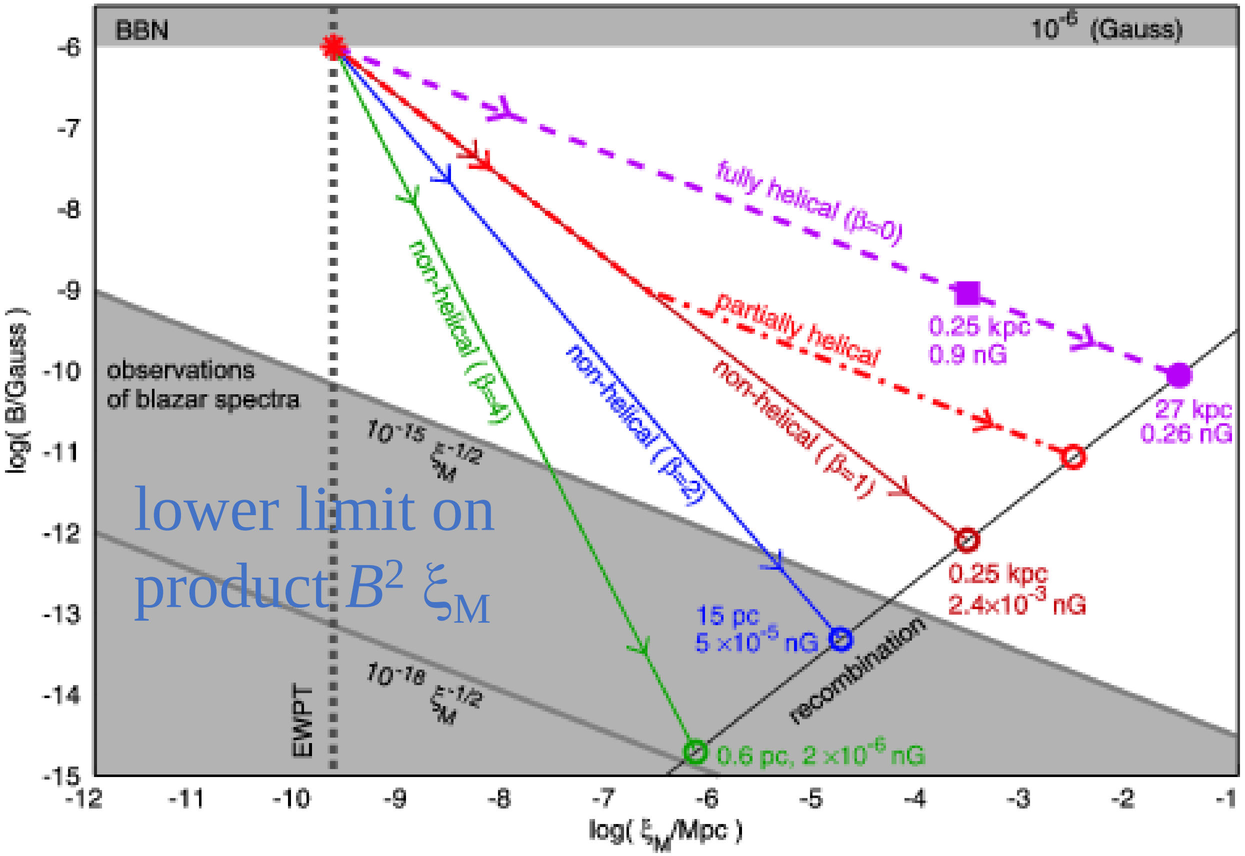
# Conservation laws

$\xi_M \sim \langle A.B \rangle \, t^{2/3}$

$\text{cm} \sim (\text{cm}^3/\text{s}^2) \, \text{s}^{2/3}$

| | | | |
|---|---|---|---|
| Magnetic helicity | $\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$ | $\text{cm}^3 \, \text{s}^{-2}$ | $\xi_M(t) \propto \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle^{1/3} t^{2/3}$ |
| Anastrophy (2-D) | $\langle A_z^2 \rangle$ | $\text{cm}^4 \, \text{s}^{-2}$ | $\xi_M(t) \propto \langle A_z^2 \rangle^{1/4} t^{1/2}$ |
| Hosking integral | $I_H$ | $\text{cm}^9 \, \text{s}^{-4}$ | $\xi_M(t) \propto I_H^{1/9} t^{4/9}$ |
| Saffman integral | $I_S$ | $\text{cm}^5 \, \text{s}^{-2}$ | $\xi_M(t) \propto I_S^{1/5} t^{2/5}$ |
| Loitsyansky integral | $I_L$ | $\text{cm}^7 \, \text{s}^{-2}$ | $\xi_M(t) \propto I_S^{1/7} t^{2/7}$ |

AB, Kahniashvili, ..., Vachaspati (2017)




Magnetic energy dependence
Parametric representation

| magnetic energy | $\kappa = p/2q$ |
|---|---|
| $\mathcal{E}_M(t) \propto \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle^{2/3} t^{-2/3}$ | $\propto \xi_M^{-1/2}$ |
| $\mathcal{E}_M(t) \propto \langle A_z^2 \rangle^{1/2} t^{-1}$ | $\propto \xi_M^{-1}$ |
| $\mathcal{E}_M(t) \propto I_H^{2/9} t^{-10/9}$ | $\propto \xi_M^{-5/4}$ |
| $\mathcal{E}_M(t) \propto I_S^{2/5} t^{-6/5}$ | $\propto \xi_M^{-3/2}$ |
| $\mathcal{E}_M(t) \propto I_S^{2/7} t^{-10/7}$ | $\propto \xi_M^{-5/2}$ |

# Hosking integral

$$\mathcal{I}_H(R \ll \xi_M) \simeq \int_{V_R} \mathrm{d}^3 r \, \langle h(\boldsymbol{x}) h(\boldsymbol{x}) \rangle \propto R^3$$

$$\mathcal{I}_H(R) = \int_0^\infty \mathrm{d}k \, w_{\mathrm{sph}}^{\mathrm{BC}}(k) \, \mathrm{Sp}\,(h)$$

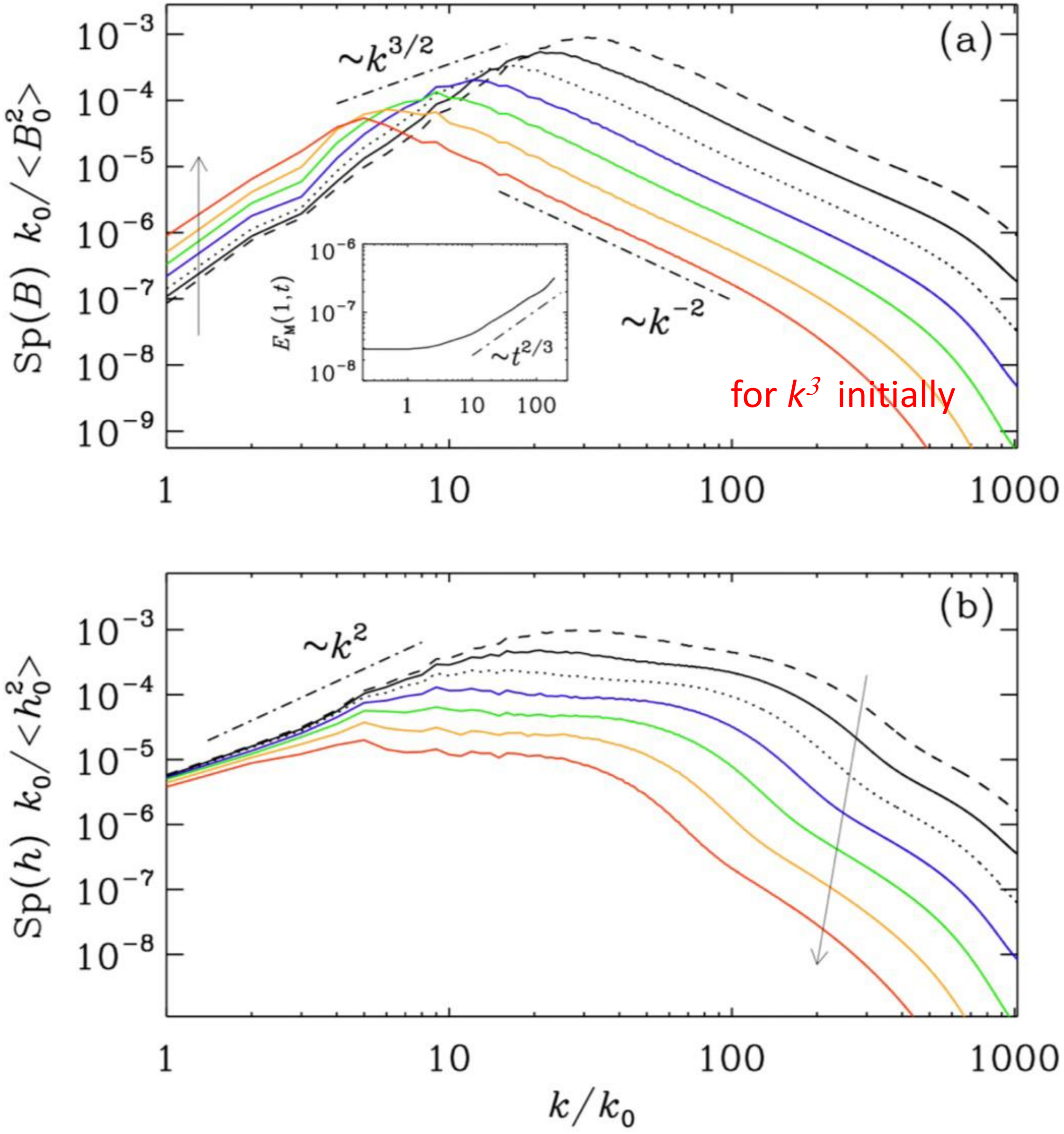
$$\mathrm{Sp}\,(h) = \frac{1}{V} \frac{k^2}{(2\pi)^3} \int_{|\boldsymbol{k}|=k} \mathrm{d}\Omega_k \, \tilde{h}^*(\boldsymbol{k}) \tilde{h}(\boldsymbol{k})$$

$$\mathrm{Sp}\,(h) = \frac{I_H}{2\pi^2} k^2 + \mathcal{O}(k^4)$$

$[I_{\mathrm{H}}] = \mathrm{cm}^9 \ \mathrm{s}^{-4}$

$\xi_M = I_{\mathrm{H}}^{\,a} \ \mathrm{t}^b$

$a$=1/9, $b$=4/9




AB, Sharma, Vachaspati (2023)

$$\xi_{\mathrm{M}}(t) \approx 0.12\, I_{\mathrm{H}}^{1/9} t^{4/9}, \quad \mathcal{E}_{\mathrm{M}}(t) \approx 3.7\, I_{\mathrm{H}}^{2/9} t^{-10/9}, \quad E_{\mathrm{M}}(k,t) \lesssim 0.025\, I_{\mathrm{H}}^{1/2} (k/k_0)^{3/2}$$

# Quantitatively

$$\xi_{\rm M}(t) = C_i^{(\xi)}\, I_i^{\sigma} t^{q}, \quad \mathcal{E}_{\rm M}(t) = C_i^{(\mathcal{E})}\, I_i^{2\sigma} t^{-p}, \quad E_{\rm M}(k) = C_i^{(E)}\, I_i^{(3+\beta)\sigma} (k/k_0)^{\beta},$$

$$I_{\rm M} = \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$$

$$\xi_{\rm M}(t) \approx 0.13\, I_{\rm M}^{1/3}\, t^{4/9}, \quad \mathcal{E}_{\rm M}(t) \approx 4.1\, I_{\rm M}^{2/3}\, t^{-10/9}, \quad E_{\rm M}(k,t) \lesssim 0.7\, I_{\rm M}.$$

$[I_{\rm H}] = {\rm cm}^9\ {\rm s}^{-4}$

$$\xi_{\rm M}(t) \approx 0.12\, I_{\rm H}^{1/9}\, t^{4/9}, \quad \mathcal{E}_{\rm M}(t) \approx 3.7\, I_{\rm H}^{2/9}\, t^{-10/9}, \quad E_{\rm M}(k,t) \lesssim 0.025\, I_{\rm H}^{1/2}\, (k/k_0)^{3/2}.$$

# *Anastrophy in 2-D*

conservation of anastrophy, $\langle A_z^2 \rangle = \text{const}$

Vector potential (0, 0, $A_z$) obeys $\frac{\mathrm{D}A_z}{\mathrm{D}t} = \eta \nabla^2 A_z$

Slightly different decay laws

$$\xi_{\mathrm{M}}(t) \approx 0.13 \, \langle A_z^2 \rangle^{1/4} \, t^{1/2}, \quad \mathcal{E}_{\mathrm{M}}(t) \approx 15 \, \langle A_z^2 \rangle^{1/2} \, t^{-1}$$

