Five Parton Scattering at Two Loops

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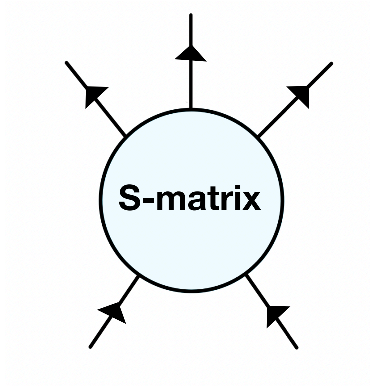

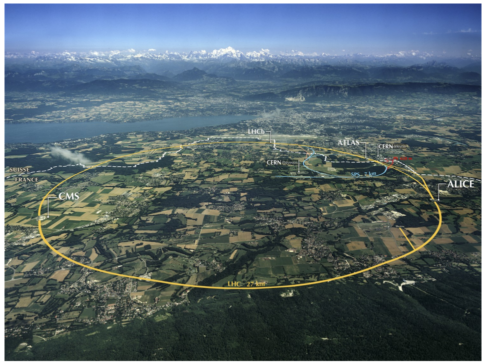

Loopfest 2024, May 20-22
Southern Methodist University
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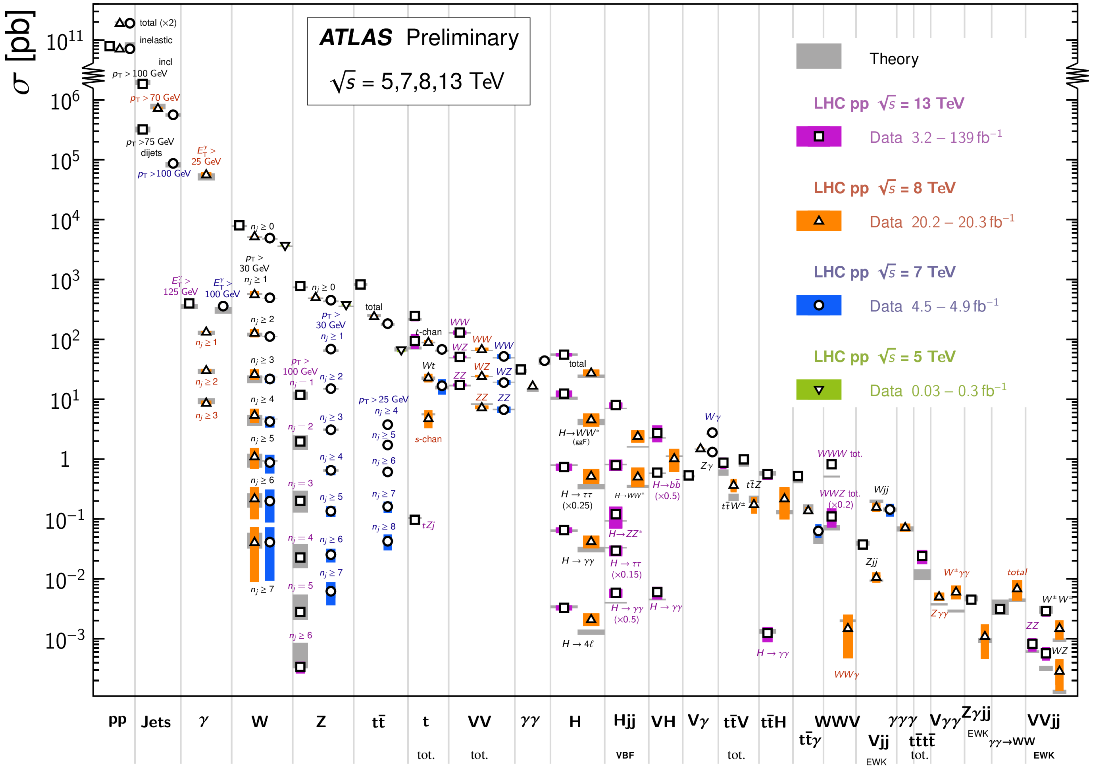
20th of May

