Cusp anomalous dimension with massive lines through four loops in QCD

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- **Cusp anomalous dimension**
- **Asymptotics at large and small velocities**
- **Approximate expressions at four loops**

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Cusp anomalous dimension

controls the infrared behavior of perturbative QCD scattering amplitudes

simplest soft anomalous dimension in QCD

an essential ingredient of all calculations of soft anomalous dimensions for processes with more complicated color structures

Wilson or eikonal lines - ordered exponentials

the path is a straight line in the direction of the parton four-velocity v

$$W(\lambda_2,\lambda_1;x) = P\exp\left(-ig\int_{\lambda_1}^{\lambda_2} d\lambda\, v\cdot A(\lambda v + x)\right)$$

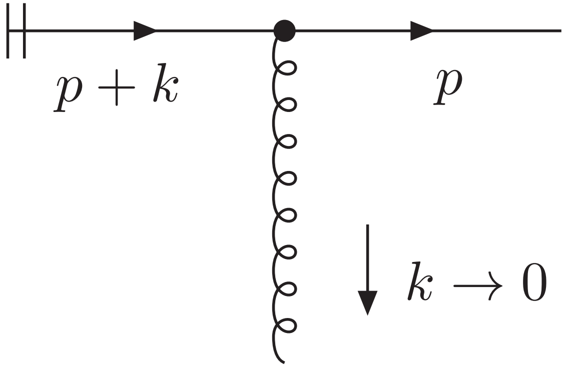
cusp angle $\theta = \cosh^{-1}(v_1 \cdot v_2/\sqrt{v_1^2 v_2^2})$

$v_1 \cdot v_2 = 1+\beta^2$ **and** $v_1^2 = v_2^2 = 1-\beta^2$**, where** $\beta = \sqrt{1-4m^2/s}$ **is the quark speed**

Thus, $\theta = \ln[(1+\beta)/(1-\beta)]$ **and** $\beta = \tanh(\theta/2)$

range $0 \le \theta < \infty$ **corresponds to** $0 \le \beta < 1$

Eikonal approximation



$$\bar{u}(p)\,(-ig_s T_F^c)\,\gamma^\mu\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon} \to \bar{u}(p)\,g_s T_F^c\,\gamma^\mu\,\frac{\not{p}+m}{2p\cdot k+i\epsilon} = \bar{u}(p)\,g_s T_F^c\,\frac{v^\mu}{v\cdot k+i\epsilon}$$

with $p \propto v$

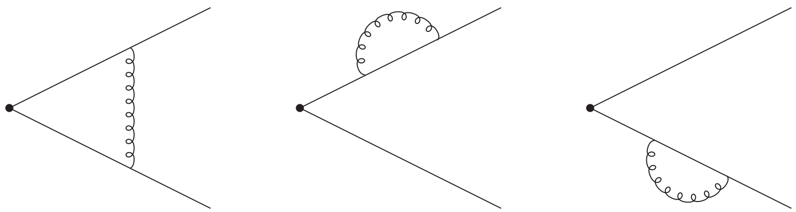
The cusp anomalous dimension at each order can be read off the coefficients of the ultraviolet poles of the corresponding eikonal loop diagrams

Perturbative series for $\Gamma_{\rm cusp}$

$$\Gamma_{\rm cusp} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \Gamma^{(n)}$$

One-loop cusp anomalous dimension **[Polyakov 1980]**

$$\Gamma^{(1)} = C_F(\theta \coth\theta - 1)$$



Noting that $\coth\theta = (1+\beta^2)/(2\beta)$, **we define**

$$L_\beta = \frac{(1+\beta^2)}{2\beta} \ln\left(\frac{1-\beta}{1+\beta}\right)$$

Then, the one-loop cusp anomalous dimension written as a function of β **is**

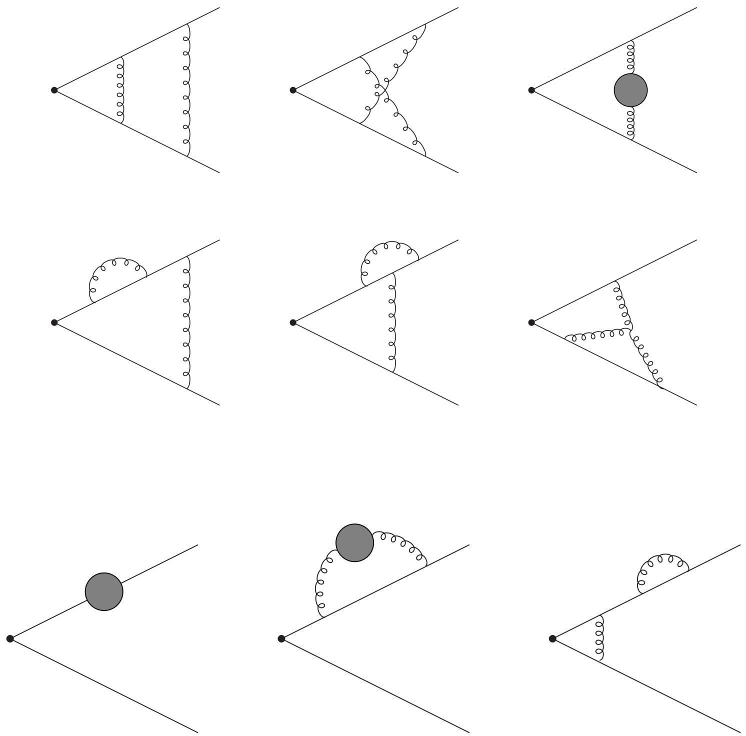
$$\Gamma^{(1)} = -C_F \left(L_\beta + 1\right)$$

At two loops, first attempt [Knauss and Scharnhorst 1984]
in terms of double and triple integrals

later two-loop result [Korchemsky and Radyushkin 1986]
in terms of three unevaluated single integrals