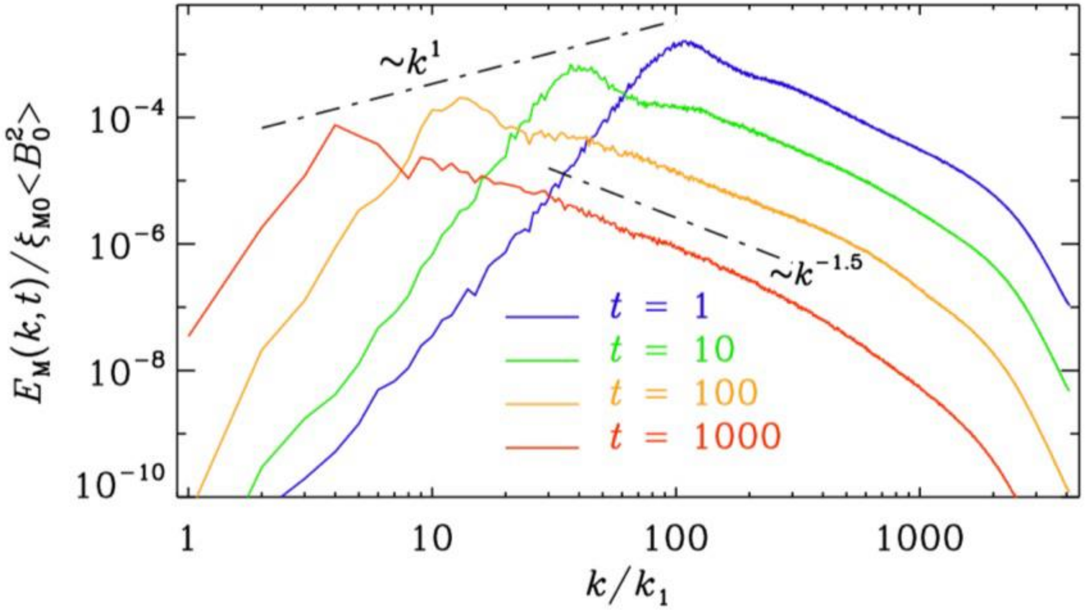
Envelope: linear increase

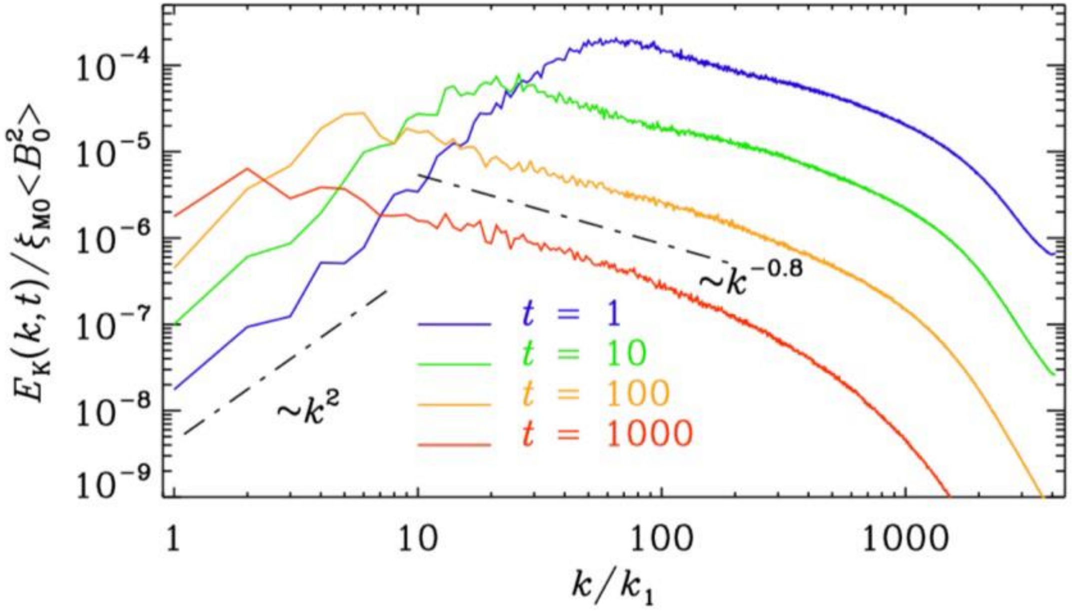
$$E_{\mathrm{M}}(k,t) \leq 60 \, \langle A_z^2 \rangle \, k$$

**Appendix A: Historical note on anastrophy**

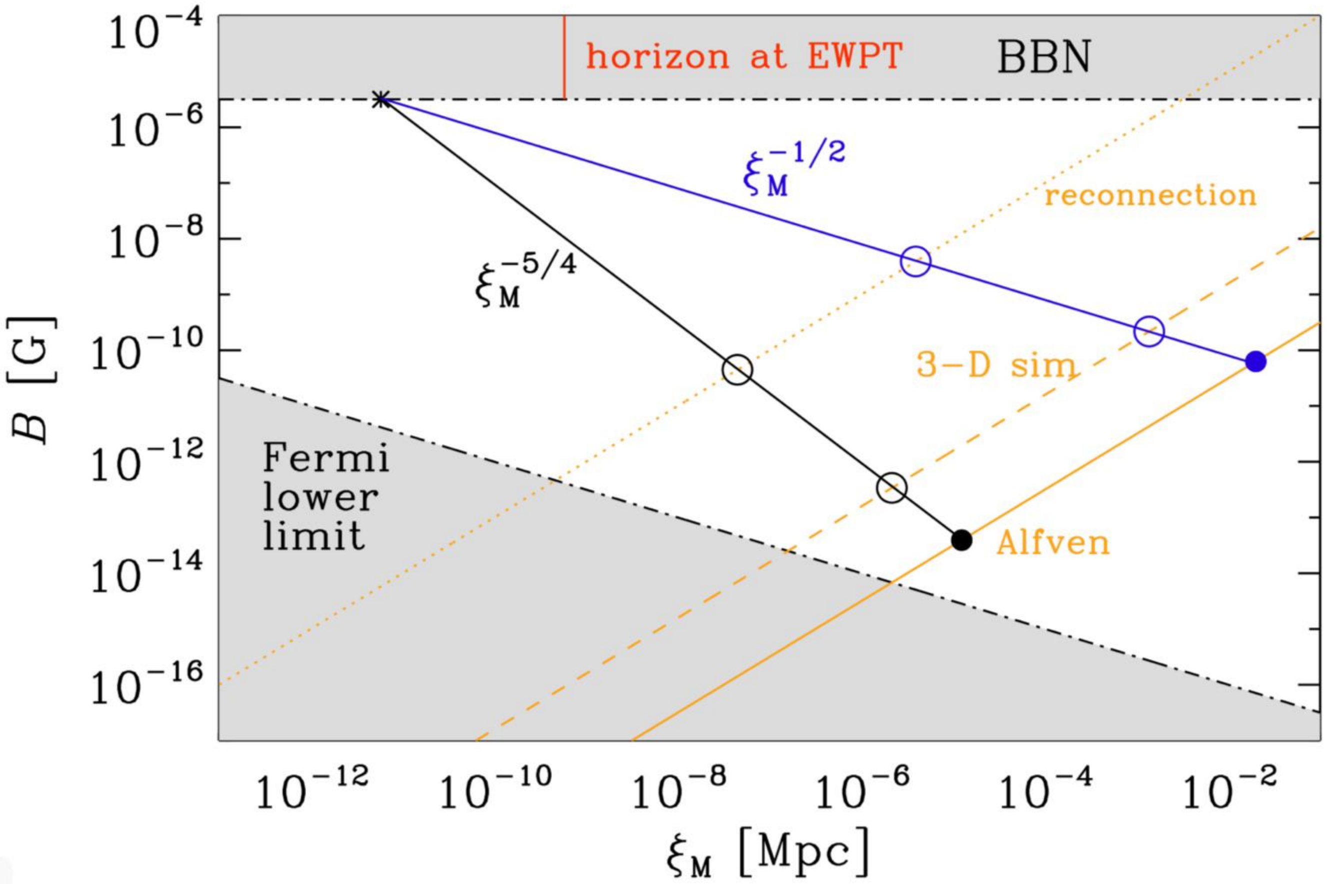
Annick Pouquet (2024, private communication) informed us now that the word may have been invented by Uriel Frisch and Nicolas Papanicolaou during a meeting on a Winter Sunday in the late 1970s at Saint-Jean-Cap-Ferrat, while she and Jacques Léorat were also present.

The word has Greek roots, where 'strophe' refers to curl or turning, and 'ana' therefore hints at the inverted curl of $\boldsymbol{B}$. This is somewhat reminiscent of the word palinstrophy, which is the mean squared double curl of the velocity, where 'palin' means again. This term was also invented by Frisch and Papanicolaou. The palinstrophy is proportional to the rate of change of enstrophy, i.e., the mean squared vorticity.



# Resistively prolonged decay during radiative era




- Endpoints under assumption that decay time = Alfven time
- Use: decay time = recombination time
- Possibility: decay time >> Alfven time
- → Premature endpoint of evolution

## Resistively controlled primordial magnetic turbulence decay

A. Brandenburg[1,2,3,4,5], A. Neronov[6,7], and F. Vazza[8,9,10]

Relation between decay time

$$\tau^{-1} = -\mathrm{d}\ln\mathcal{E}_\mathrm{M}/\mathrm{d}t$$

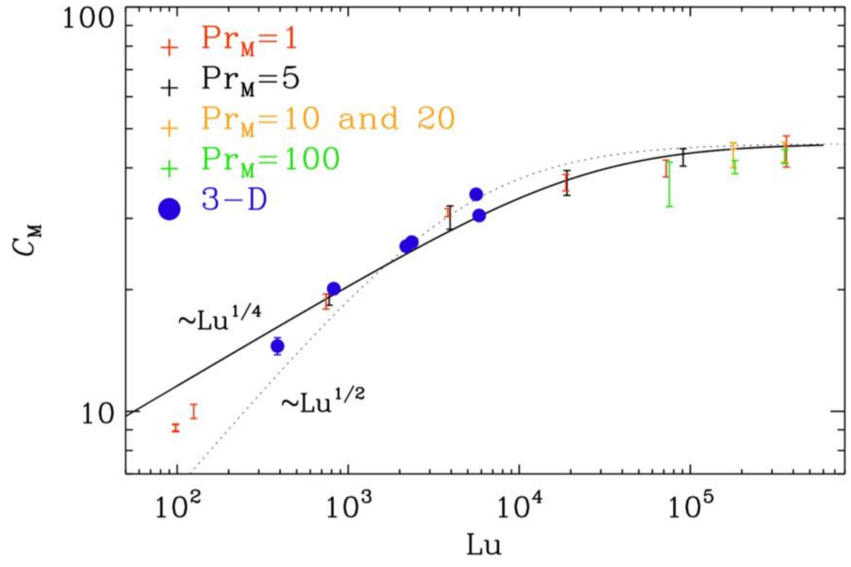
and Alfven time

$$\tau_\mathrm{A} = \xi_\mathrm{M}/v_\mathrm{A}$$

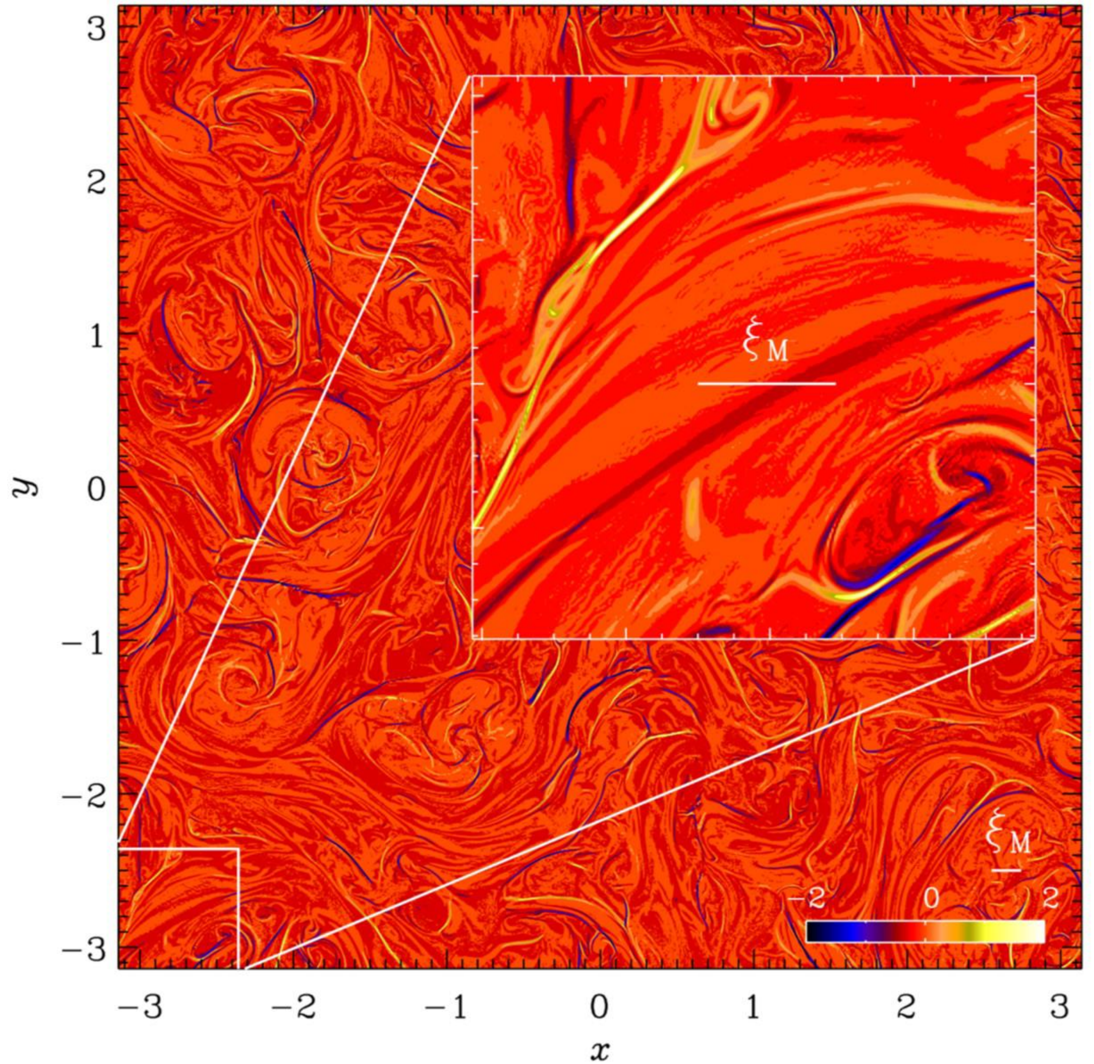
$$\mathcal{E}_\mathrm{M} = B_\mathrm{rms}^2/2\mu_0 = \rho v_\mathrm{A}^2/2$$

Determine $C_\mathrm{M}$ in relation:

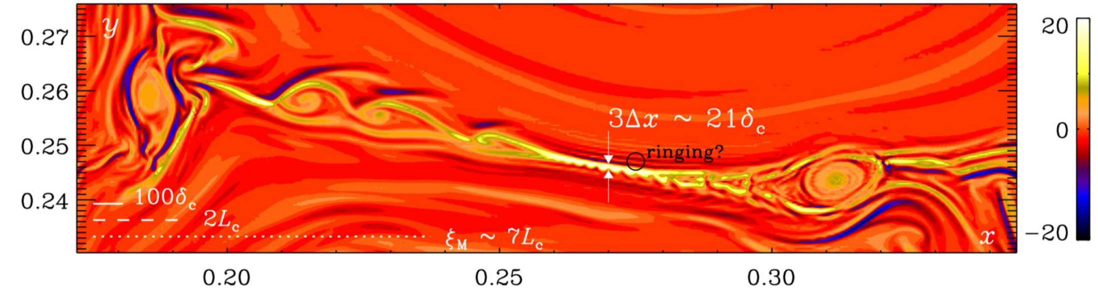
$$\tau = C_\mathrm{M}\xi_\mathrm{M}/v_\mathrm{A}$$