Motivation

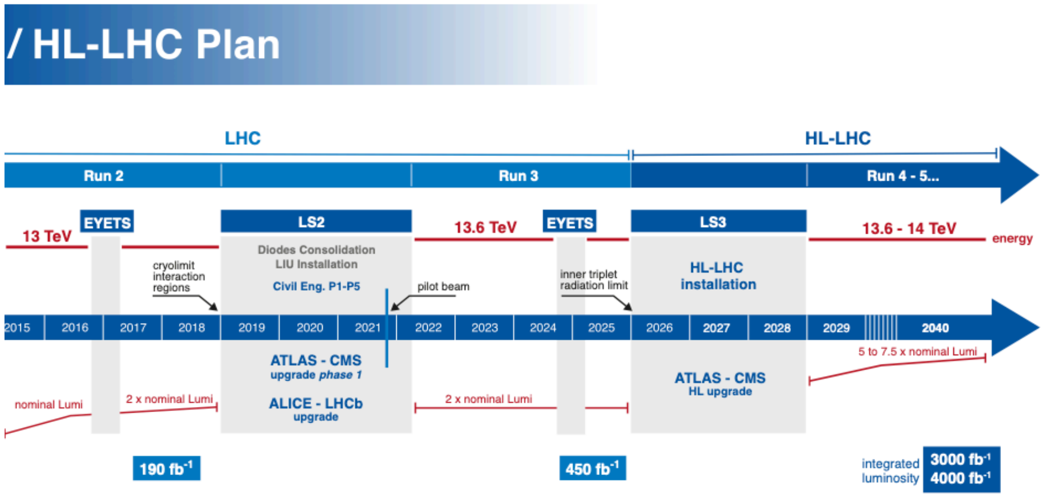
- Impressive understanding of Standard Model at high-energy collisions

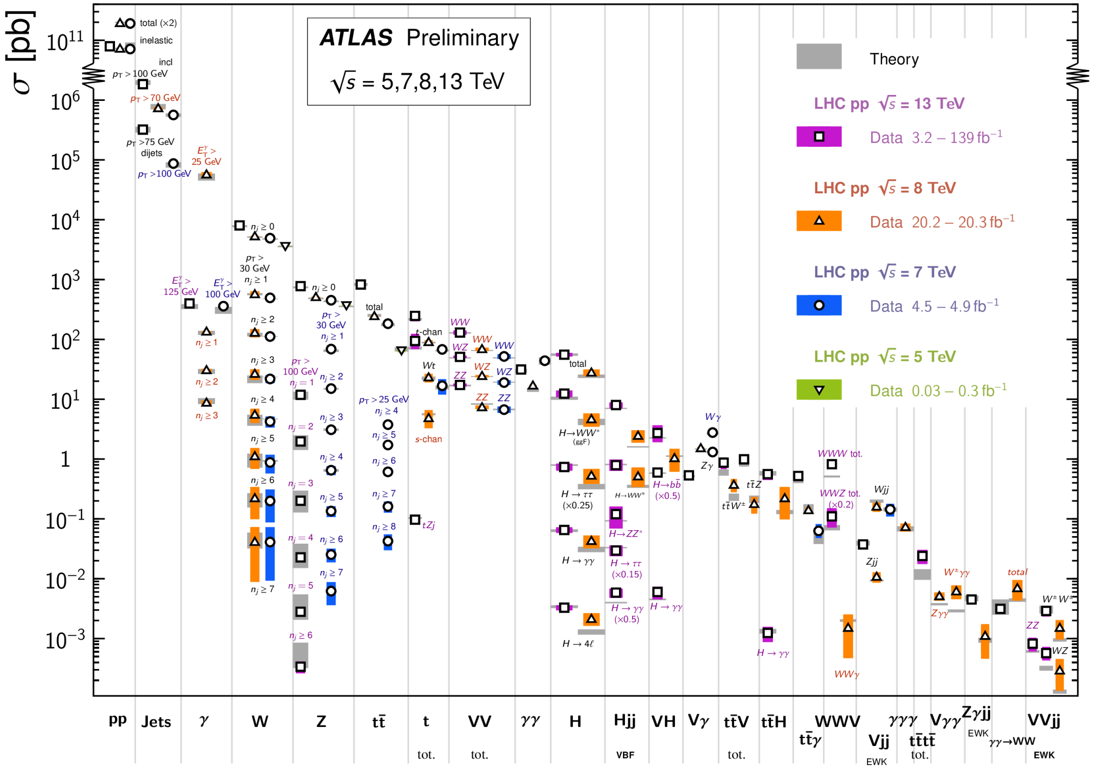


[ATL-PHYS-PUB-2022-009, February 2022]