Fully analytical result at two loops [NK 2009] **derived independently**

in terms of β and also reexpressed in terms of θ



Two-loop cusp anomalous dimension [NK 2009]

$$
\begin{aligned}
\Gamma^{(2)} \quad = \quad & K_2\,\Gamma^{(1)} + C_F C_A \left\{ \frac{1}{2} + \frac{\zeta_2}{2} + \frac{1}{2}\ln^2\left(\frac{1-\beta}{1+\beta}\right) \right. \\
& + \frac{(1+\beta^2)}{4\beta}\left[\zeta_2 \ln\left(\frac{1-\beta}{1+\beta}\right) - \ln^2\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{3}\ln^3\left(\frac{1-\beta}{1+\beta}\right) - \mathrm{Li}_2\left(\frac{4\beta}{(1+\beta)^2}\right)\right] \\
& + \frac{(1+\beta^2)^2}{8\beta^2}\left[-\zeta_3 - \zeta_2 \ln\left(\frac{1-\beta}{1+\beta}\right) - \frac{1}{3}\ln^3\left(\frac{1-\beta}{1+\beta}\right) \right. \\
& \left.\left. - \ln\left(\frac{1-\beta}{1+\beta}\right)\mathrm{Li}_2\left(\frac{(1-\beta)^2}{(1+\beta)^2}\right) + \mathrm{Li}_3\left(\frac{(1-\beta)^2}{(1+\beta)^2}\right)\right]\right\}
\end{aligned}
$$

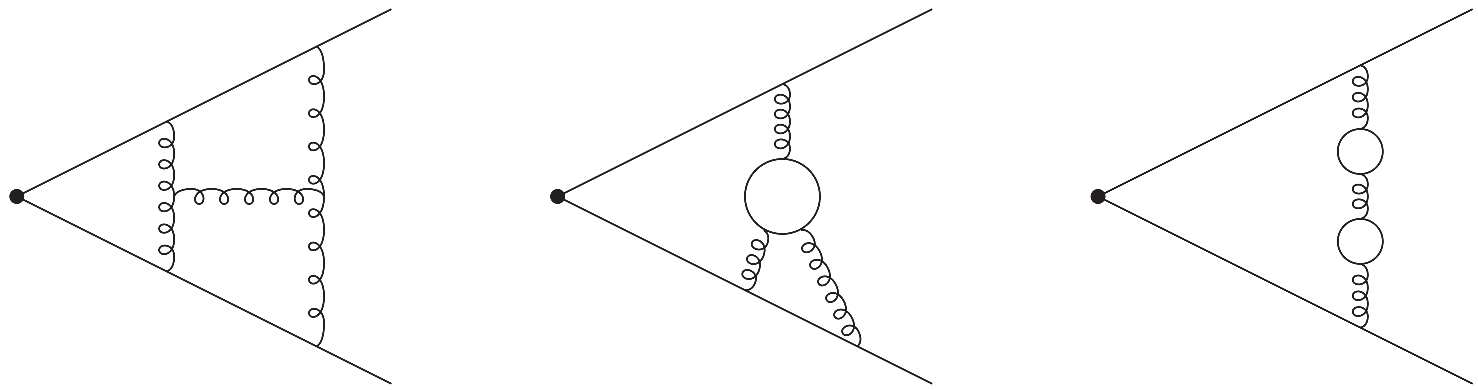
where $K_2 = C_A(67/36 - \zeta_2/2) - 5n_f/18$

In terms of θ [NK 2009]

$$
\begin{aligned}
\Gamma^{(2)} \quad = \quad & K_2\,\Gamma^{(1)} + C_F C_A \left\{ \frac{1}{2} + \frac{\zeta_2}{2} + \frac{\theta^2}{2} - \frac{1}{2}\coth\theta\left[\zeta_2\theta + \theta^2 + \frac{\theta^3}{3} + \mathrm{Li}_2\left(1-e^{-2\theta}\right)\right] \right. \\
& \left. + \frac{1}{2}\coth^2\theta\left[-\zeta_3 + \zeta_2\theta + \frac{\theta^3}{3} + \theta\,\mathrm{Li}_2\left(e^{-2\theta}\right) + \mathrm{Li}_3\left(e^{-2\theta}\right)\right]\right\}
\end{aligned}
$$

Three-loop cusp anomalous dimension [Grozin, Henn, Korchemsky, Marquard 2015]

some representative diagrams



Further study and reexpressions in [NK 2016, 2023]

$$\Gamma^{(3)} \quad = \quad K_3 \Gamma^{(1)} + 2K_2 \left(\Gamma^{(2)} - K_2 \Gamma^{(1)}\right) + C^{(3)}$$

where K_3 and $C^{(3)}$ have long expressions

Large θ and β asymptotics of $\Gamma_{\rm cusp}$

Large-θ behavior of $\Gamma_{\rm cusp}$

$$\lim_{\theta\to\infty} \Gamma^{(n)} = A^{(n)} \lim_{\theta\to\infty} \theta + R_n$$

where $A^{(n)} = C_F K_n$ is the lightlike cusp anomalous dimension and known through four loops

Large-β behavior of $\Gamma_{\rm cusp}$

$$\lim_{\beta\to 1} \Gamma^{(n)} = K_n \lim_{\beta\to 1} \Gamma^{(1)} + P_n = -C_F K_n \lim_{\beta\to 1} \ln\left(\frac{1-\beta}{2}\right) + R_n$$

where $R_n = P_n - C_F K_n$, and the constants P_n at one, two, and three loops are given by

$P_1 = 0$,

$P_2 = (1/2) C_F C_A (1-\zeta_3)$,

and

$$P_3 = K_2 C_F C_A (1-\zeta_3) + C_F C_A^2 \left(-\frac{1}{2} + \frac{3}{4}\zeta_2 - \frac{\zeta_3}{4} + \frac{9}{8}\zeta_5 - \frac{3}{4}\zeta_2\zeta_3\right)$$

