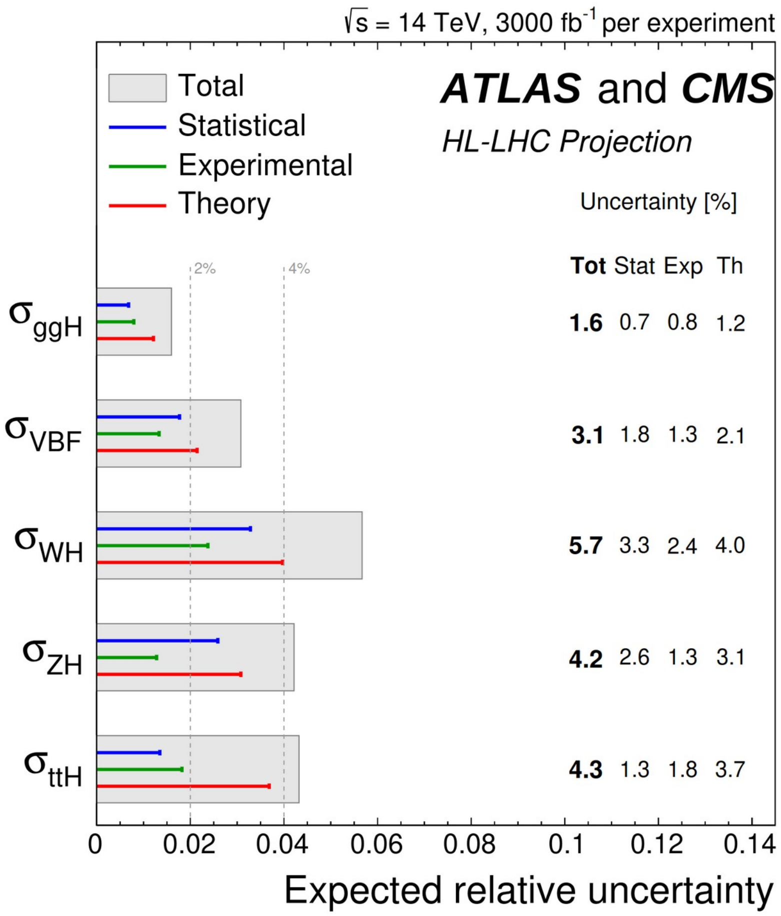
# A first look at two-loop master integrals for ttH

**Manfred Kraus**
In Collaboration: F. Febres Cordero, G. Figueiredo, B. Page, L. Reina
LoopFest 2024, SMU, Dallas
20. May 2024

# Motivation




[Yellow Report, CERN-2019-007]

$t\bar{t}H$ allows for a direct measurement of

$$y_t = \frac{\sqrt{2} m_t}{v}$$

Exp. Precision: $\mathcal{O}(20\%) \to \mathcal{O}(2\%)$

Approximate NNLO:

| $\sigma$ [pb] | $\sqrt{s} = 13\,\text{TeV}$ | $\sqrt{s} = 100\,\text{TeV}$ |
|---|---|---|
| $\sigma_{\text{LO}}$ | $0.3910\,^{+31.3\%}_{-22.2\%}$ | $25.38\,^{+21.1\%}_{-16.0\%}$ |
| $\sigma_{\text{NLO}}$ | $0.4875\,^{+5.6\%}_{-9.1\%}$ | $36.43\,^{+9.4\%}_{-8.7\%}$ |
| $\sigma_{\text{NNLO}}$ | $0.5070\,(31)^{+0.9\%}_{-3.0\%}$ | $37.20(25)\,^{+0.1\%}_{-2.2\%}$ |

[Catani et al., arXiv:2210.07846]

The missing ingredient:

Two-Loop Virtual Amplitudes

# Status of NNLO 2 -> 3 with tops

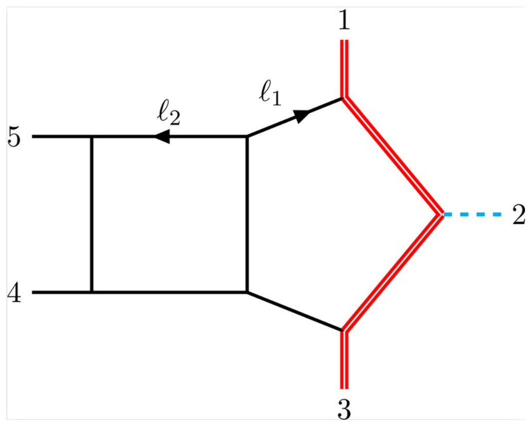
$pp \to t\bar{t}j$

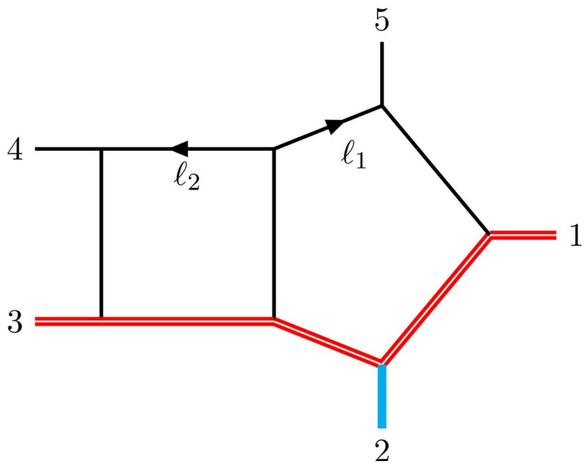
- One-loop Amplitudes at $@ \, \mathcal{O}(\epsilon^2)$
  [Badger, Bechetti, Chaubey, Marzucca, Sarandrea, arXiv:2201.12188]
- A planar two-loop integral family
  [Badger, Bechetti, Chaubey, Marzucca, arXiv:2210.17477]
- Leading-color two-loop integrals
  [Badger, Bechetti, Giraudo, Zoia, arXiv:2404.12325]

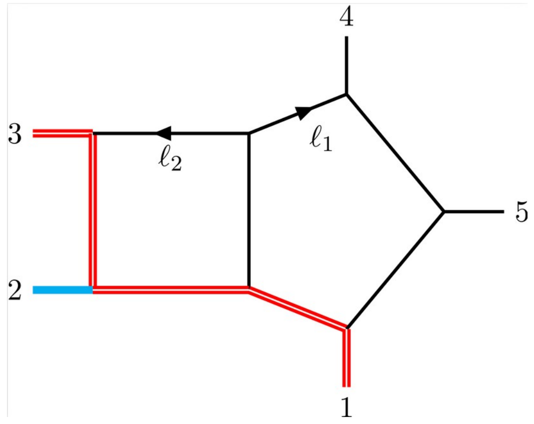
$pp \to t\bar{t}H$

- IR divergences of two-loop amplitudes
  [Chen, Ma, Wang, Yang, Ye, arXiv:2202.02913]
- Two-loop amplitudes in soft Higgs/boosted limit
  [Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini, arXiv:2210.07846] [Wang, Xia, Yang, Ye, arXiv:2401.07632]
- Two-loop planar master integrals for light-quark loop
  [Febres Cordero, Figueiredo, MK, Page, Reina, arXiv:2312.08131]
- One-loop $gg \to t\bar{t}H \; @ \, \mathcal{O}(\epsilon^2)$
  [Buccioni, Kreer, Liu, Tancredi, arXiv:2312.10015]
- Two-loop $q\bar{q} \to t\bar{t}H$ amplitudes $N_f$ contribution
  [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson, arXiv:2402.03301]

# Two-loop ttH @ Leading Color




111 MIs

Nested root

156 MIs

More nested roots ...

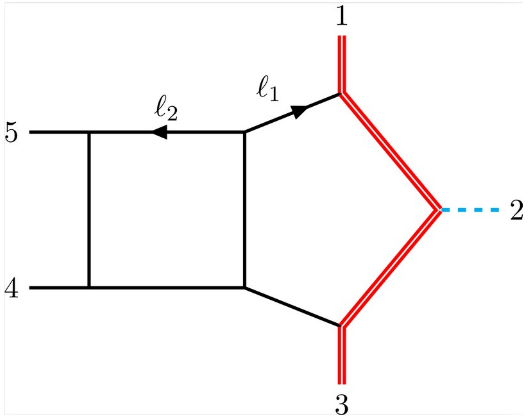
Elliptic sectors

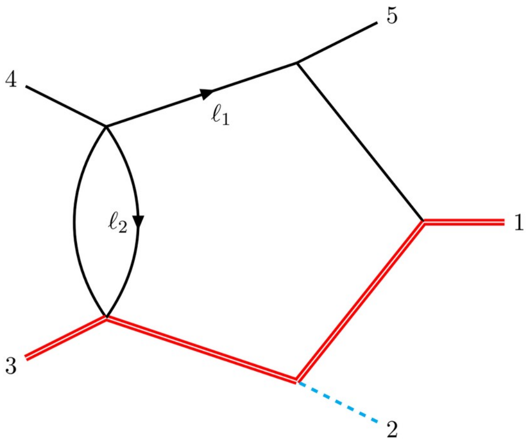
186 MIs

Even more nested roots ...