Motivation

- Ten-fold increase in data at LHC experiments





[ATL-PHYS-PUB-2022-009, February 2022]

- Physics goal: Higgs couplings 2-4%, W mass, top mass, $\sin\theta_w$, multi W/Z,…

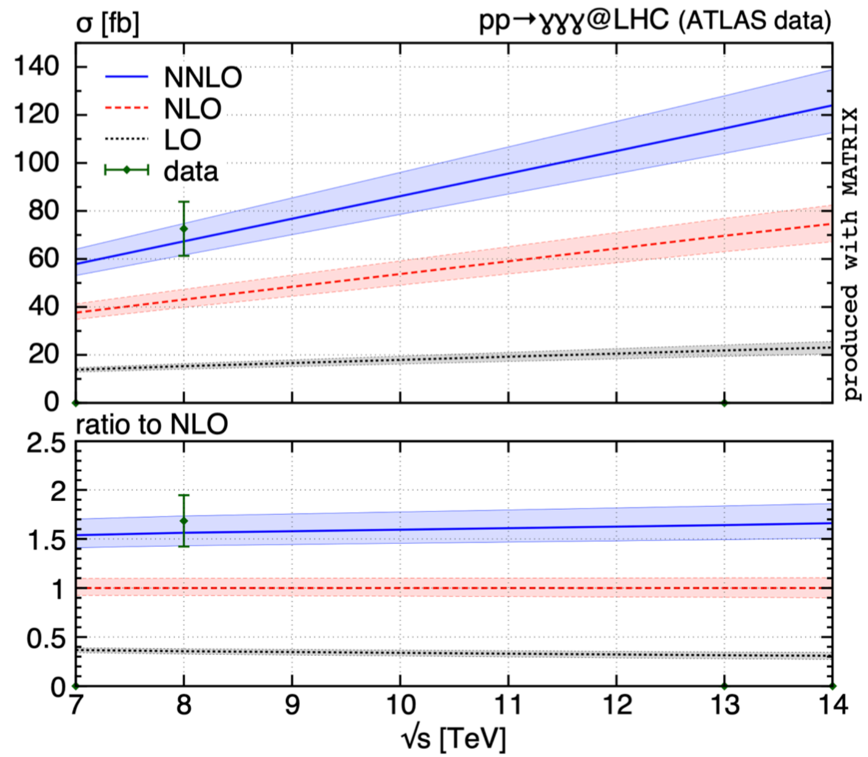
- Theory goal: few-percent precision for many observables:

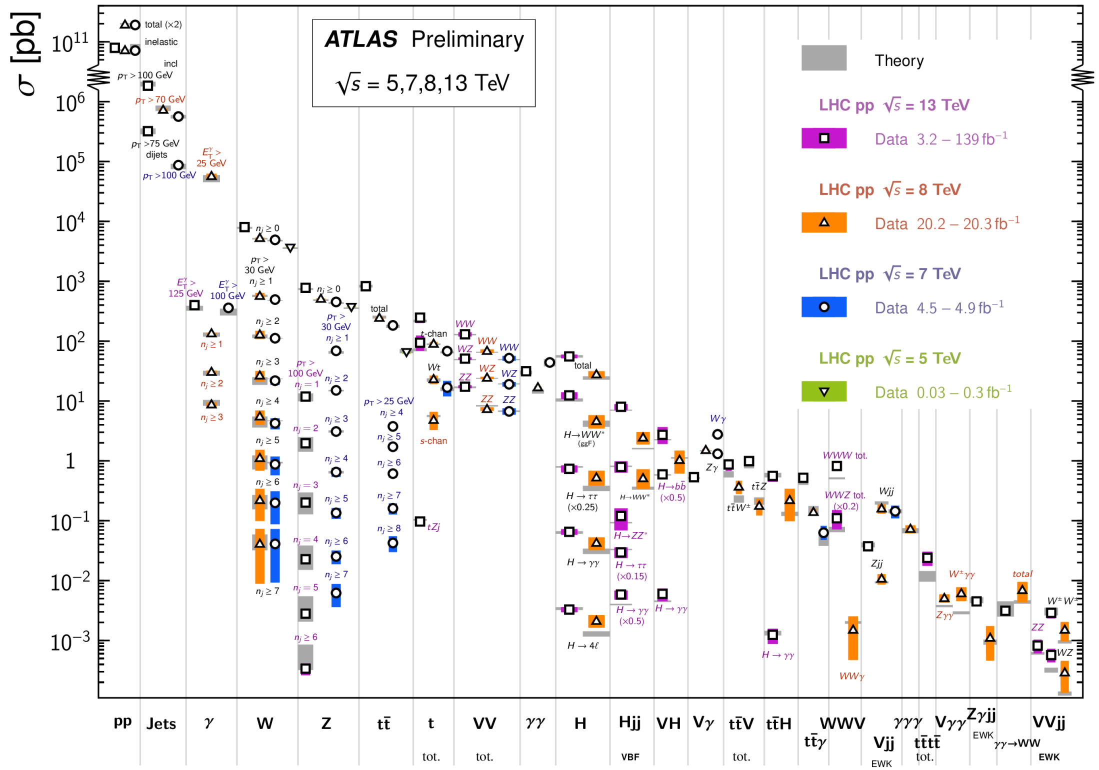
$$\sigma = \sigma_{LO} + \alpha_s \, \Delta\sigma_{nlo}^{qcd} + \alpha_s^2 \, \Delta\sigma_{nnlo}^{qcd} + \alpha_f \, \Delta\sigma_{nlo}^{ew} + \dots$$