3-D




2-D



# Resistive limitations also in driven turbulence

**Magnetic helicity in stellar dynamos: new numerical experiments**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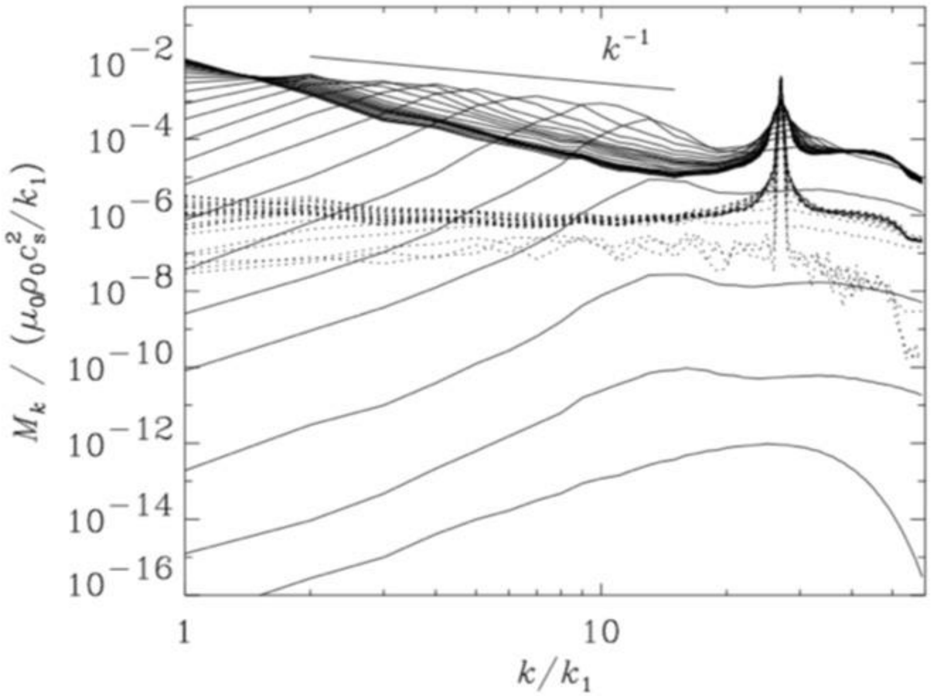
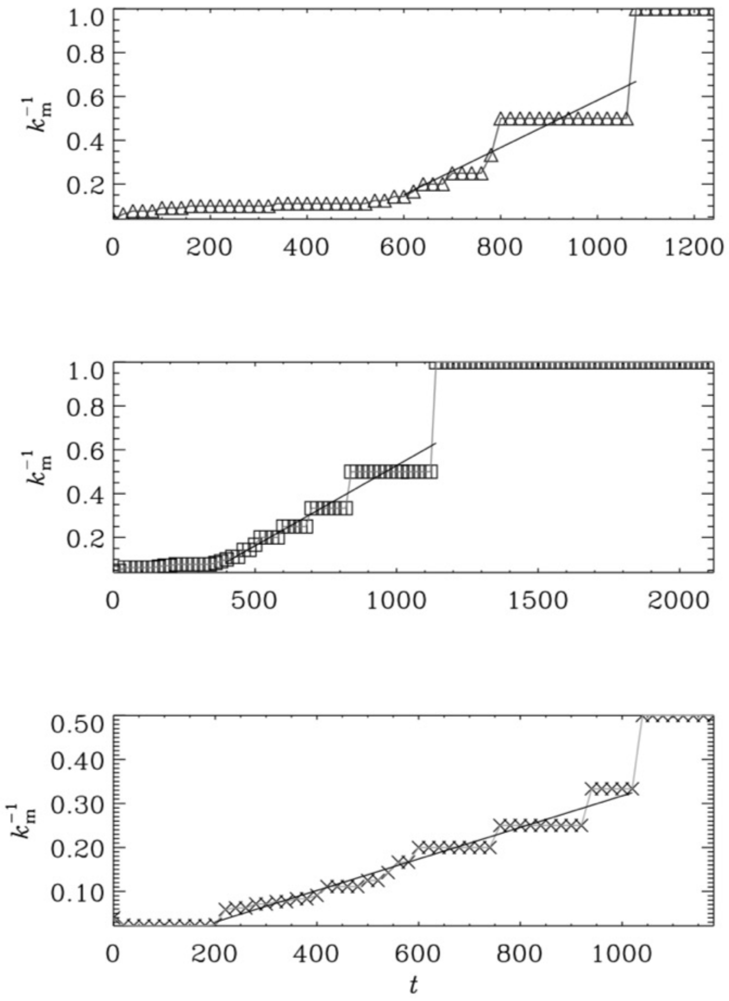
Received 2001 November 30; accepted 2002 February 5

# Travel speed of the bump to the left

bers. In order to check whether or not the speed of the spectral bumps in the different runs shown in Figs 7 and 10 also decreases with decreasing magnetic diffusivity we plot in Fig. 11 the position of the secondary bump in the power spectrum as a function of time. It is evident that the slope, which signifies the speed of the bump, decreases with decreasing magnetic diffusivity. A reasonable fit to this migration is given by

$$k_{\rm max}^{-1} = \alpha_{\rm trav}(t - t_{\rm sat}), \quad (53)$$

where the parameter $\alpha_{\rm trav}$ characterises the speed at which this secondary bump travels. The values obtained from fits to various runs are also listed in Table 1. Following arguments given by Pouquet et al. (1976), $\alpha_{\rm trav}$ obtained in this way is actually a measure of the $\alpha$ effect. The decrease of $\alpha_{\rm trav}$ with decreasing magnetic diffusivity adds to the evidence that the $\alpha$ effect is quenched in a $R_{\rm m}$ dependent fashion.

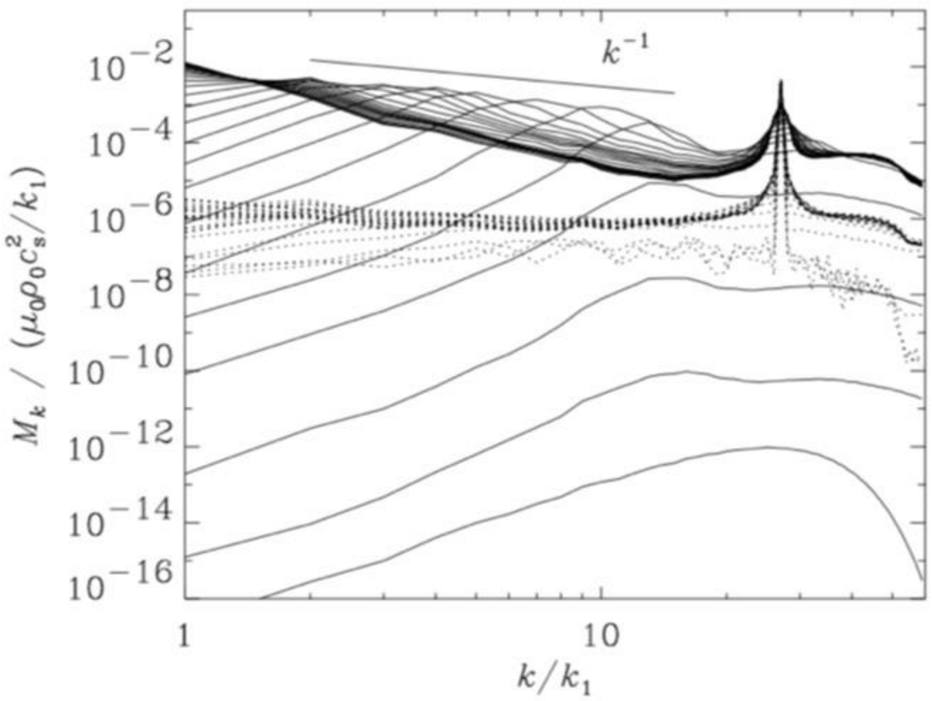
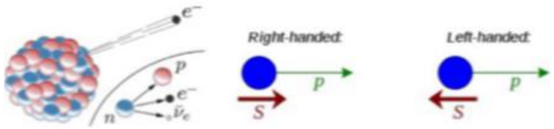
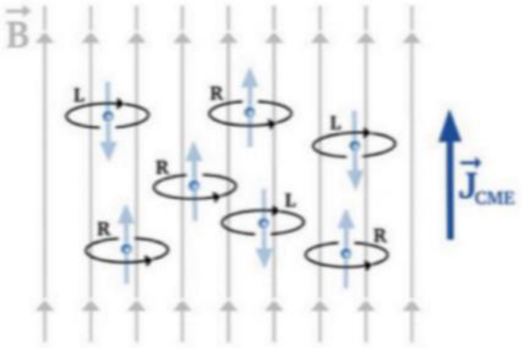



**Table 1.** Summary of runs, some of which were already presented in BS02. The runs with $n = 2$ have hyperdiffusivity, so the effective diffusivity at the forcing wavenumber, $k_{\rm f}$, and at the Nyquist wavenumber, $k_{\rm Ny} = \pi/\delta x = N/2$, are different. $N^3$ is the total number of mesh points, and $\delta x$ is the mesh spacing. The parameter $\ell_{\rm skin}$ gives an approximate upper bound for $|H|/(2\mu_0 M)$.

| Run | $N$ | $n$ | $\eta_n$ | $\eta_n k_{\rm f}^{2n-2}$ | $\eta_n k_{\rm Ny}^{2n-2}$ | $\gamma$ | $k_{\rm f}$ | $\alpha_{\rm trav}$ | $\lvert H\rvert/(2\mu_0 M)$ | $\ell_{\rm skin}$ | $S/(2M)\rvert_{t=1000}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | $120^3$ | 1 | $10^{-4}$ | $10^{-4}$ | $10^{-4}$ | 0.047 | 27 | $1.1 \times 10^{-3}$ | 0.035 | 0.065 | 0.00075 |
| B | $120^3$ | 2 | $3 \times 10^{-8}$ | $2 \times 10^{-5}$ | $10^{-4}$ | 0.070 | 27 | $7.3 \times 10^{-4}$ | 0.018 | 0.025 | 0.00035 |
| C | $120^3$ | 2 | $10^{-8}$ | $7 \times 10^{-6}$ | $4 \times 10^{-5}$ | 0.082 | 27 | $3.6 \times 10^{-4}$ | 0.005 | 0.013 | 0.00020 |

# Two examples of magnetogenesis in cosmology



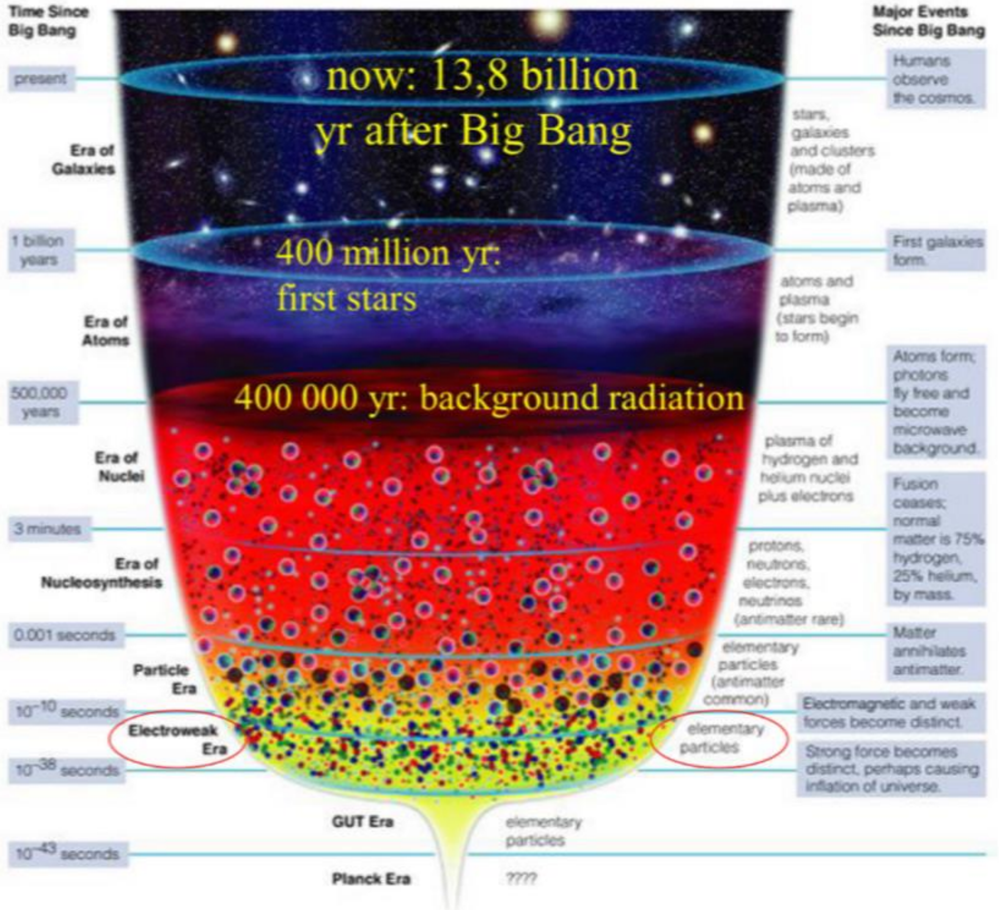




Back in time

temperaturen 4000 Kelvin

opaque

1 petakelvin
$= 10^{15}$ K
~ 100 GeV




"Battery" still needed

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{c}{q n_e} \nabla p_e,$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{c^2}{\sigma}(\mu_5 \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{u} \times \boldsymbol{B}$$

$$\mu_5 = 24\, \alpha_{\rm em}\, (n_{\rm L} - n_{\rm R})\, (\hbar c / k_{\rm B} T)^2$$

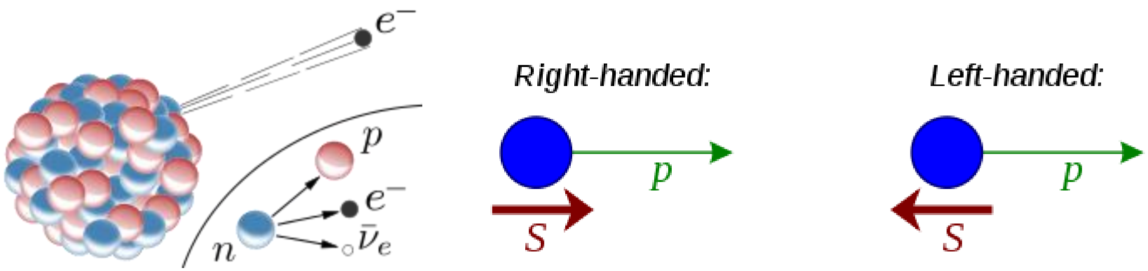
$$f^2 F_{\mu\nu} F^{\mu\nu}$$

$$\tilde{\mathbf{A}}'' + \left(\mathbf{k}^2 - \frac{f''}{f}\right) \tilde{\mathbf{A}} = 0$$