Small β and θ asymptotics of $\Gamma_{\rm cusp}$

Small-β expansion of $\Gamma_{\rm cusp}$ through four loops

$$\Gamma^{(n)} = \Gamma^{(n)}_{\beta^2} + \Gamma^{(n)}_{\beta^4} + \mathcal{O}(\beta^6)$$

and we find at one loop

$$\Gamma^{(1)}_{\beta^2} = \frac{4}{3} C_F \beta^2 \qquad \qquad \Gamma^{(1)}_{\beta^4} = \frac{8}{15} C_F \beta^4$$

and at two loops

$$\Gamma^{(2)}_{\beta^2} = \beta^2 \left[C_F C_A \left(\frac{94}{27} - \frac{4}{3}\zeta_2 \right) - \frac{20}{27} C_F n_f T_F \right]$$

$$\Gamma^{(2)}_{\beta^4} = \beta^4 \left[C_F C_A \left(\frac{64}{45} - \frac{8}{15}\zeta_2 \right) - \frac{8}{27} C_F n_f T_F \right]$$

longer expressions at three loops [NK 2016, 2023]

recently derived at four loops in [NK 2023]
based on the small-θ expansions in [Grozin, Lee, Pikelner 2022]

$$\Gamma^{(n)} = \Gamma^{(n)}_{\theta^2} + \Gamma^{(n)}_{\theta^4} + \mathcal{O}(\theta^6)$$

using $\theta = 2\beta + (2/3)\beta^3 + \mathcal{O}(\beta^5)$

Approximate expression for $\Gamma_{\rm cusp}$ from asymptotics

Expressions for $\Gamma_{\rm cusp}$ from asymptotics at two loops [NK 2009], **three loops** [NK 2016], **and four loops** [NK 2023]

start with the small-β expansion of $\Gamma_{\rm cusp}$, with $\Gamma^{(n)}_{\beta^{2,4}}=\Gamma^{(n)}_{\beta^2}+\Gamma^{(n)}_{\beta^4}$, then add $K_n\Gamma^{(1)}$ and subtract its small-β expansion

$$\Gamma_A^{(n)}=\Gamma^{(n)}_{\beta^{2,4}}-K_n\,\Gamma^{(1)}_{\beta^{2,4}}+K_n\,\Gamma^{(1)}$$

The last two terms on the right cancel against each other at small β

The first two terms on the right cancel against each other at large β

Equivalently, $$\Gamma_A^{(n)}=\Gamma^{(n)}_{\beta^{2,4}}-C_F K_n\left(\frac{4}{3}\beta^2+\frac{8}{15}\beta^4+L_\beta+1\right)$$

The approximation works due to the small range of β

It does not work using θ expansions due to the infinite range of θ

Of course, the approximate results in β can later be reexpressed in terms of θ

Approximate expression for $\Gamma_{\rm cusp}$ from asymptotics

Results at each perturbative order through four loops

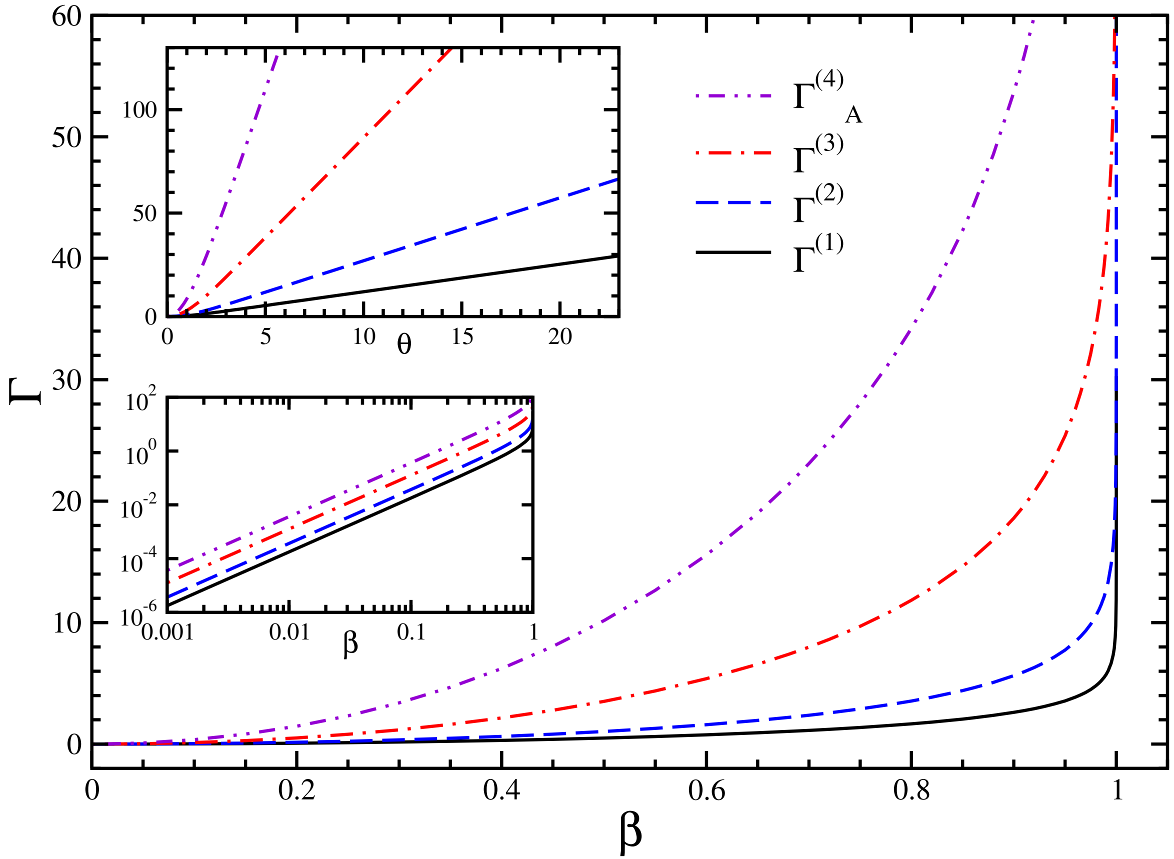
$$\Gamma_A^{(1)} = \Gamma^{(1)}$$

$$\Gamma_A^{(2)} = -0.386490845\,\beta^2 - 0.036077819\,\beta^4 + (3.115932233 - 0.277777778\,n_f)\,\Gamma^{(1)}$$