Even more elliptics ...

# Two-loop ttH @ Leading Color




111 MIs

Nested root

19 MIs

No suprises :)

All Master Integrals for Leading-Color + closed light fermion for ttH

# Scattering Kinematics

Channel: $g(p_4)g(p_5) \to t(p_1)H(p_2)\bar{t}(p_3)$

Five-point, two massless legs, two on-shell tops, generic Higgs

$$p_1^2 = p_3^2 = m_t^2\,, \qquad p_2^2 = q^2\,, \qquad p_4^2 = p_5^2 = 0$$

7 independent kinematic invariants

$$\vec{s} = \{v_{12}, v_{23}, v_{34}, v_{45}, v_{15}, m_t^2, q^2\}, \qquad v_{ij} = 2(p_i \cdot p_j)$$

11 square roots:

6 – Gram determinants 2 – mod. Cayleys 3 – Maximal cut

2 nested roots:

$$\sqrt{N_\pm} = \sqrt{q^2\left(N_b \pm \sqrt{N_b^2 - N_c}\right)}$$

$N_b$ degree 3 polynomial in 5 variables

# General Strategy

1. Setup differential equations for system of master integrals

$$\frac{d\vec{I}}{dx} = B_x \, \vec{I}$$

[Kotikov, ‘91; Remiddi ‘97; Gehrman, Remiddi ‘01]

2. Find a basis in $\epsilon$-factorized form

$$\vec{J} = T\vec{I} \qquad \frac{d\vec{J}}{dx} = \epsilon A_x \vec{J}$$

[Henn ‘13]

3. Rewrite differential matrix in terms of basis of one-forms

$$A_x = \frac{d}{dx} \sum_\alpha M_\alpha \mathrm{d} \log(\omega_\alpha)$$

4. Numerical solutions via series expansions

[Moriello ‘19; Hidding ‘20, Liu; Ma ‘22]

# Construction of $\epsilon$-factorized Basis

Construction of a canonical Basis is non-trivial!

**Our Approach:**

1. Find initial choice of masters such that

$$B(\epsilon, \vec{s}) = B^{(0)}(\vec{s}) + \epsilon B^{(1)}(\vec{s})$$

2. Change basis $\vec{J} = T\vec{I}$ , where T satisfies homogeneous DE

$$dT_{ij} = B^{(0)}_{ik}\, T_{kj}$$

3. If necessary compute subtraction terms for off-shell corrections by explicit integration

Computation in Finite-Fields – Analytics only where necessary!

# The Easy Sectors

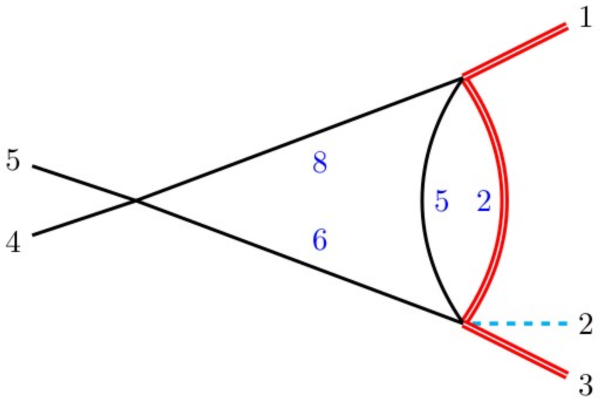
$$B^{(0)} = \begin{pmatrix} \star & 0 & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{pmatrix}$$