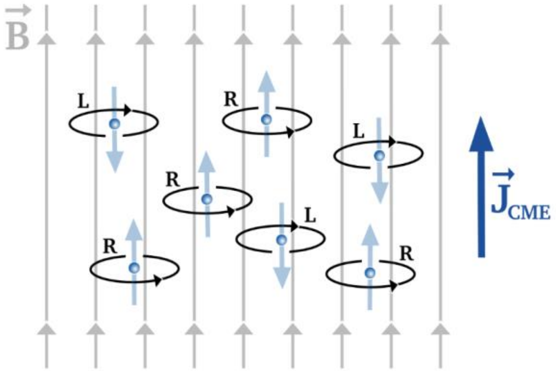
# Chiral magnetic effect: introduces pseudoscalar

- Mathematically identical to α effect in mean-field dynamos
- Comes from chiral chemical potential μ (or $\mu_5$)
- Number differences of left- & right-handed fermions

$$\mu_5 = 24\,\alpha_{\rm em}\,(n_{\rm L} - n_{\rm R})\,(\hbar c/k_{\rm B}T)^2,$$




- In the presence of a magnetic field, particles of opposite charge have momenta
- → electric current
- Self-excited dynamo
- But depletes μ




$$\frac{\partial \boldsymbol{A}}{\partial t} = \eta(\mu \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{U} \times \boldsymbol{B}$$

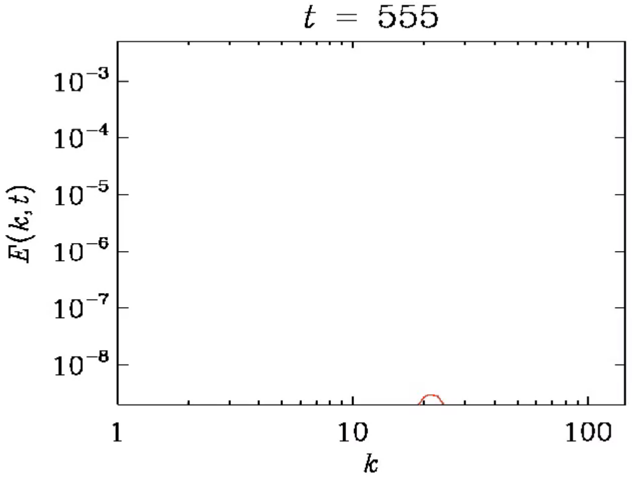
$$\sigma = |\mu k| - \eta k^2$$

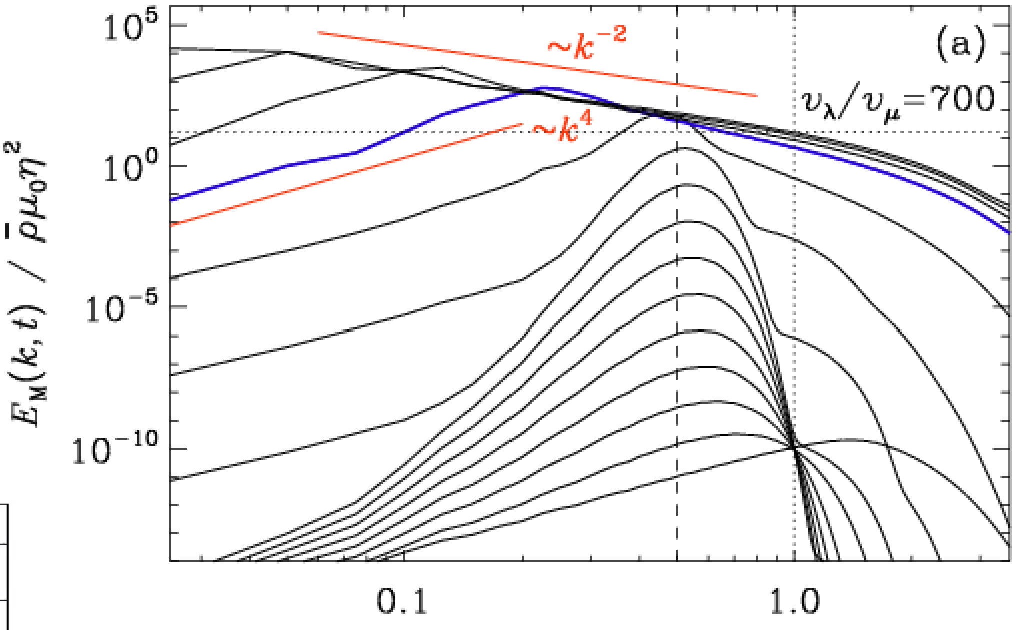
$\boldsymbol{B}$=curl$\boldsymbol{A}$

Discovered originally by Vilenkin (1980); application to magnetogenesis in early Universe by Joyce & Shaposhnikov (1997)

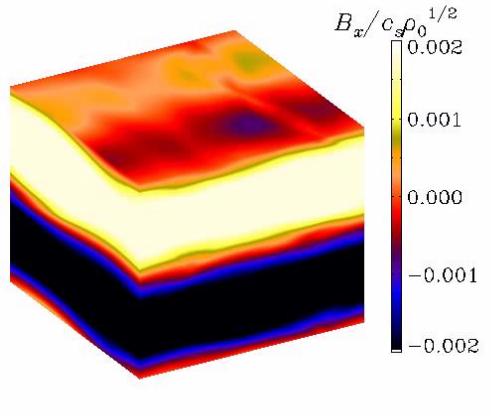
# Time dependence from chiral magnetic effect (CME)

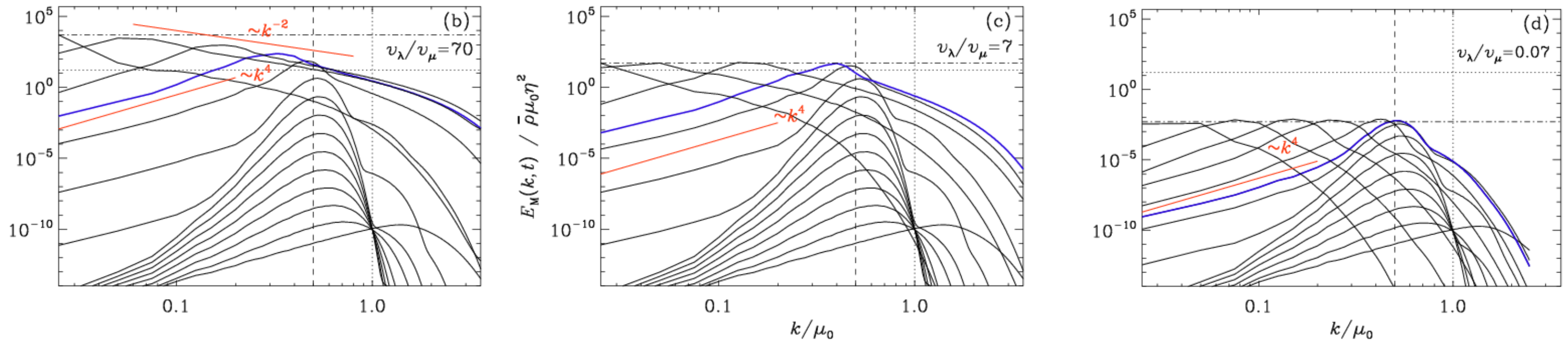
- Exponential growth at one k
- Subsequent inverse cascade
- Always fully helical



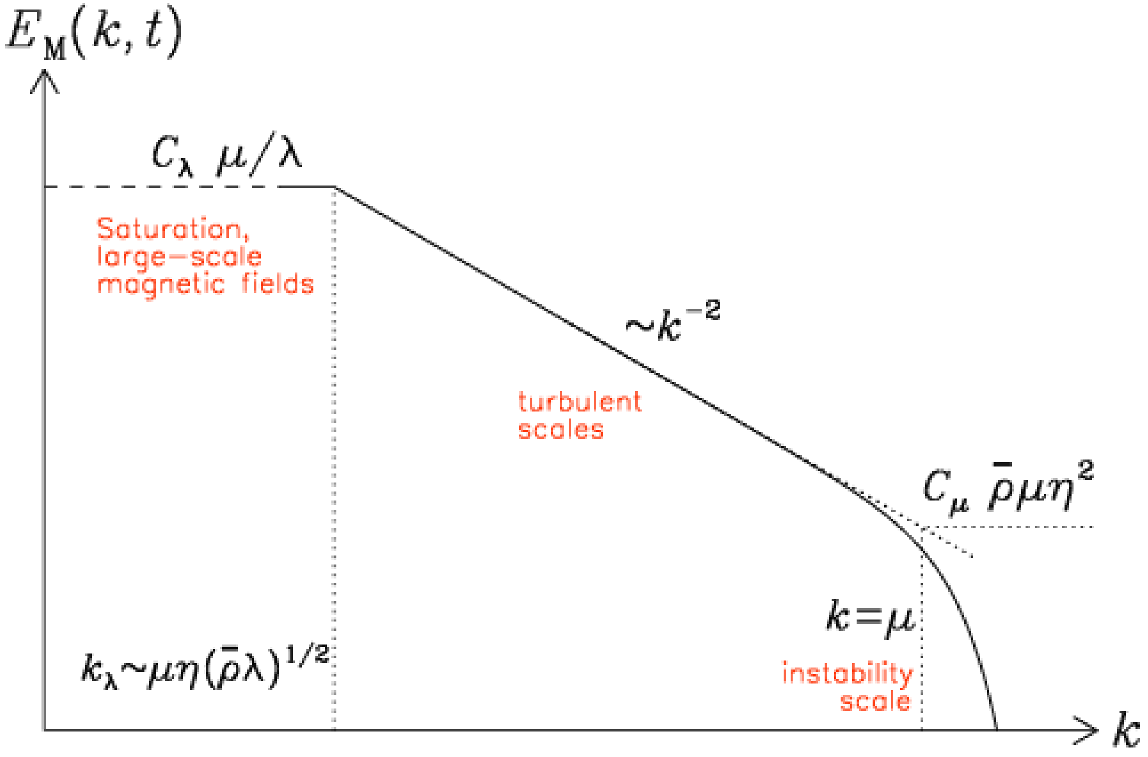

Growth at one wavenumber
Then: saturation caused by initial chemical potential




Brandenburg et al. (2017, ApJL 845, L21)

# Many details are known by now




- Instability just η dependant
- Saturation governed by λ

- Regime I is when turbulent subrange is long
- In regime II, just inverse cascading

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{u} \times \boldsymbol{B} + \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J})], \quad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B},$$

$$\frac{\mathrm{D}\mu_5}{\mathrm{D}t} = -\lambda\,\eta\,(\mu_5 \boldsymbol{B} - \boldsymbol{J}) \cdot \boldsymbol{B} + D_5 \nabla^2 \mu_5 - \Gamma_{\mathrm{f}} \mu_5,$$

$$v_\lambda = \mu_{50}/\lambda^{1/2}, \qquad v_\mu = \mu_{50}\eta. \qquad (6)$$

We recall that we have used here dimensionless quantities. We can identify two regimes of interest:

$$\eta k_1 < v_\mu < v_\lambda \quad \text{(regime I)}, \qquad (7)$$

$$\eta k_1 < v_\lambda < v_\mu \quad \text{(regime II)}, \qquad (8)$$

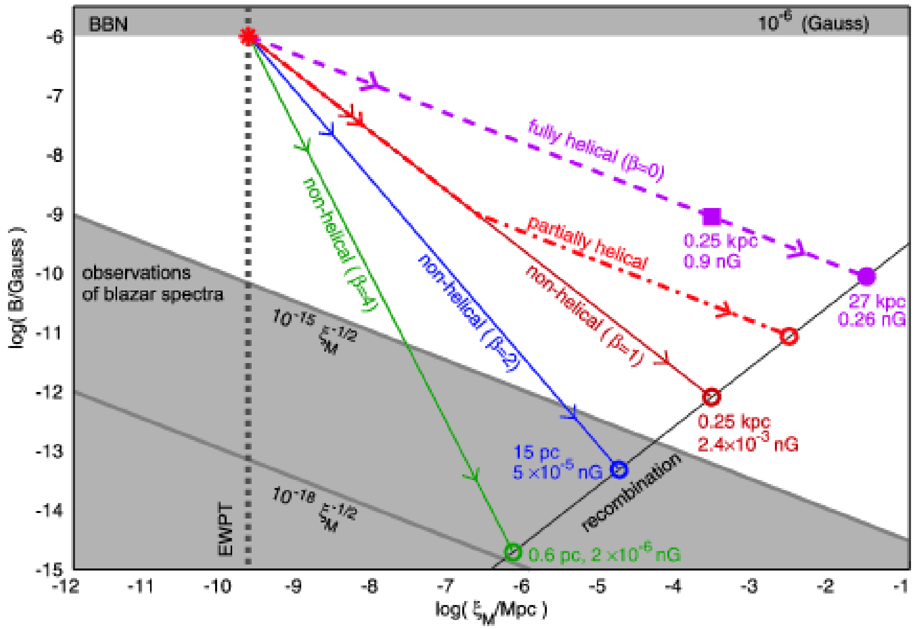
# Strength of chiral magnetic effect

- Inverse turbulent cascade
  - $\langle \boldsymbol{B}^2 \rangle \sim t^{-2/3}$ length scale: $\xi_M \sim t^{+2/3}$
- Dimensional arguments give

$$\langle \boldsymbol{B}^2 \rangle \, \xi_{\rm M} = \epsilon \, (k_{\rm B} T_0)^3 (\hbar c)^{-2},$$

- Inserting T=3K gives $10^{-18}$ G on 1 Mpc
- Consequence of conservation law

$$(n_{\rm L} - n_{\rm R}) + \frac{4\alpha_{\rm em}}{\hbar c} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = {\rm const.}$$



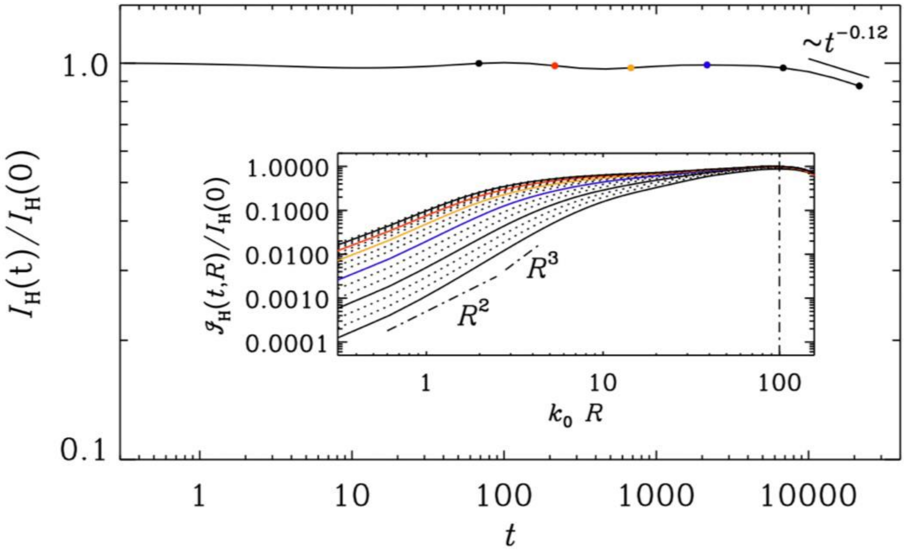

- But starting length scale very small → 12 cm
- Compared with horizon scale at that time (electroweak) of ~1 AU
- Other dimensional argument:

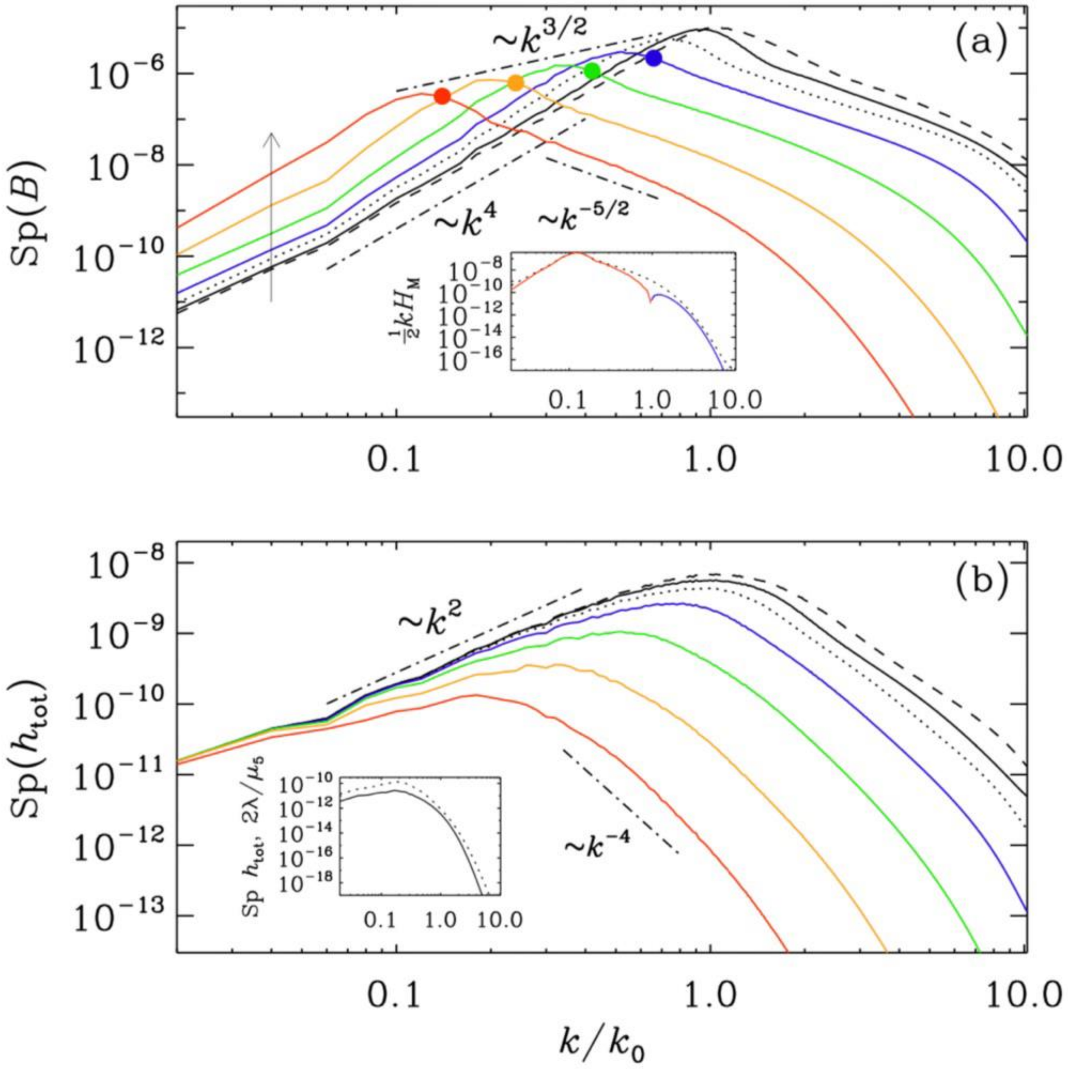
$$\langle \boldsymbol{B}^2 \rangle \xi_{\rm M} \lesssim \epsilon_3 (a_\star / a_0)^3 G^{-3/2} \hbar^{-1/2} c^{11/2},$$

# Decay law of magnetic turbulence with helicity balanced by chiral fermions

Axel Brandenburg,[1,2,3,4] Kohei Kamada,[5] and Jennifer Schober[6]

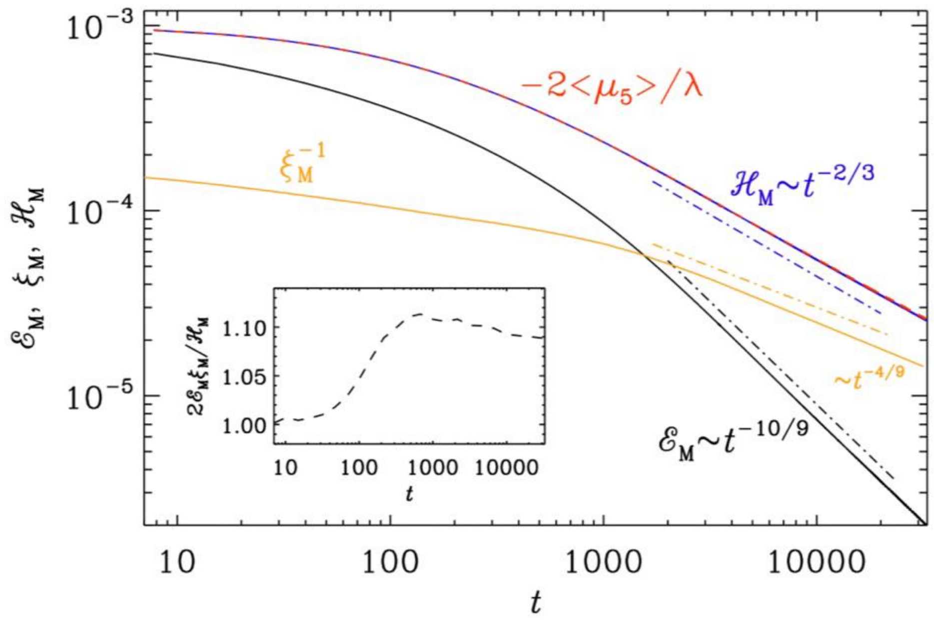
$$h_{\rm tot} \equiv \boldsymbol{A} \cdot \boldsymbol{B} + 2\mu_5/\lambda$$

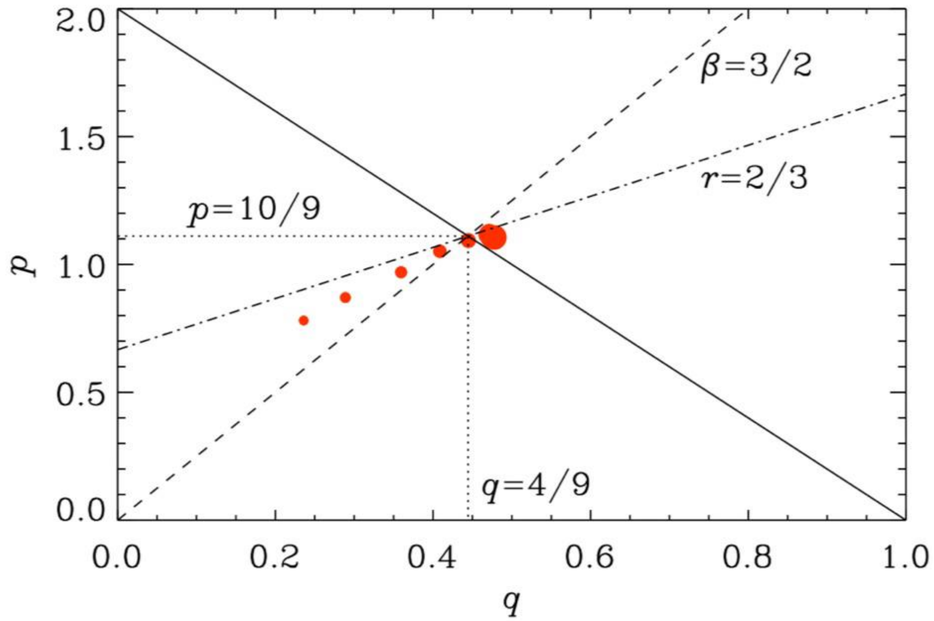



- Fully helical, but balanced chirality
- Hosking scaling applies
- Same scaling as for nonhelical turbulence
- But magnetic helicity not conserved: power law!

# Algebraic decay of helicity




$$2\mathcal{E}_M\xi_M/\mathcal{H}_M \approx 1$$

$$|\mathcal{H}_M| \propto \ \mu_5\rangle| \propto t^{-r}$$

$$r = p - q = 2/3$$

- Helicity decays not exponentially,
- But algebraically: 10/9 – 4/9 = 6/9
- Important for baryogenesis

# Gravitational waves & polarization

$$\left(\partial_t^2 + 3H\partial_t - c^2\nabla^2\right) h_{ij}(\boldsymbol{x},t) = \frac{16\pi G}{c^2}\, T_{ij}^{\mathrm{TT}}(\boldsymbol{x},t)$$

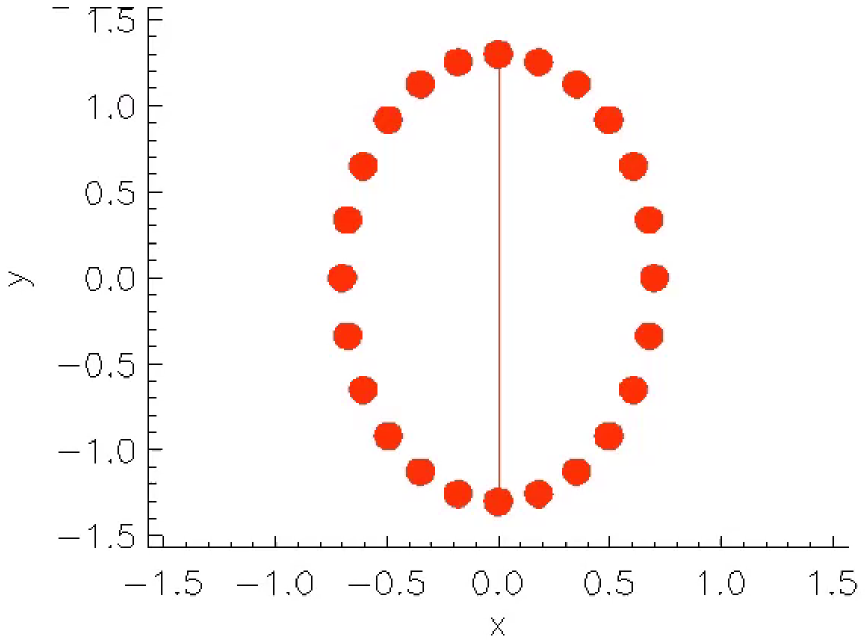
$$T_{ij}(\boldsymbol{x},t) = \left(p/c^2 + \rho\right)\gamma^2 u_i u_j - B_i B_j + (\boldsymbol{B}^2/2 + p)\delta_{ij}$$

Example

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ \sigma \sin kx \\ \cos kx \end{pmatrix} \longrightarrow \boldsymbol{\nabla}\times\boldsymbol{B} = \begin{pmatrix} \partial_x \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \sin kx \\ \cos kx \end{pmatrix} = k \begin{pmatrix} 0 \\ \sin kx \\ \cos kx \end{pmatrix} = k\boldsymbol{B}$$

Traceless-transverse

$$T_{ij}(x) = \mathcal{E}_{\mathrm{M}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos 2kx & \sigma\sin 2kx \\ 0 & \sigma\sin 2kx & \cos 2kx \end{pmatrix}$$




GW energy dependence on magnetic energy and wavenumber k0.

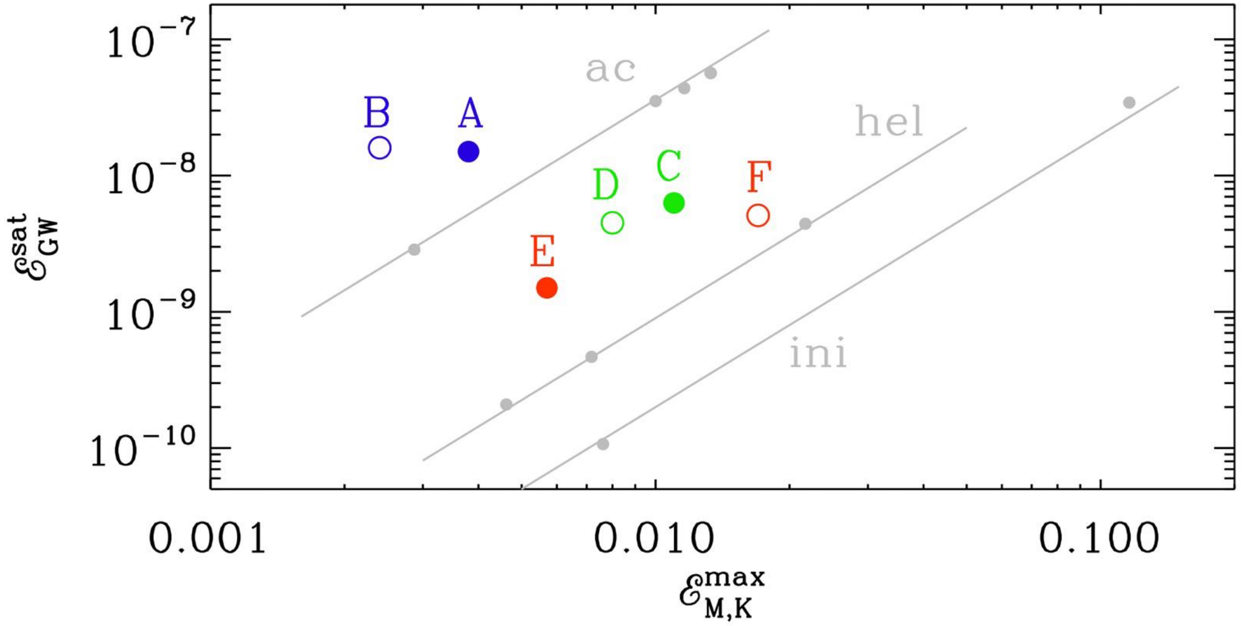
$$\bar{\Omega}_{\mathrm{GW}} = \frac{3H_*^2}{c^2 k_0^2}\,\Omega_{\mathrm{M}}^2$$