$\mathcal{O}(10\%)$ $\quad$ $\mathcal{O}(1-5\%)$ $\quad$ $\mathcal{O}(1\%)$

Motivation

- NNLO QCD calculations: large K factors



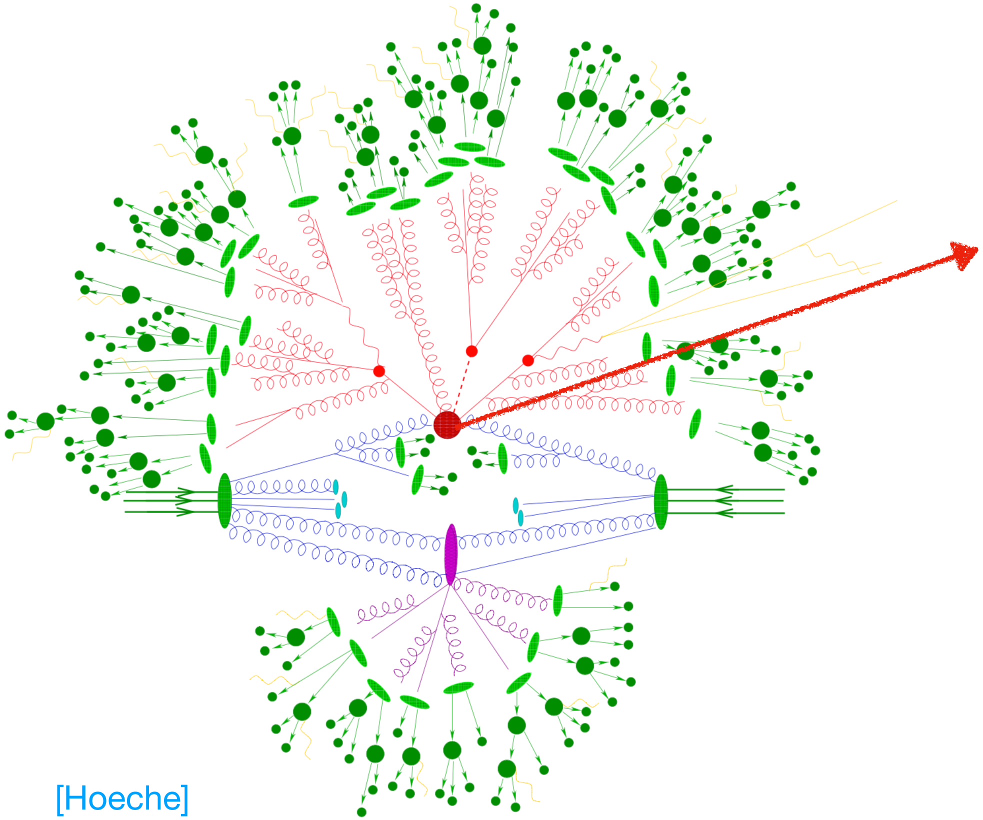


[Kallweit, Sotnikov, Wiesemann, 20]