1. First remove the diagonal elements via

$$T_{ii} = \exp\left(-\int dx B_{ii}^{(0)}\right)$$

2. Then remove off-diagonal elements

$$T_{ij} = -\int dx B_{ij}^{(0)}$$

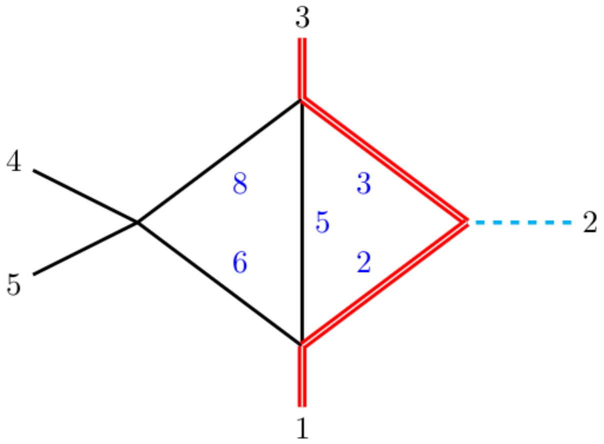



$$\mathcal{N}_{31}^{(1)} = \epsilon^3 \sqrt{\Delta_3^{(3)}} \frac{1}{\rho_2},$$

$$\mathcal{N}_{32}^{(1)} = \epsilon^3 \sqrt{\Delta_3^{(3)}} \frac{1}{\rho_5},$$

$$\mathcal{N}_{33}^{(1)} = \epsilon^2 \left[ \frac{m_t^2 v_{45}}{\rho_2 \rho_6} + \epsilon(q^2 + v_{23} - v_{45}) \left( \frac{1}{\rho_2} + \frac{1}{2\rho_5} \right) \right]$$

# The fun-stuff: Kite




This sector has 7 master integrals

Using the previous methods we can reduce the problem to a coupled 3x3 system

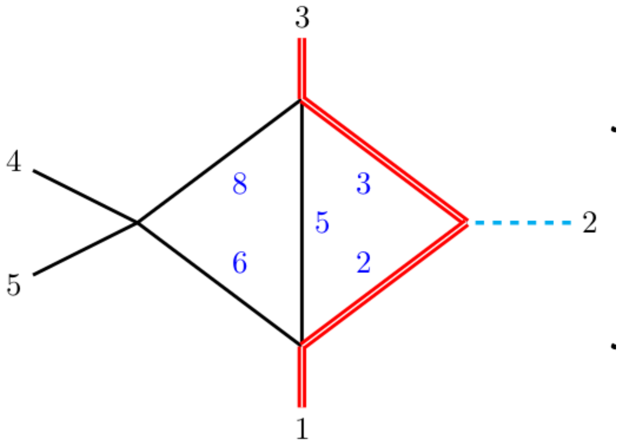
$$B^{(0)} = \begin{pmatrix} 0 & \star & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix}$$

Needs Magnus expansion of entire 3x3 system!

$$dT = \begin{pmatrix} 0 & \frac{1}{4} d\log(A) & 0 \\ \frac{1}{4} d\log(A) & 0 & 0 \\ x & y & 0 \end{pmatrix} T$$

A - algebraic

# The fun-stuff: Kite




$$dT = \begin{pmatrix} 0 & \frac{1}{4} d\log(A) & 0 \\ \frac{1}{4} d\log(A) & 0 & 0 \\ x & y & 0 \end{pmatrix} T$$

A - algebraic

Solution introduces a nested root in the integral basis

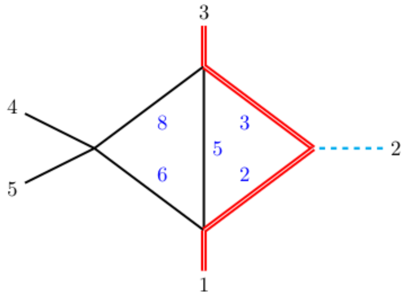
$$\sqrt{N_\pm} = \sqrt{q^2 \left( N_b^2 \pm \sqrt{N_b^2 - N_c} \right)}$$

$$N_b = q^2 \left[ (v_{14} + v_{15})^2 + (v_{34} + v_{35})^2 \right] - 2m_t^2 (v_{24} + v_{25})^2$$

$$N_c = q^2 (q^2 - 4m_t^2)(v_{12} - v_{23})^2 (v_{24} + v_{25} + 2v_{45})^2$$

New analytic structure!