Roper Pol et al. (2020, GAFD 114, 130)

Polarization in turbulent cases:
Kahniashvili et al. (2021, PRR 3, 013193)

# GW energy depends quadratically on energy input & scale

Brandenburg+21




$$\mathcal{E}_{\mathrm{GW}}^{\mathrm{sat}} \approx (q\mathcal{E}_{\mathrm{M}}^{\max}/k_{\mathrm{peak}})^2$$

Acoustic turbulence more efficient ($q$~30)

Vortical turbulence less efficient ($q$<5)

Helical MHD turbulence least efficient

- Large-scale fields → more GW energy
- Generation at electroweak era: need strong fields
- Generation during inflation & reheating

# GW energy depends quadratically on energy input & scale

Rogachevskii, Boyarsky+17

Chiral magnetic effect (CME), use $[\mu_5]=[k]$

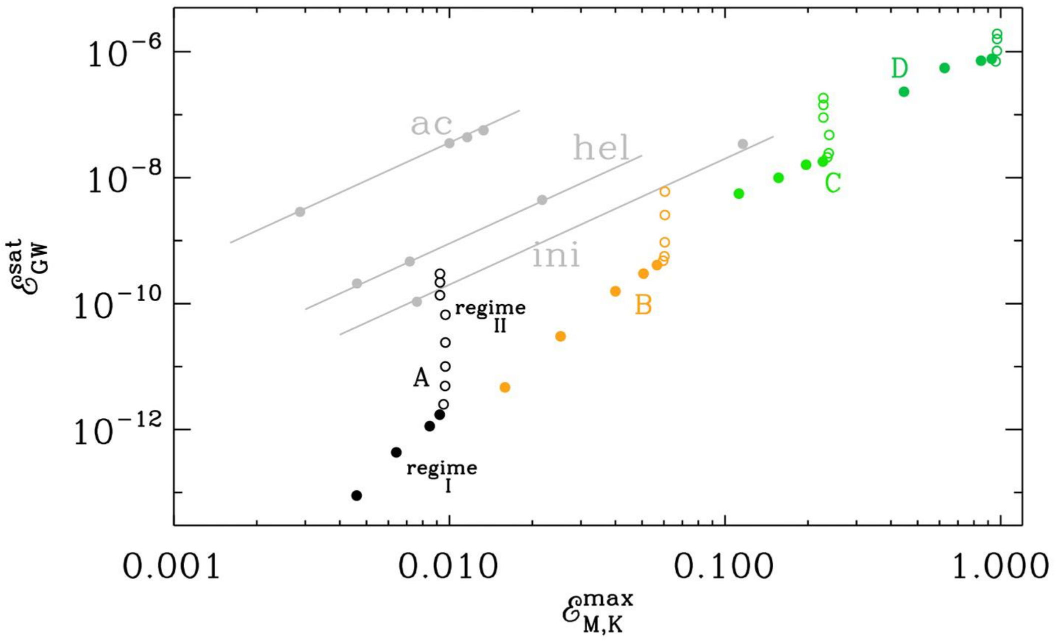
$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{u} \times \boldsymbol{B} + \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J})], \quad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$$

$$\frac{D\mu_5}{Dt} = -\lambda\, \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J}) \cdot \boldsymbol{B} + D_5 \nabla^2 \mu_5 - \Gamma_{\mathrm{f}} \mu_5$$

$$v_\lambda = \mu_{50}/\lambda^{1/2}, \qquad v_\mu = \mu_{50}\eta.$$

$$\eta k_1 < v_\mu < v_\lambda \quad \text{(regime I)},$$

$$\eta k_1 < v_\lambda < v_\mu \quad \text{(regime II)},$$

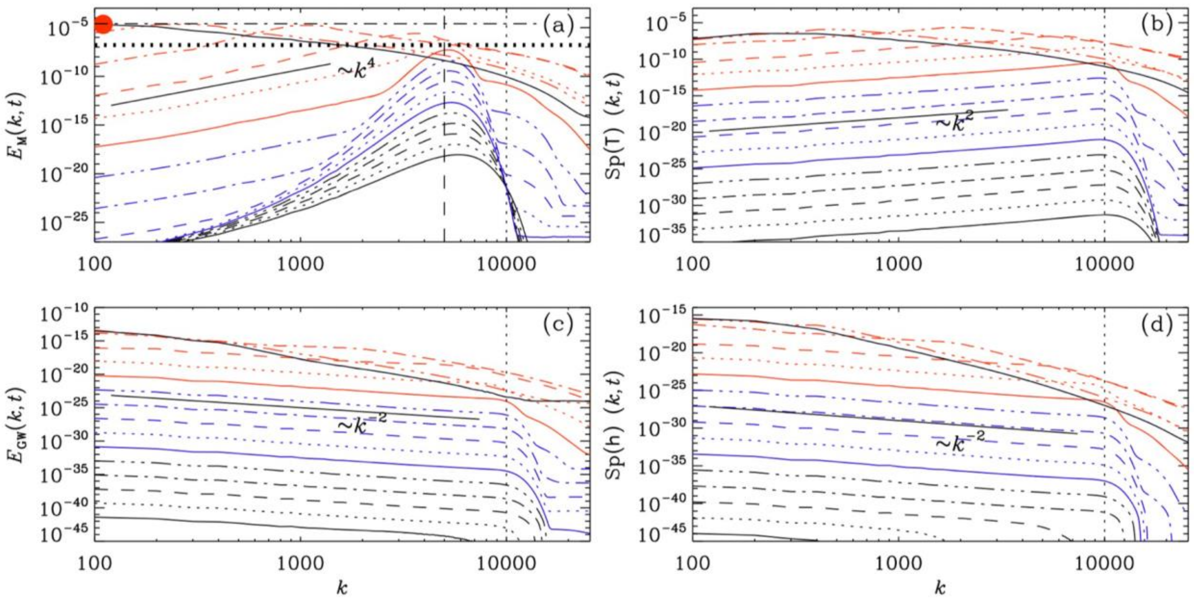



Brandenburg+21

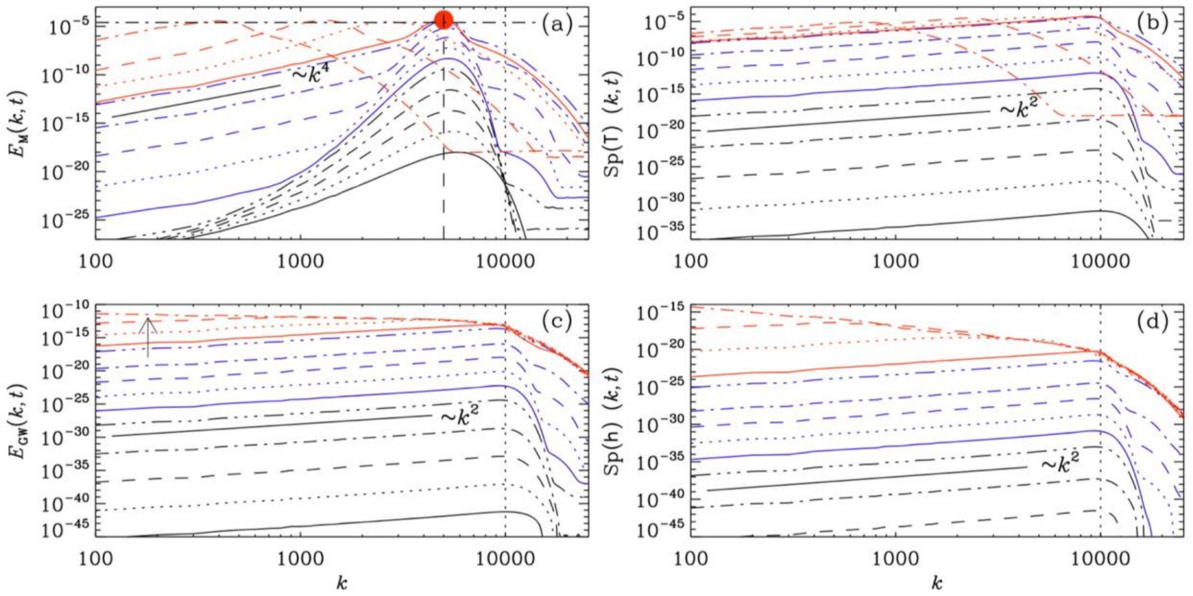
- Regime II: is more resistive → unrealistic, but large GW energies
- Regime I: → realistic, but small scales & less GW energy

# GW energy depends quadratically on energy input & scale

Regime I




Regime II

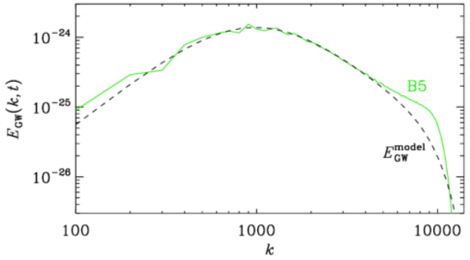



Brandenburg+21

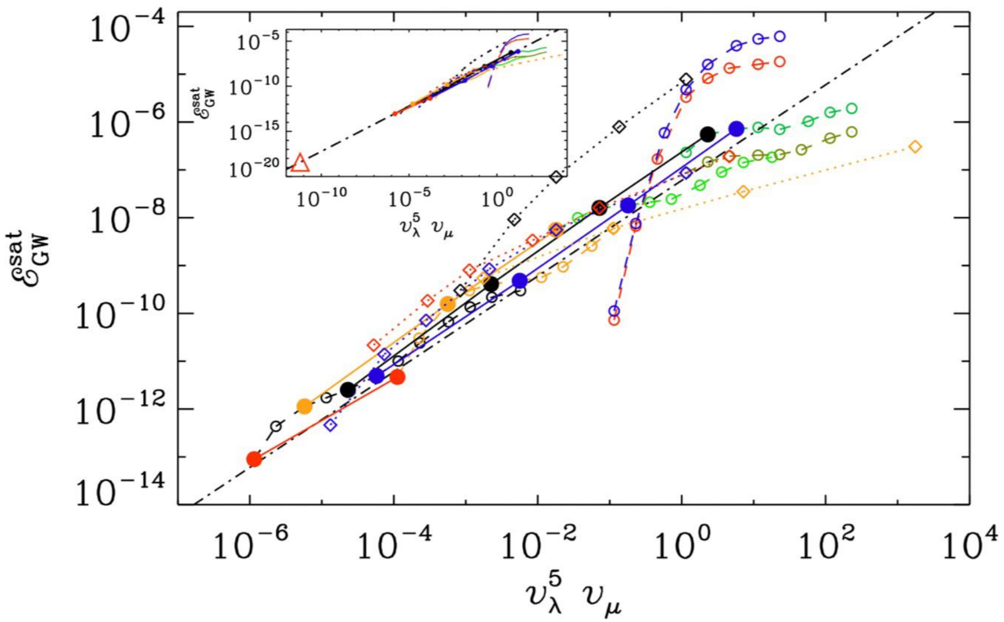
$$\tilde{h}(k,\,t) = \frac{6\tilde{T}_0(k)}{4\gamma_0^2 + k^2}\left[e^{2\gamma_0\tau} - \cos k\tau - \frac{2\gamma_0}{k}\sin k\tau\right]_{\tau=t-1}$$

Kinematic phase:
Peak determined
By growth rate

$$\gamma_0 = \eta\mu_{50}^2/4$$



# GW energy depends quintically on limiting CME speed $v_\lambda$




$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{u} \times \boldsymbol{B} + \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J})], \quad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$$

$$\frac{D\mu_5}{Dt} = -\lambda\, \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J}) \cdot \boldsymbol{B} + D_5 \nabla^2 \mu_5 - \Gamma_f \mu_5$$

$$v_\lambda = \mu_{50}/\lambda^{1/2}, \qquad v_\mu = \mu_{50}\eta.$$

$$\eta k_1 < v_\mu < v_\lambda \quad \text{(regime I)},$$
$$\eta k_1 < v_\lambda < v_\mu \quad \text{(regime II)},$$

Brandenburg+21

- For realistic parameters → very weak GW energy
- Need larger length scales

Magnetic energy also weak

$$\langle \boldsymbol{B}^2 \rangle\ \xi_M = \epsilon\, (k_B T_0)^3 (\hbar c)^{-2},$$

Brandenburg+17

# Inflationary magnetogenesis

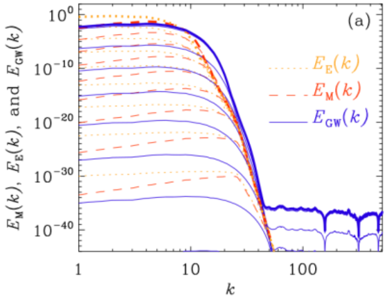
- Early Universe Turbulence
  - Source of gravitational waves
  - Information from young universe
- Magnetogenesis
  - Inflation/reheating
  - No particles yet, no conductivity
  - Coupling with electromagn field

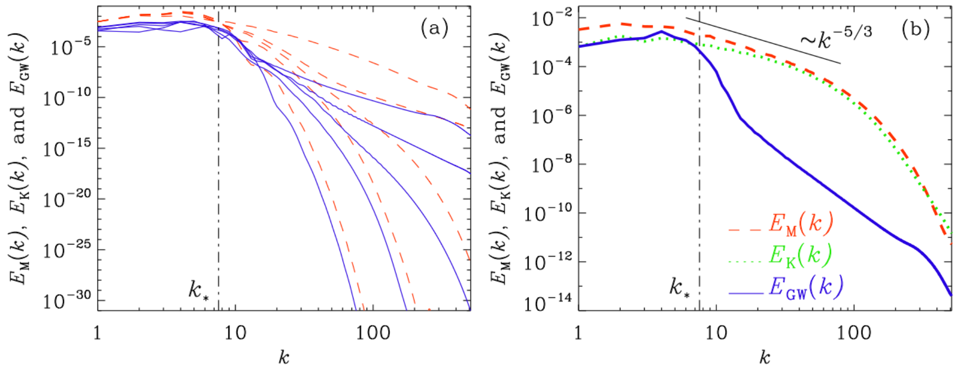
$$f^2 F_{\mu\nu} F^{\mu\nu}$$

  - Breaking of conformal invariance
  - Quantum fluct → field stretched

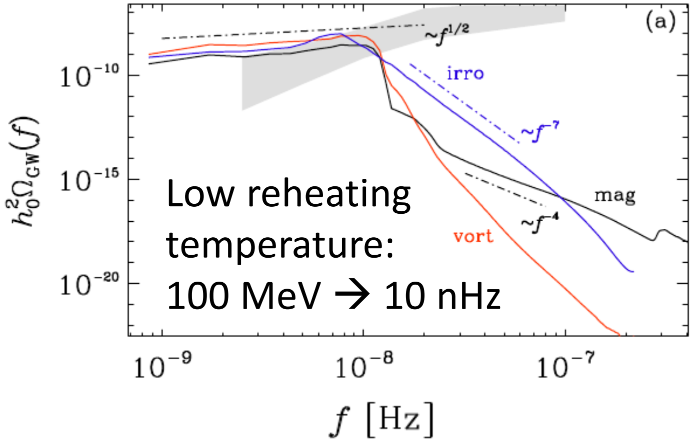
$$\tilde{\mathbf{A}}'' + \left(\mathbf{k}^2 - \frac{f''}{f}\right)\tilde{\mathbf{A}} = 0,$$