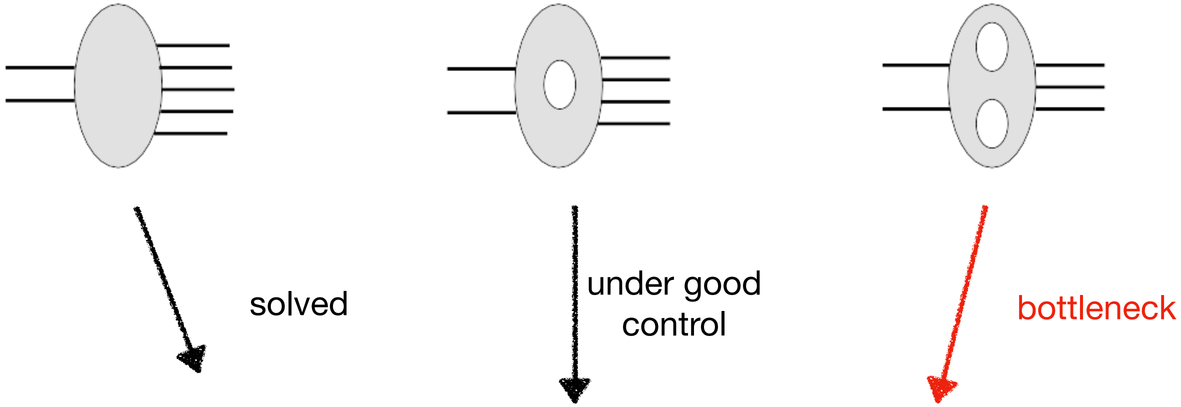
[see also Chawdhry, Czakon, Mitov, Poncelet,19]

Motivation

- Scattering amplitudes for NNLO QCD cross sections to five-point processes:



[Hoeche]



$$\sigma_{nnlo}^{qcd} = \sigma_{rr} + \sigma_{rv} + \sigma_{vv}$$

Status: Five-Point Two-Loop Amplitudes

	Comment	Complete analytic results	Public numerical code	Cross sections
$pp \to \gamma\gamma\gamma$		[1–3]	[1, 3]	[4, 5]
$pp \to \gamma\gamma j$	l.c.⋆	[6, 7]	[6]	[8]
$pp \to \gamma\gamma j$		**[9]**		
$gg \to \gamma\gamma g$	NLO loop induced	[10]	[10]	[11]
$pp \to \gamma j j$		**[12]**		**[12]**
New $pp \to jjj$	l.c.	[13], **[14–16]**	[13],**[16]**	[17, 18]
$pp \to Wb\bar{b}$	l.c.⋆, on-shell W	[19, 20]		
$pp \to W(l\nu)b\bar{b}$	l.c.	[21, 22]		[22]
$pp \to W(l\nu)jj$	l.c.	[21]		
$pp \to Z(l\bar{l})jj$	l.c.⋆	[21]		
$pp \to W(l\nu)\gamma j$	l.c.⋆	[23]		
$pp \to Hb\bar{b}$	l.c., b-quark Yukawa	[24]		
$pp \to Ht\bar{t}$	approx. 2-loop ampl.			[25]

Table 1: Known two-loop QCD corrections for five-point scattering processes at hardon colliders. "l.c." refers to the calculations in the leading-color approximation; "l.c.⋆" means that in addition non-planar l.c. contributions are omitted. All public codes employ `PentagonFunctions++` [26, 27] for numerical evaluation of special functions.

…. full color

7

- New: five-carton non-planar amplitudes [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23], [De Laurentis, HI, Klinkert, Sotnikov '23]
- New: five-point one-mass non-planar integral library [Abreu, Chicherin, HI, Page, Sotnikov, Tschernow, Zoia '24]
- Future demand: multi-scale processes with 5-10 kinematic scales

  [5 x 4 (mom.) + 5 (masses) -
  - 5 (mass shell) - 4 (mom. cons.) - 6 (Lorentz) = 10]

Adapted from [Sotnikov '22; Abreu '22]

See also Les Houches Standard-Model Precision Wishlist [Huss, Huston, Jones, Pellen '22]