$$\begin{aligned}\Gamma_A^{(3)} = {} & (-0.981370903 + 0.214717136\,n_f)\,\beta^2 + (-0.141381392 + 0.020043233\,n_f)\,\beta^4 \\ & + (13.76833912 - 2.146727700\,n_f - 0.009259259\,n_f^2)\,\Gamma^{(1)}\end{aligned}$$

$$\begin{aligned}\Gamma_A^{(4)} = {} & (-3.749290323 + 1.186688634\,n_f - 0.022664587\,n_f^2)\,\beta^2 \\ & + (-0.290594150 + 0.156331101\,n_f - 0.002115675\,n_f^2)\,\beta^4 \\ & + (60.65142489 - 15.15209803\,n_f + 0.572980154\,n_f^2 + 0.009586947\,n_f^3)\,\Gamma^{(1)}\end{aligned}$$

Cusp anomalous dimension $n_f = 3$

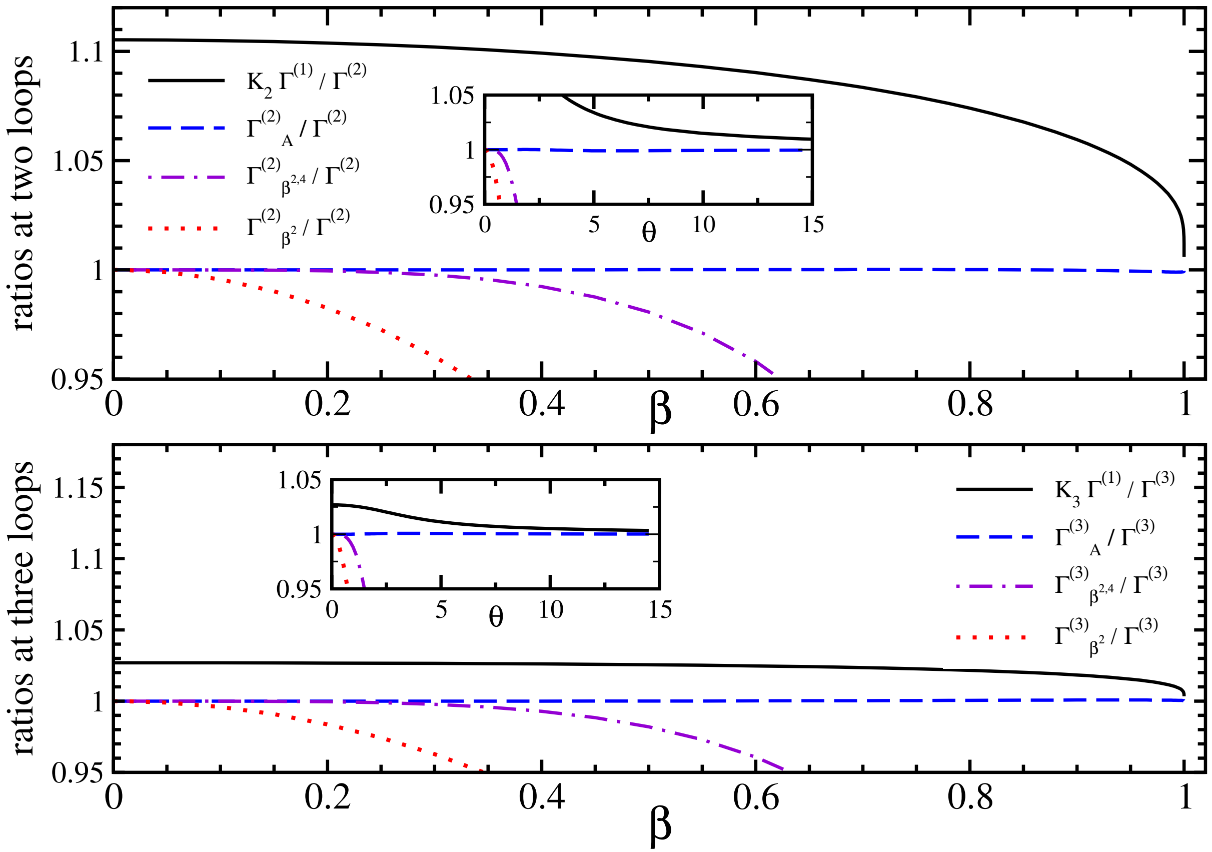


three alternative ways to plot the result