# The fun-stuff: Kite




$$\mathcal{N}_{60}^{(1)} = \epsilon^4 \sqrt{r_1}\,,$$

$$\mathcal{N}_{61}^{(1)} = \epsilon^3 v_{45} \sqrt{\Delta_3^{(2)}}\,\frac{1}{\rho_8}\,,$$

$$\mathcal{N}_{62}^{(1)} = \epsilon^3 v_{45} \sqrt{\Delta_3^{(1)}}\,\frac{1}{\rho_6}\,,$$

$$\mathcal{N}_{63}^{(1)} = \epsilon^3 \sqrt{\mathcal{C}_2}\,\frac{1}{\rho_5}\,,$$

$$\mathcal{N}_{64}^{(1)} = \epsilon^3 \left[\frac{\sqrt{N_+}}{2}\left(\frac{1}{\rho_3} - \frac{1}{\rho_2}\right) + \frac{\sqrt{\mathcal{C}_1}\sqrt{N_-}}{2q^2}\left(\frac{1}{\rho_3} + \frac{1}{\rho_2}\right)\right],$$

$$\mathcal{N}_{65}^{(1)} = \epsilon^3 \left[\frac{\sqrt{N_-}}{2}\left(\frac{1}{\rho_3} - \frac{1}{\rho_2}\right) + \frac{\sqrt{\mathcal{C}_1}\sqrt{N_+}}{2q^2}\left(\frac{1}{\rho_3} + \frac{1}{\rho_2}\right)\right],$$

$$\begin{aligned}
\mathcal{N}_{66}^{(1)} = {} & \epsilon^2 \frac{m_t^2 v_{45}(q^2+v_{12})(q^2+v_{23})}{2q^2+v_{12}+v_{23}}\left(\frac{1}{\rho_2\rho_6} + \frac{1}{\rho_3\rho_8}\right) \\
& + \epsilon^3 \left(C_{66}^{(1)}\frac{1}{\rho_5} + C_{66}^{(2)}\frac{1}{\rho_6} + C_{66}^{(3)}\frac{1}{\rho_8}\right) \\
& + \epsilon^3 \left(C_{66}^{(4)}\left(\frac{1}{\rho_3} + \frac{1}{\rho_2}\right) + C_{66}^{(5)}\left(\frac{1}{\rho_3} - \frac{1}{\rho_2}\right)\right) \\
& + C_{66}^{(6)}\left[\rho_2\,\mathcal{N}_{36}^{(1)} - \rho_3\,\mathcal{N}_{33}^{(1)} + \rho_2\rho_8\,\mathcal{N}_{3}^{(1)}\right] + C_{66}^{(7)}\left[\rho_5\,\mathcal{N}_{28}^{(1)}\right] \\
& + C_{66}^{(8)}\left[\rho_6\,\mathcal{N}_{26}^{(1)} - \rho_8\,\mathcal{N}_{13}^{(1)}\right] + C_{66}^{(9)}\left[\rho_3\rho_6\,\mathcal{N}_{10}^{(1)} + \rho_2\rho_8\,\mathcal{N}_{2}^{(1)}\right] \\
& + C_{66}^{(10)}\left[\rho_3\rho_6\,\mathcal{N}_{9}^{(1)}\right].
\end{aligned}$$

# Reconstruction of DEs

We follow the random direction DE approach: [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

$$(\vec{r}\cdot\nabla_{\vec{s}})\vec{I} = C(\epsilon,\vec{s})\vec{I}\,, \qquad C(\epsilon,\vec{s}) = \epsilon\sum_{\alpha} M_\alpha(\vec{r}\cdot\nabla_{\vec{s}})\omega_\alpha$$

From which we obtain the number of linear independent basis functions:

$$\mathcal{C} = \{C(\epsilon,\vec{s}_1),\ldots,C(\epsilon,\vec{s}_N)\} \quad \to \quad \mathrm{Rank}[\mathcal{C}] = \min\{N,\dim(\omega)\} \to \dim(\omega) = 152$$

Now we can build an Ansatz Matrix:

$$W_{\alpha n} \equiv (\vec{r}\cdot\nabla_{\vec{s}})\omega_\alpha(\vec{s})\Big|_{\vec{s}=\vec{s}_n}$$

And fit only the coefficient matrices of the entire DE:

$$M_\alpha = \frac{1}{\epsilon}\sum_{n=1}^{152} W^{-1}_{\alpha n}\cdot C(\epsilon,\vec{s}_n)$$

Problem reduced to computation of basis functions

# Alphabet

PentaBox Topology has 152 letters

- Much larger than single family for $pp \to Wjj$
- 9 Letters are irrelevant -> only arise at @ $\mathcal{O}(\epsilon^5)$

122 relevant letters without nested roots

| Mass dimension | 1 | 2 | 3 | 4 | 5 | 6 | $\Sigma$ |
|---|---|---|---|---|---|---|---|
| # Polynomial Letters | 19 | 10 | 8 | 5 | 0 | 1 | 43 |
| # Algebratic, single odd | 14 | 12 | 15 | 0 | 0 | 1 | 42 |
| # Algebratic, double odd | 5 | 21 | 10 | 1 | 0 | 1 | 37 |

Remaining 21 relevant letters depend on nested roots!