Status: Five-Point Two-Loop Amplitudes

[1] S. Abreu, B. Page, E. Pascual and V. Sotnikov, *Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders*, *JHEP* **01** (2021) 078 [2010.15834]. (page 2)

[2] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-color helicity amplitudes for three-photon production at the LHC*, *JHEP* **06** (2021) 150 [2012.13553]. (page 2)

[3] S. Abreu, G. De Laurentis, H. Ita, M. Klinkert, B. Page and V. Sotnikov, *Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders*, 2305.17056. (page 2)

[4] H.A. Chawdhry, M.L. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, *JHEP* **02** (2020) 057 [1911.00479]. (page 2)

[5] S. Kallweit, V. Sotnikov and M. Wiesemann, *Triphoton production at hadron colliders in NNLO QCD*, *Phys. Lett. B* **812** (2021) 136013 [2010.04681]. (page 2)

[6] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-loop leading colour QCD corrections to $q\bar{q} \to \gamma\gamma g$ and $qg \to \gamma\gamma q$*, *JHEP* **04** (2021) 201 [2102.01820]. (page 2)

[7] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-colour QCD helicity amplitudes for two-photon plus jet production at the LHC*, *JHEP* **07** (2021) 164 [2103.04319]. (page 2)

[8] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to diphoton production with an additional jet at the LHC*, *JHEP* **09** (2021) 093 [2105.06940]. (page 2)

[9] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-Loop Helicity Amplitudes for Diphoton Plus Jet Production in Full Color*, *Phys. Rev. Lett.* **127** (2021) 262001 [2105.04585]. (pages 2 and 3)

[10] S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H.B. Hartanto, J. Henn et al., *Virtual QCD corrections to gluon-initiated diphoton plus jet production at hadron colliders*, *JHEP* **11** (2021) 083 [2106.08664]. (pages 2, 3, and 5)

[11] S. Badger, T. Gehrmann, M. Marcoli and R. Moodie, *Next-to-leading order QCD corrections to diphoton-plus-jet production through gluon fusion at the LHC*, *Phys. Lett. B* **824** (2022) 136802 [2109.12003]. (page 2)

[12] S. Badger, M. Czakon, H.B. Hartanto, R. Moodie, T. Peraro, R. Poncelet et al., *Isolated photon production in association with a jet pair through next-to-next-to-leading order in QCD*, *JHEP* **10** (2023) 071 [2304.06682]. (page 2)

[13] S. Abreu, F.F. Cordero, H. Ita, B. Page and V. Sotnikov, *Leading-color two-loop QCD corrections for three-jet production at hadron colliders*, *JHEP* **07** (2021) 095 [2102.13609]. (pages 2 and 7)

[14] B. Agarwal, F. Buccioni, F. Devoto, G. Gambuti, A. von Manteuffel and L. Tancredi, *Five-parton scattering in QCD at two loops*, *Phys. Rev. D* **109** (2024) 094025 [2311.09870]. (page 2)

[15] G. De Laurentis, H. Ita, M. Klinkert and V. Sotnikov, *Double-virtual NNLO QCD corrections for five-parton scattering. I. The gluon channel*, *Phys. Rev. D* **109** (2024) 094023 [2311.10086]. (page 2)

[16] G. De Laurentis, H. Ita and V. Sotnikov, *Double-virtual NNLO QCD corrections for five-parton scattering. II. The quark channels*, *Phys. Rev. D* **109** (2024) 094024 [2311.18752]. (page 2)

Status: Five-Point Two-Loop Amplitudes

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[18] X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, *Automation of antenna subtraction in colour space: gluonic processes*, 2203.13531. (page 2)

[19] S. Badger, H.B. Hartanto and S. Zoia, *Two-Loop QCD Corrections to Wbb⁻Production at Hadron Colliders*, *Phys. Rev. Lett.* **127** (2021) 012001 [2102.02516]. (pages 2, 3, 4, and 8)

[20] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *Flavour anti-k_T algorithm applied to $Wb\bar{b}$ production at the LHC*, 2209.03280. (page 2)

[21] S. Abreu, F. Febres Cordero, H. Ita, M. Klinkert, B. Page and V. Sotnikov, *Leading-color two-loop amplitudes for four partons and a W boson in QCD*, *JHEP* **04** (2022) 042 [2110.07541]. (pages 2, 3, and 5)