$$\theta = \ln\left(\frac{1+\beta}{1-\beta}\right)$$

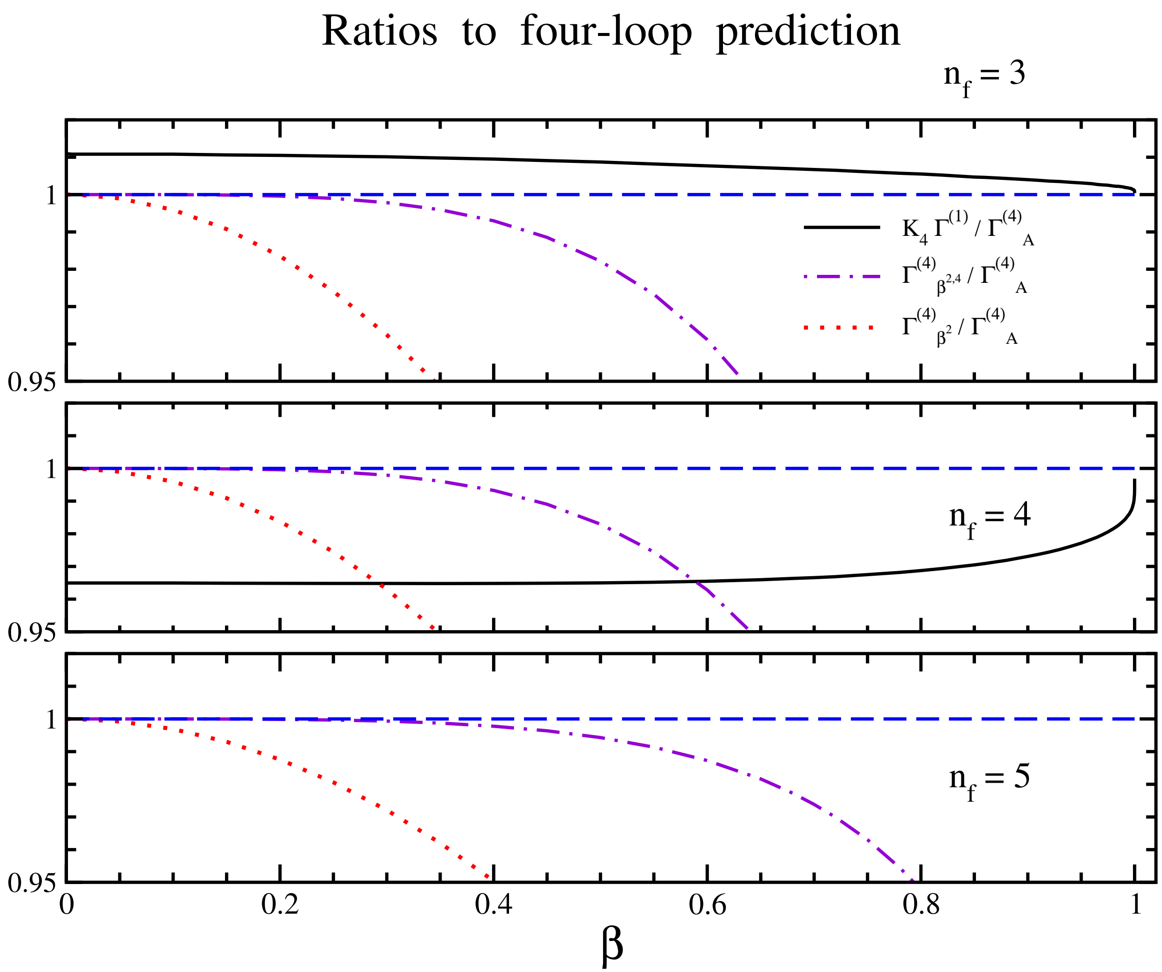
$\beta = 0$ **corresponds to** $\theta = 0$ **while** $\beta = 0.99999$ **corresponds to** $\theta \approx 12.2$

Ratios to exact results for $n_f = 3$



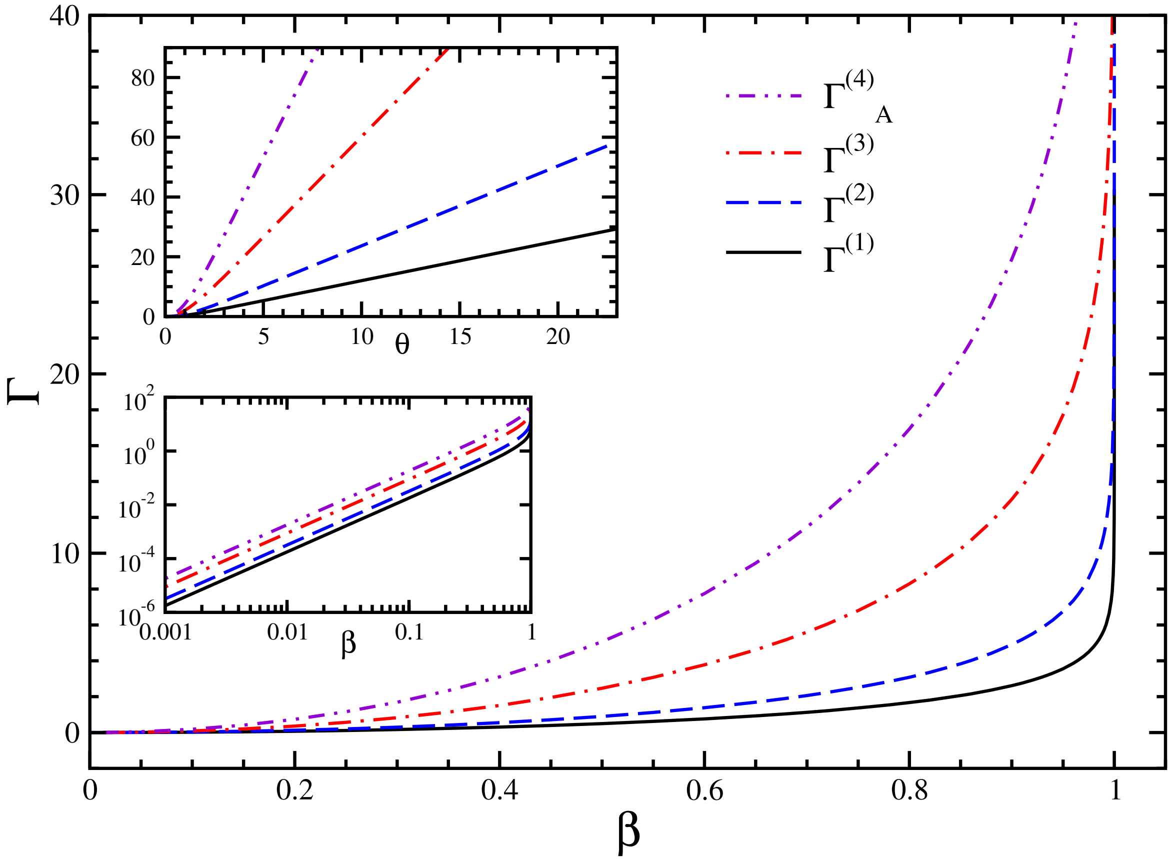
The difference between $\Gamma^{(2)}_A$ and $\Gamma^{(2)}$ is less than 1 per mille over the entire β range from 0 to 1. It is less than one part per million from $\beta = 0$ up to $\beta \approx 0.17$; and better than 0.1 per mille for $\beta = 0$ to $\beta \approx 0.6$, and $\beta = 0.8$ to $\beta = 0.9$, and for $\beta \approx 1$.