# Letters with nested roots

Many entries can be put into dLog form

$$d\log\left(\frac{q^2[v_{45}+s_{13}-q^2]-\sqrt{N_+}}{q^2[v_{45}+s_{13}-q^2]+\sqrt{N_+}}\right), \quad d\log\left(\frac{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)+\sqrt{r_1}\sqrt{N_+}}{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)-\sqrt{r_1}\sqrt{N_+}}\right)$$

We were unable to find dLog forms in 4 cases generated by

$$\omega^E = \frac{\Omega^E}{m_t^2(q^2-v_{23})\sqrt{G(p_2,p_3)}\sqrt{N_+}\sqrt{N_b^2-N_cW_{32}}}$$

Generates only single-poles when expanded around zeros of denominator

Not clear, if can be brought into dLog form. Lack of decision procedure

# Numerical Solutions - I

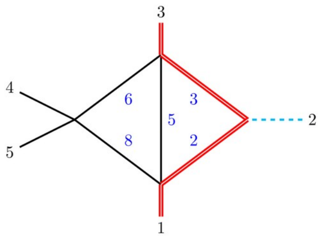
Use AMFlow to compute 100-digit physical region boundary constants

[Liu,Ma, arXiv:2201.11669]

Solve DE using generalized series expansions techniques

[Moriello, arXiv:1907.13234]

Public implementations: No nested roots -> use auxiliary basis



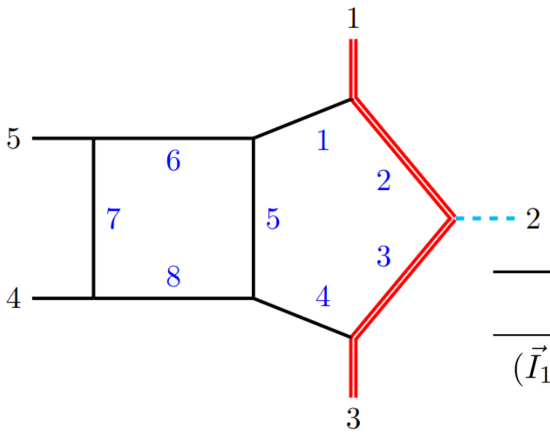

$$= \left\{ \epsilon^3 (q^2)^2 \left( \frac{1}{\rho_3} + \frac{1}{\rho_2} \right), \epsilon^3 (q^2)^2 \left( \frac{1}{\rho_3} - \frac{1}{\rho_2} \right) \right\}$$

Proof of concept DiffExp implementation in ancillary files

[Febres Cordero, Figueiredo, MK, Page, Reina, arXiv:2312.08131] [Hidding, arXiv:2006.05510]

# Numerical Solutions - II

$$\vec{s}_1 = \left\{ \frac{19}{3}, \frac{46}{3}, -\frac{24}{7}, \frac{383}{5}, -\frac{61}{28}, \frac{25}{118}, \frac{97}{896} \right\},$$




| | $\mathcal{O}(\epsilon^0)$ | $\mathcal{O}(\epsilon^1)$ | $\mathcal{O}(\epsilon^2)$ | $\mathcal{O}(\epsilon^3)$ | $\mathcal{O}(\epsilon^4)$ |
|---|---|---|---|---|---|
| $(\vec{I}_1)_{109}$ | 0 | 0 | 0 | $-3.703380133$ $+5.885655074\,i$ | $2.149576969$ $-10.432322830\,i$ |
| $(\vec{I}_1)_{110}$ | 0 | 0 | 0 | 0 | 0 |
| $(\vec{I}_1)_{111}$ | 0 | 0 | $-1.306045093$ $-12.647039669\,i$ | $2.05552771$ $+25.35139955\,i$ | $-85.55528965$ $-75.93834102\,i$ |

Solving DE is approximately 2 orders of magnitude faster than AMFlow

# Summary & Outlook

**Summary:**

- We computed the first set of master integrals for

$$pp \to t\bar{t}H \quad @ \quad \text{NNLO}$$

- We provide differential equations in $\epsilon$-factorized form
- We find a new analytic structure: Nested square roots!

**Outlook:**

- Computation of one-fold integral solution of DEs

following [Chicherin, Sotnikov '20]

- Finding canonical form of remaining two elliptic PentaBox families