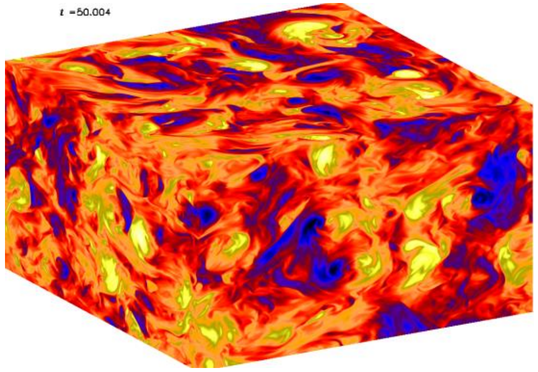
$$\tilde{h}''_{+/\times} + \left(\mathbf{k}^2 - \frac{a''}{a}\right)\tilde{h}_{+/\times} = \frac{6}{a}\tilde{T}_{+/\times},$$




Gravitational waves:



Code (Pencil Code Collaboration et al. 2021), where

$$\mathbf{A}' = \mathbf{u} \times \mathbf{B} + \sigma^{-1}\nabla^2\mathbf{A} \quad (8)$$

is solved in real space, which includes the induction effect from the velocity and the finite conductivity. It is solved together with (Brandenburg et al. 1996)

$$\mathbf{u}' = -\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{4}\nabla\ln\rho + \frac{3}{4\rho}\mathbf{J}\times\mathbf{B} + \mathcal{F}_\nu + \mathcal{F}, \quad (9)$$

$$(\ln\rho)' = -\frac{4}{3}(\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\ln\rho) + \mathcal{H}, \quad (10)$$

where $\mathcal{F} = (\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\ln\rho)\mathbf{u}/3 - [\mathbf{u}\cdot(\mathbf{J}\times\mathbf{B}) + \mathbf{J}^2/\sigma]\mathbf{u}/\rho$, and $\mathcal{H} = [\mathbf{u}\cdot(\mathbf{J}\times\mathbf{B}) + \mathbf{J}^2/\sigma]\rho$ are higher

*Software and Data Availability*. The source code used for the simulations of this study, the Pencil Code (Pencil Code Collaboration et al. 2021), is freely available on https://github.com/pencil-code/. The doi of the code is 10.5281/zenodo.2315093 (Brandenburg 2018). The simulation setup and the corresponding data are freely available from doi:10.5281/zenodo.4448211, see also http://www.nordita.org/~brandenb/projects/GWfromCME/ for easier access.

Brandenburg & Sharma 2106:03857

# Circular polarization in chiral inflationary magnetogenesis

$$\iota f^2 F_{\mu\nu}{}^*F^{\mu\nu}$$

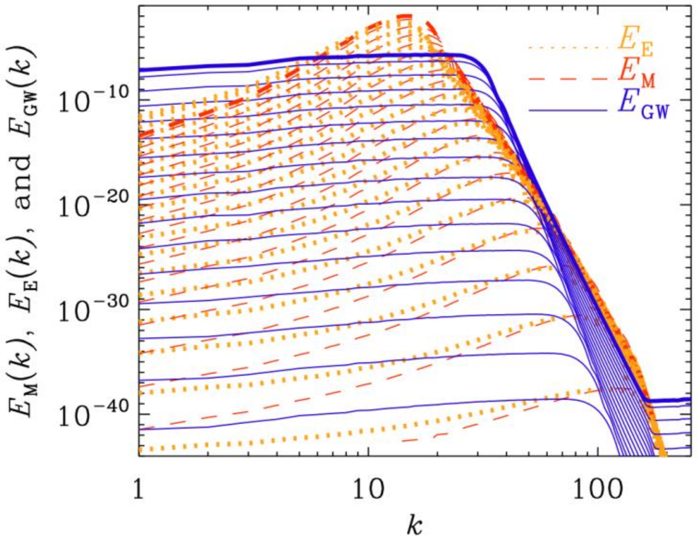
$$f(a) = a^{-\beta}, \quad \text{where } a = (\eta+1)^2/4$$

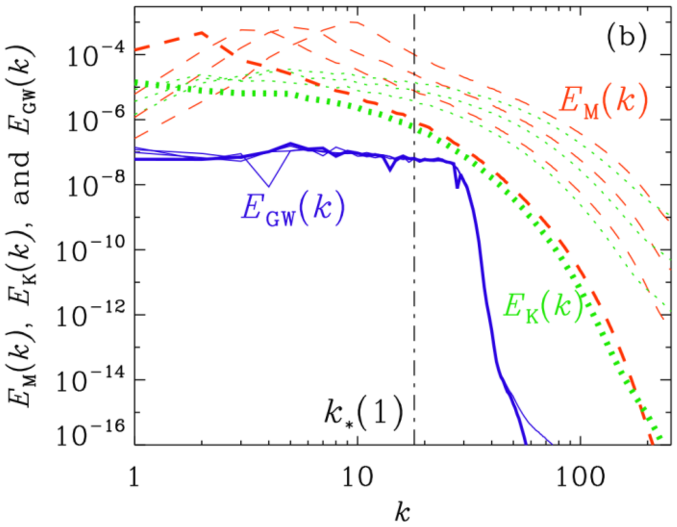
$$\tilde{A}''_\pm + \left(k^2 \pm 2\iota k \frac{f'}{f} - \frac{f''}{f}\right)\tilde{A}_\pm = 0,$$

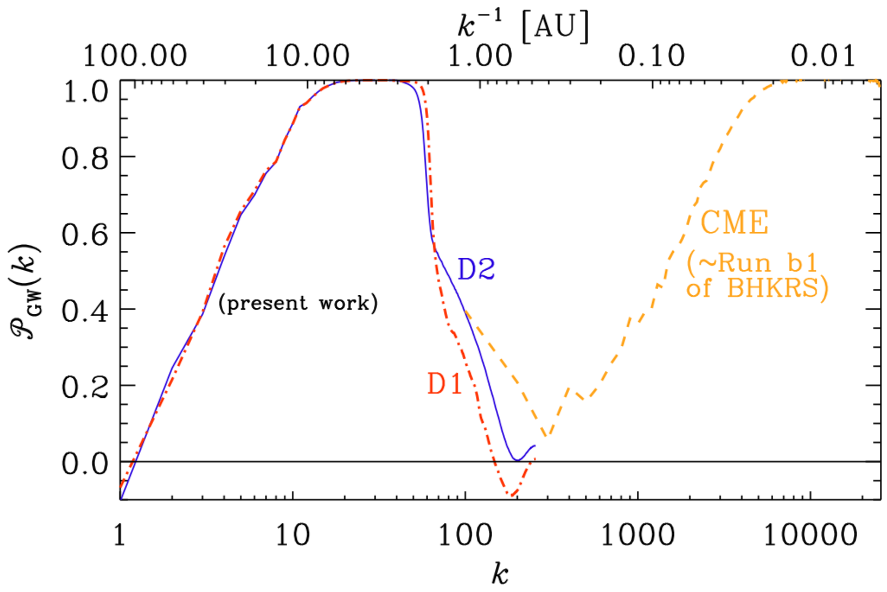
$$\frac{f'}{f} = -\frac{2\beta}{\eta+1}, \qquad \frac{f''}{f} = \frac{2\beta(2\beta+1)}{(\eta+1)^2}.$$

- Step I: spectra peaked
- Step II: Inverse cascade
- GW: circularly polarized

$$\mathcal{P}(k) = \int 2\,\mathrm{Im}\,\tilde{h}_+\tilde{h}_\times^*\,k^2\mathrm{d}\Omega_k \Big/ \int \left(|\tilde{h}_+|^2 + \tilde{h}_\times|^2\right) k^2\mathrm{d}\Omega_k$$

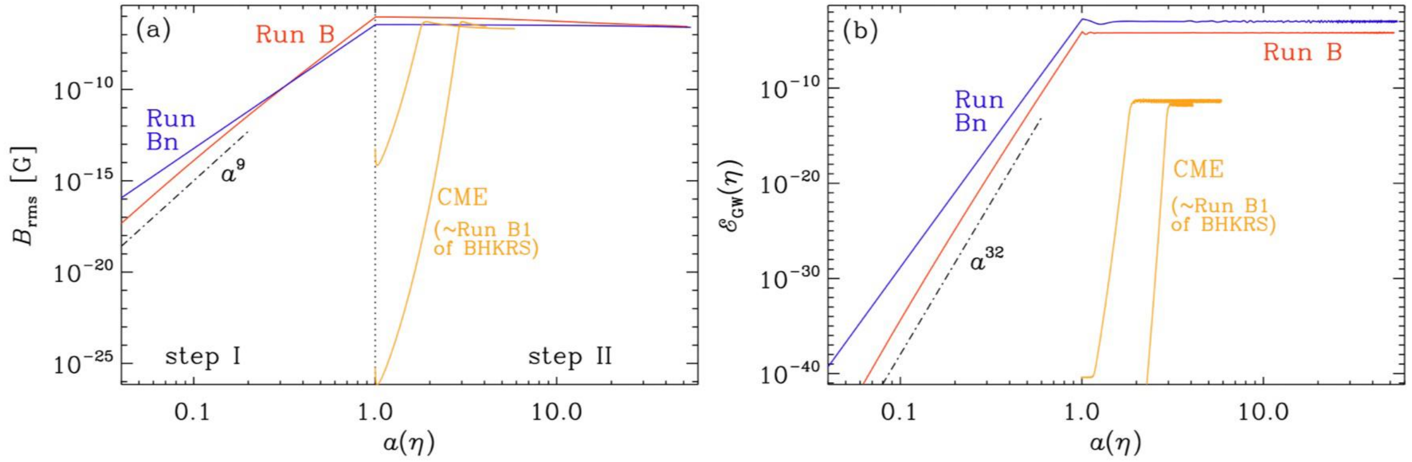


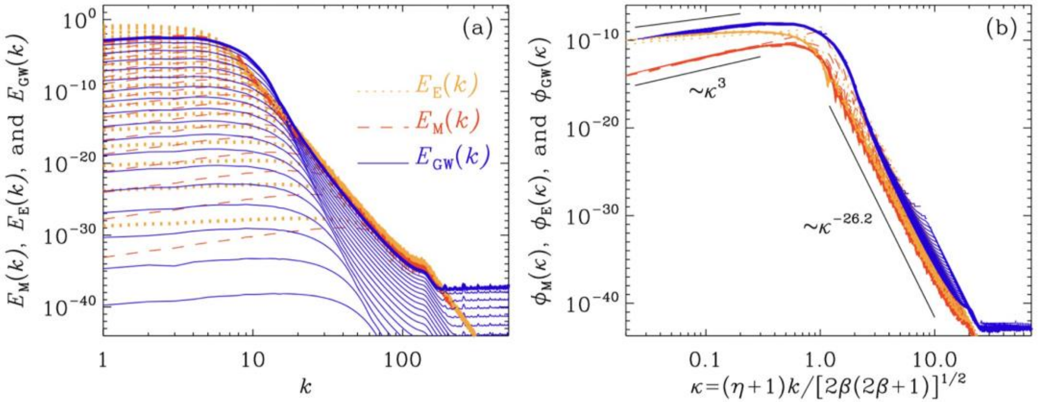




Helical field from CME or inflation: Always ~100% circular polarized

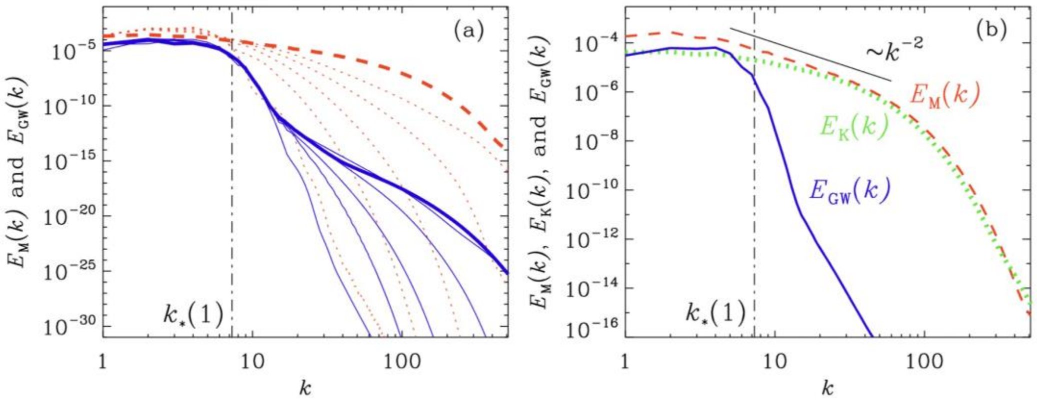
# Inflationary growth & magnetic decay




Inflationary growth: electric, magnetic, and GW grow




Lorentz force drives smaller scales: surprisingly weak



# Flow of energy in inflationary magnetogenesis

$f^2 F_{\mu\nu} F^{\mu\nu}$,

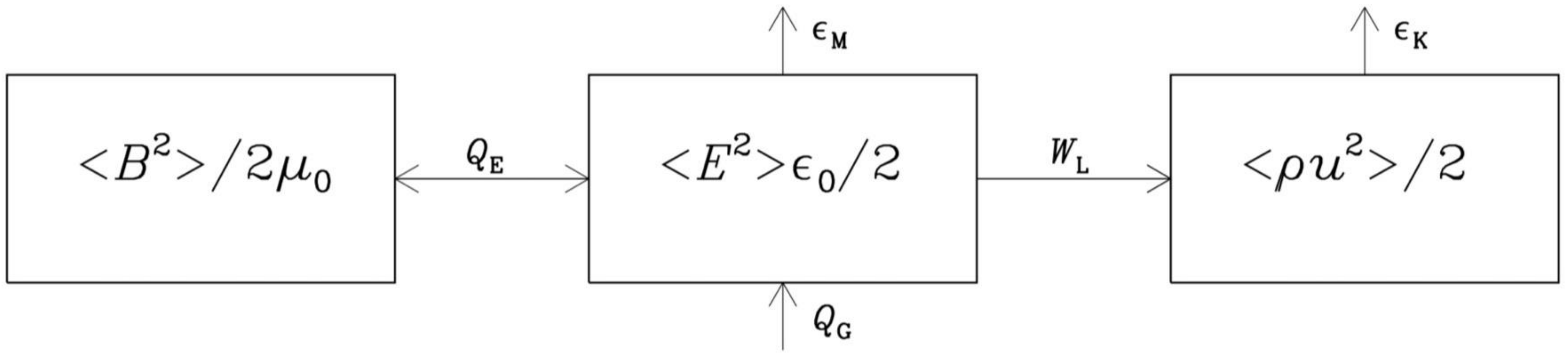
$$\frac{1}{c^2}\left(\frac{\partial \boldsymbol{E}}{\partial t} + 2\frac{f'}{f}\boldsymbol{E}\right) = \boldsymbol{\nabla} \times \boldsymbol{B} - \mu_0 \boldsymbol{J}$$