The difference between $\Gamma^{(3)}_A$ and $\Gamma^{(3)}$ is well below 1 per mille everywhere. It is less than one part per million from $\beta = 0$ to $\beta \approx 0.16$; and better than 0.1 per mille from $\beta = 0$ to $\beta \approx 0.5$, and for $\beta \approx 1$.



behavior of the small-β expansion at four loops
is similar to the two-loop and three-loop cases

Cusp anomalous dimension $n_f = 4$

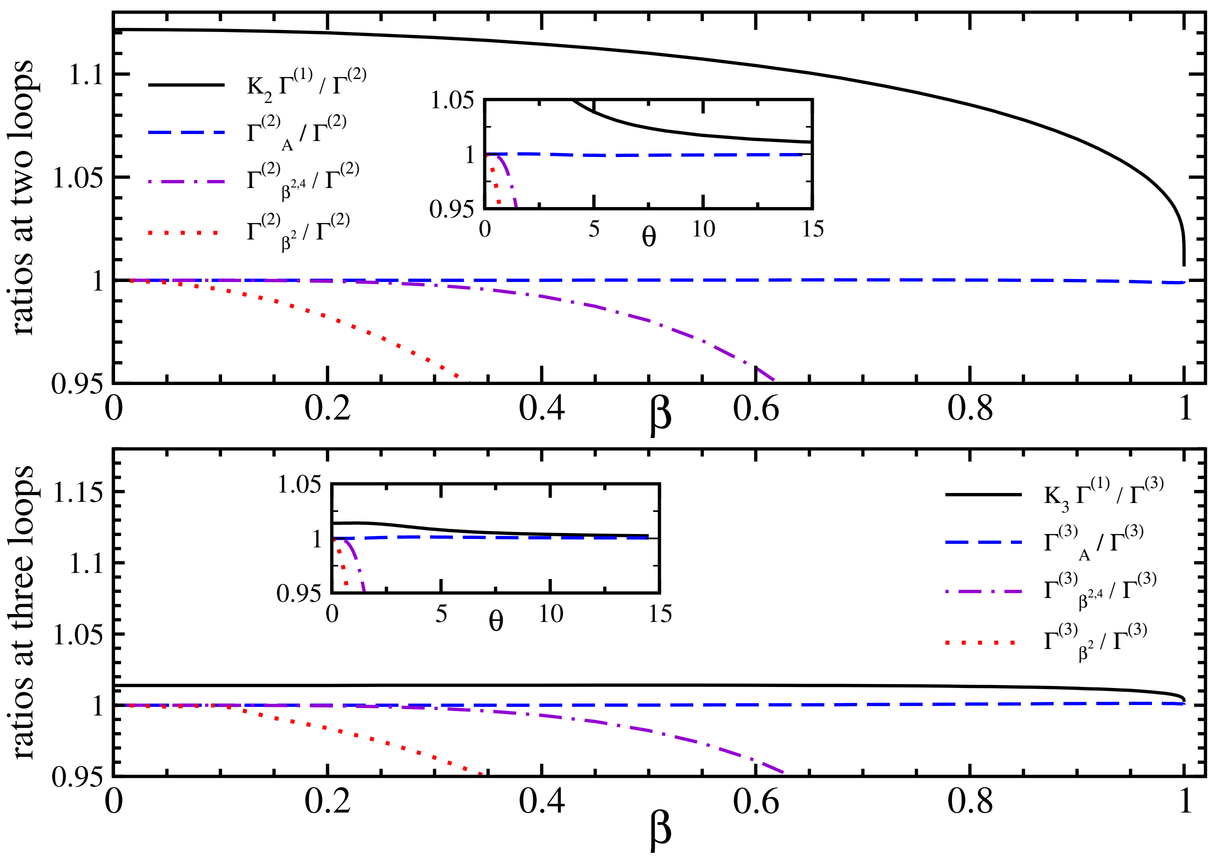


three alternative ways to plot the result

$$\theta = \ln\left(\frac{1+\beta}{1-\beta}\right)$$

$\beta = 0$ **corresponds to** $\theta = 0$ **while** $\beta = 0.99999$ **corresponds to** $\theta \approx 12.2$

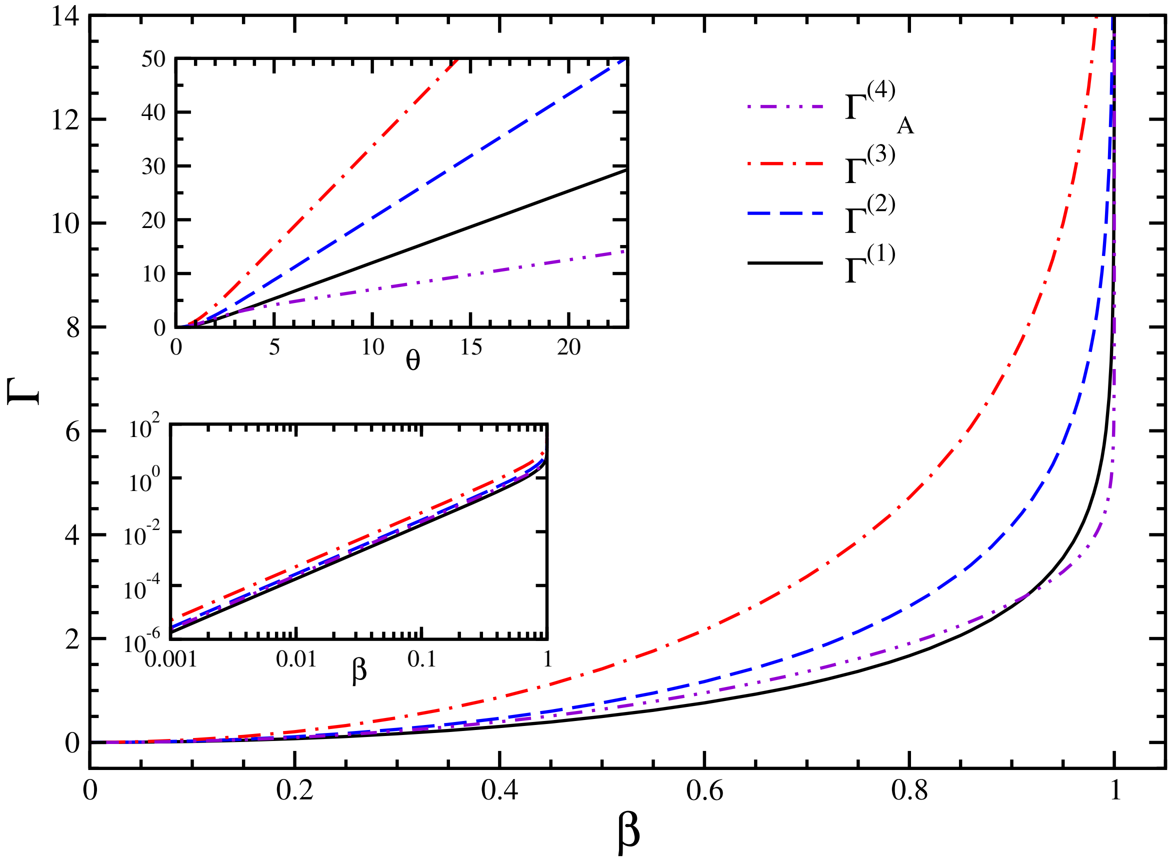
Ratios to exact results for $n_f = 4$



The difference between $\Gamma_A^{(2)}$ and $\Gamma^{(2)}$ is 1 per mille or less over the entire β range from 0 to 1. It is less than one part per million from $\beta = 0$ up to $\beta \approx 0.16$, and 0.1 per mille or better for $\beta = 0$ to $\beta \approx 0.6$, and $\beta = 0.8$ and $\beta = 0.9$, and for $\beta \approx 1$

The difference between $\Gamma_A^{(3)}$ and $\Gamma^{(3)}$ is 1 per mille or better everywhere. It is less than one part per million from $\beta = 0$ up to $\beta \approx 0.15$, and 0.1 per mille or better from $\beta = 0$ to $\beta \approx 0.5$ and for $\beta \approx 1$.

Cusp anomalous dimension $n_f = 5$

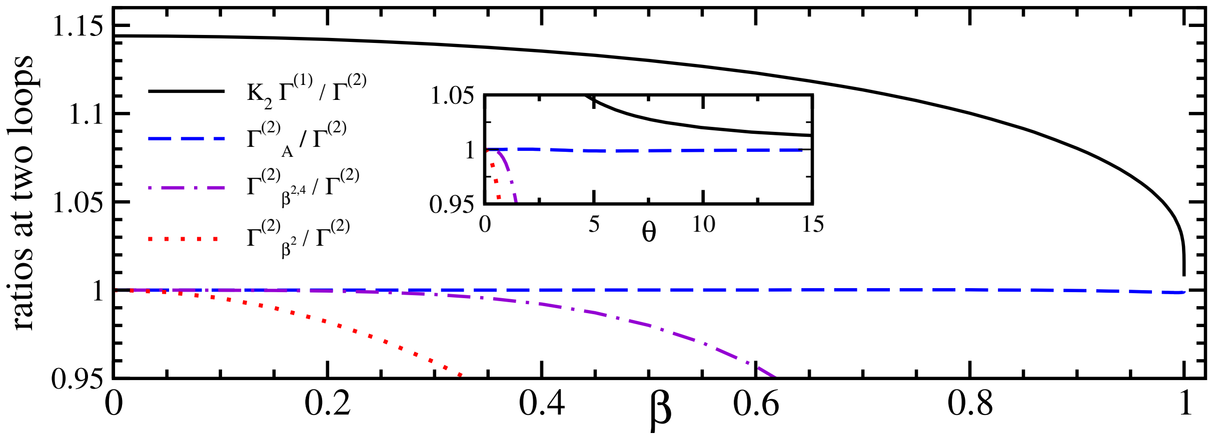


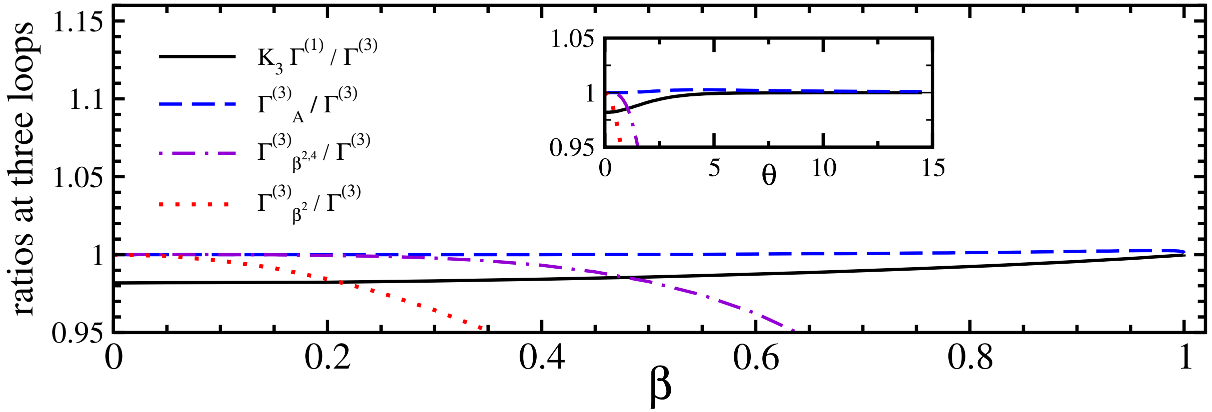
three alternative ways to plot the result

$$\theta = \ln\left(\frac{1+\beta}{1-\beta}\right)$$

$\beta = 0$ **corresponds to** $\theta = 0$ **while** $\beta = 0.99999$ **corresponds to** $\theta \approx 12.2$