$$f \propto a^{-\beta} \propto t^{-2\beta}$$

$$f'/f = -2\beta/t$$

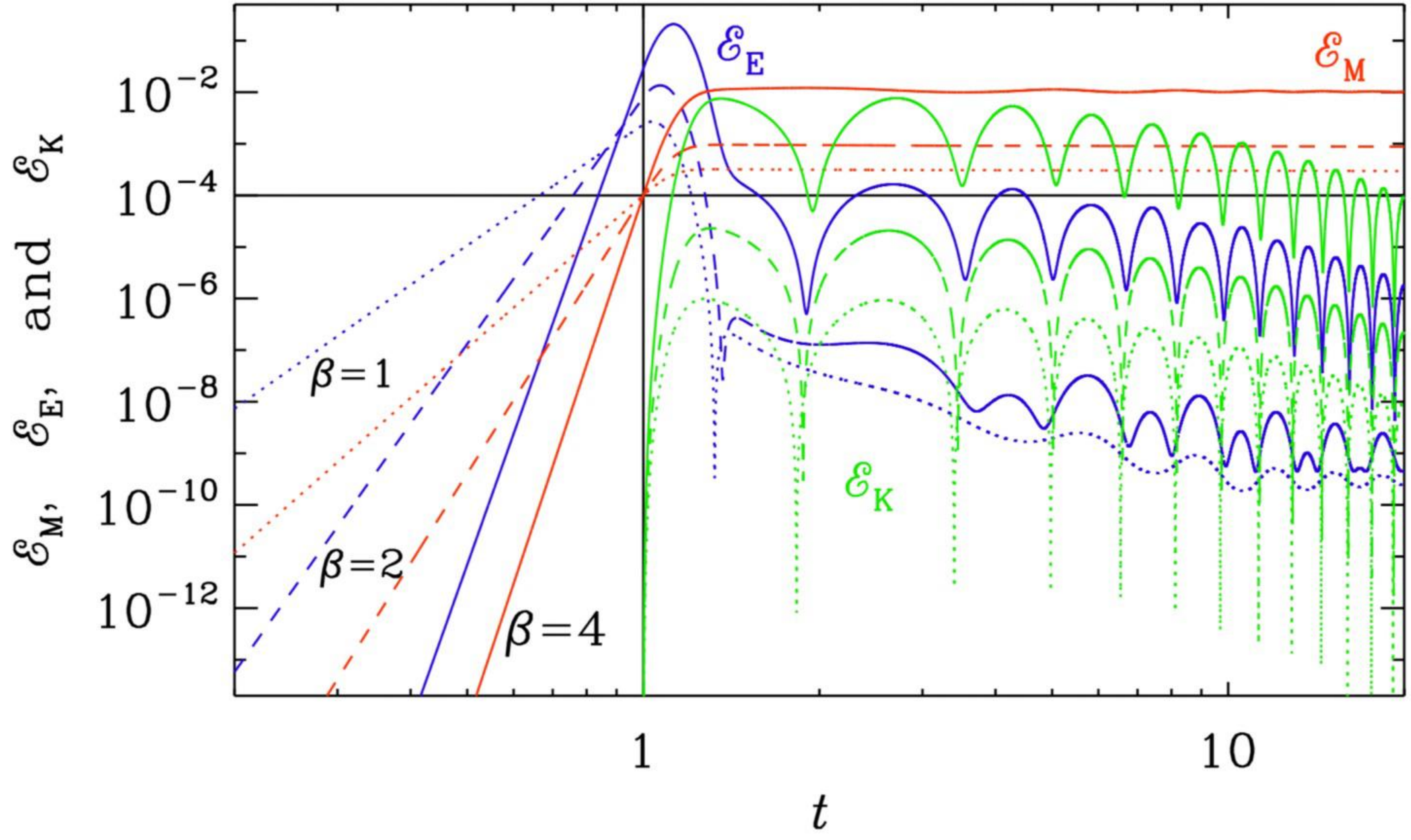
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \epsilon_0 \boldsymbol{E}^2/2 \right\rangle = -2(f'/f)\left\langle \epsilon_0 \boldsymbol{E}^2 \right\rangle + \langle \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}/\mu_0 \rangle - \langle \boldsymbol{J} \cdot \boldsymbol{E} \rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \boldsymbol{B}^2/2\mu_0 \right\rangle = -\langle \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{B} \rangle/\mu_0$$

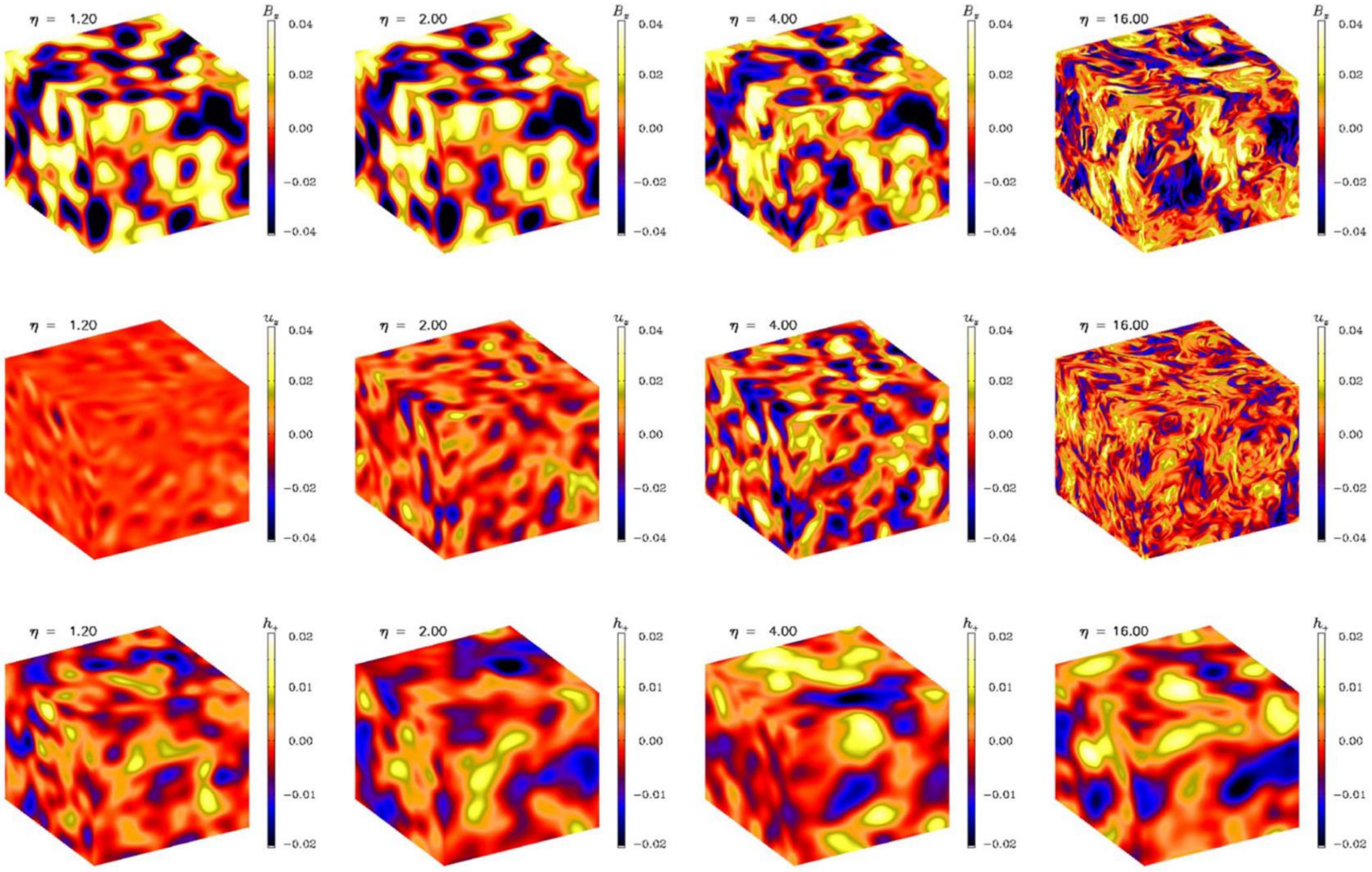



$$Q_G = -4(f'/f)\mathcal{E}_E$$

# Electric fields converted to kinetic energy



# No small scales in GW field



# Note on the Pencil Code

- 2001 started at Summer School
- 2004 First User Meeting
  - Annually since then
- 2016 Steering Committee
- 2020 Special Issue in GAFD
- 2020 Newletter
  - 4 newsletters since then
- 2020 Office hours
  - Second Friday of the month
- JOSS=Journal for Open Source Software: code rather than paper

Open code: will one be scooped?
Negative press? Mistakes traced back..

The Journal of Open Source Software


**DOI:** 10.21105/joss.02807

**Software**
- Review
- Repository
- Archive

**Editor:** Arfon Smith
**Reviewers:**
- @zingale
- @rtfisher

**Submitted:** 17 September 2020
**Published:** 21 February 2021




## The Pencil Code, a modular MPI code for partial differential equations and particles: multipurpose and multiuser-maintained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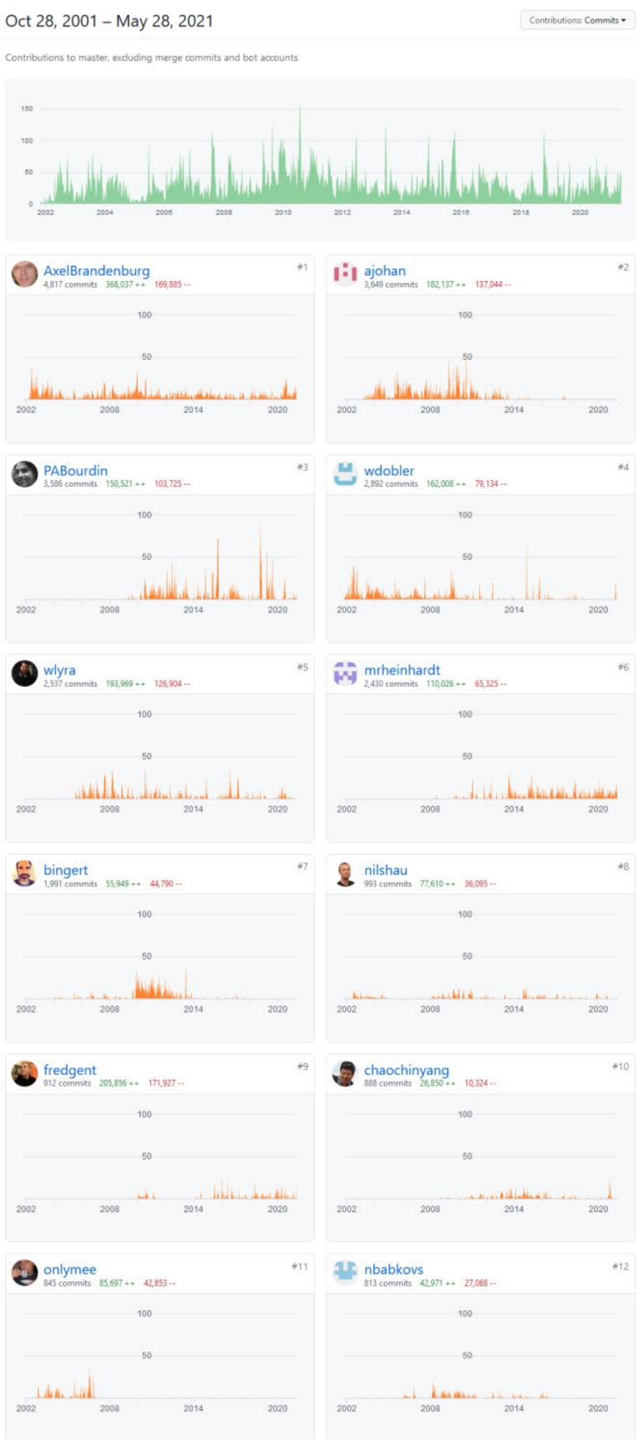
### Summary


The Pencil Code is a highly modular physics-oriented simulation code that can be adapted to a wide range of applications. It is primarily designed to solve partial differential equations (PDEs) of compressible hydrodynamics and has lots of add-ons ranging from astrophysical magnetohydrodynamics (MHD) (A. Brandenburg & Dobler, 2010) to meteorological cloud microphysics (Li et al., 2017) and engineering applications in combustion (Babkovskaia et al., 2011). Nevertheless, the framework is general and can also be applied to situations not related to hydrodynamics or even PDEs, for example when just the message passing interface or input/output strategies of the code are to be used. The code can also evolve Lagrangian (inertial and noninertial) particles, their coagulation and condensation, as well as their interaction with the fluid. A related module has also been adapted to perform ray tracing


H=37 people have done > 37 commits



# Conclusions

- Selfsimilar decay
  - Magnetic helicity plays a role even when it vanishes on average!
  - Hosking integral conserved relevant for early universe
  - Perhaps also for galaxy clusters (after mergers)
- Universe as a whole → primordial (non-astrophysical) fields
  - Decay till recombination: ~ nG fields, 30 kpc scales at best
  - If nonhelical: Hosking integral conserved
  - Also applies to fully helical, if balanced by fermion chirality
- Inflationary: large scales, often helical
  - Electric energy → kinetic energy
  - Circularly polarized waves
- What next?
  - Reconnection
  - Rm dependence
  - magnetic helicity fluxes