Ratios to exact results for $n_f = 5$





The difference between $\Gamma_A^{(2)}$ and $\Gamma^{(2)}$ is 1 per mille or less over the entire β range from 0 to 1. It is less than one part per million from $\beta = 0$ to $\beta \approx 0.16$, and better than 0.1 per mille for $\beta = 0$ to above $\beta \approx 0.5$, and $\beta = 0.8$ to $\beta = 0.9$, and for $\beta \approx 1$

The difference between $\Gamma_A^{(3)}$ and $\Gamma^{(3)}$ is well below 3 per mille everywhere. It is less than one part per million from $\beta = 0$ to $\beta \approx 0.14$, and 0.1 per mille or better from $\beta = 0$ to $\beta \approx 0.5$ and for $\beta \approx 1$

$\Gamma_{\rm cusp}$ with other values of n_f

calculated the cusp anomalous dimension for integer values of n_f from 0 to 10

results are remarkably consistent, in that the formula always provides an excellent approximation to the exact results at two and three loops, throughout the β range

robust results at four loops

Extensions of the expressions and method

One obvious extension is to include more (or fewer) terms in the small-β expansion:

$$\Gamma_A^{(n)} = \Gamma^{(n)}_{\rm small-\beta} - K_n\, \Gamma^{(1)}_{\rm small-\beta} + K_n\, \Gamma^{(1)}$$

additional terms have negligible impact

Another possible extension is to include further exact results (in addition to the exact terms already present) for some color structures and/or other combinations of terms (when those are known) in the approximate expression.

For example, at three loops we can include the full two-loop results in our expression and only have a small-β expansion in $C^{(3)}$; i.e., we could consider the alternative expression

$$2K_2\left(\Gamma^{(2)} - K_2\Gamma^{(1)}\right) + C^{(3)}_{\beta^2,4} + K_3\Gamma^{(1)}$$

This, again, makes a negligible difference over the entire β range, at the level of parts per million for much of it, with details depending on the number of flavors.

The method is also clearly applicable to higher numbers of loops, and it could be utilized when the necessary information becomes available. For example, for a five-loop prediction, we would need to know the small-β expansion of the cusp anomalous dimension at five loops as well as the result for the lightlike K_5.

Further study of color structures

Study of the approximation separately for each color structure in the cusp anomalous dimension at each perturbative order.

At two loops, the $C_F C_A$ terms are not exact in $\Gamma_A^{(2)}$ while the $C_F n_f$ terms are exact. The approximation from asymptotics just for the $C_F C_A$ terms alone gives excellent agreement with the exact result for those terms, better than 1 per mille everywhere in the β range, and much smaller than that for most of the range.

At three loops, the $C_F^2 n_f$ and the $C_F n_f^2$ terms are exact in $\Gamma_A^{(3)}$, but the $C_F C_A^2$ and $C_F C_A n_f$ terms are not exact. The approximation from asymptotics provides excellent agreement with the exact result for both the $C_F C_A^2$ and $C_F C_A n_f$ terms, within a fraction of one per mille everywhere in the β range, smaller than 0.1 per mille for the majority of the β range, and smaller than one part per million at small speeds.

At four loops, the $C_F^3 n_f$, $C_F^2 n_f^2$, and $C_F n_f^3$ terms in $\Gamma_A^{(4)}$ are exact, but all the rest of the terms in $\Gamma_A^{(4)}$, i.e. the $C_F C_A^3$, $C_F^2 C_A n_f$, $C_F C_A^2 n_f$, $C_F C_A n_f^2$, $d_F d_F$, and $d_F d_A$ terms, are not exact.

Exact (conjectured) results for $C_F^2 C_A n_f$ terms and $C_F C_A n_f^2$ terms in $\Gamma^{(4)}$ are in superb agreement with my results from asymptotics. The difference is at the level of parts per million up to $\beta \approx 0.3$, less than 0.03 per mille for the vast majority of the β range, and less than a small fraction of one per mille (0.3 per mille for $C_F^2 C_A n_f$, and 0.2 per mille for $C_F C_A n_f^2$) for all β.

Furthermore, even though the $d_F d_F$ exact results are very complicated, one can investigate further known terms of this color structure at small speeds. We find that the contribution of the β^6 terms in the small-β expansion of the $d_F d_F$ color structure at four loops do not materially change the four-loop prediction: a difference of less than one part per million for much of the β range, and everywhere less than 0.02 per mille for $n_f = 3$, 0.05 per mille for $n_f = 4$, and 0.7 per mille for $n_f = 5$. Once again, this highlights the robustness of our approach and the reliability of our method.

Finally, we can also investigate the effect of including the exact form of the conjectured $C_F^2 C_A n_f$ and $C_F C_A n_f^2$ terms in our four-loop expression. Again, we find remarkable robustness in our method, consistent with all the previous checks. The difference between the results is negligible, of the order of parts per million for much of the β range (with exact numbers depending on the number of flavors) and at the level of per mille for the entirety of the β range.

Thus, the four-loop result is very robust and precise, and the inclusion of any future exact results or more terms in the small-β expansion would make very little numerical difference.

Summary

- cusp anomalous dimension at higher loops
- asymptotics at large and small β
- an approximate formula through four loops
- studies for various n_f and separate color structures
- robust and precise results
- method can be extended in a number of directions and to higher loops