[22] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *NNLO QCD corrections to $Wb\bar{b}$ production at the LHC*, 2205.01687. (pages 2, 3, and 4)

[23] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading colour helicity amplitudes for $W\gamma + j$ production at the LHC*, *JHEP* **05** (2022) 035 [2201.04075]. (pages 2 and 3)

[24] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading-colour QCD helicity amplitudes for Higgs boson production in association with a bottom-quark pair at the LHC*, *JHEP* **11** (2021) 012 [2107.14733]. (pages 2 and 3)

[25] S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli and C. Savoini, *$t\bar{t}H$ production in NNLO QCD*, 2210.07846. (page 2)

Feynman-Integrals for massless propagators

[26] D. Chicherin and V. Sotnikov, *Pentagon Functions for Scattering of Five Massless Particles*, *JHEP* **12** (2020) 167 [2009.07803]. (pages 2, 3, 6, 7, and 8)

[27] D. Chicherin, V. Sotnikov and S. Zoia, *Pentagon functions for one-mass planar scattering amplitudes*, *JHEP* **01** (2022) 096 [2110.10111]. (pages 2, 4, 6, and 9)

[28] D. Chicherin, T. Gehrmann, J. Henn, P. Wasser, Y. Zhang and S. Zoia, *All Master Integrals for Three-Jet Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **123** (2019) 041603 [1812.11160]. (pages 2 and 3)

[29] S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, *Two-Loop Integrals for Planar Five-Point One-Mass Processes*, *JHEP* **11** (2020) 117 [2005.04195]. (pages 2, 3, and 7)

New

[30] S. Abreu, D. Chicherin, H. Ita, B. Page, V. Sotnikov, W. Tschernow et al., *All Two-Loop Feynman Integrals for Five-Point One-Mass Scattering*, *Phys. Rev. Lett.* **132** (2024) 141601 [2306.15431].

Amplitude Computation

- Feynman diagrams:

$$A = \sum_{i \in \text{all integrals}} I_i(\epsilon, \vec{p}) \,,$$

$$I_i(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{n_i(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2}$$

- Integration-by-parts relations (IBP): [Chetyrkin, Tkachov 81; Laporta 00]

$$\int d^D\ell \, d^D\tilde{\ell} \, \frac{\partial}{\partial \ell_\mu} \left[\frac{v^\mu(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2} \right] = 0$$

$$\rightarrow \sum_{i \in \text{all integrals}} b_i(\epsilon, \vec{p}) \, I_i(\epsilon, \vec{p}) = 0$$

↪ find basis of integrals by solving linear system

- Sum of master integrals:

$$A = \sum_{i \in \text{basis}} c_i(\epsilon, \vec{p}) \, I_i(\epsilon, \vec{p})$$

- Integration:

$$I_i(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{n_i(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2}$$

↪ gives 11-dimensional integrals at two loop five-point

- Differential equations method: [Kotikov 91; Bern, Dixon, Kosower 93; Remiddi 97, Gehrmann, Remiddi 99,…]

$$\frac{\partial}{\partial s_{ij}} I_k(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{\partial}{\partial s_{ij}} \left[\frac{n_k(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2} \right] = \sum_{k,j} m_{kj}(\epsilon, \vec{p}) I_j(\epsilon, \vec{p})$$

↪ IBP reduction

↪ solve multi-variate differential equation & boundary conditions

- Integral functions in Laurent expansion in ϵ :

$$I_i(\epsilon, \vec{p}) = \sum \epsilon^k h_{ik}(\vec{p})$$

$h_i(\vec{p}) \in \{1, \ln(s_{12}), \ldots\}$ … integral functions

- Integrated amplitude:

$$A = \sum_{i \in \text{basis}} \epsilon^j \, d_{ij}(\vec{p}) \, h_{ij}(\vec{p})$$

Amplitude Computation

- Computational steps well established, but very complex:

  - number of diagrams/terms

  - number of variables in linear system: momenta & masses

  - multi-dimensional integration

- Keys to progress:

  - advance methods, new ideas

  - examples and structural understanding

- Simplicity of analytic results:

  - indicates mathematical & physical properties of amplitudes, which may lead to better ways to compute

Feynman diagrams

Integral reduction using
integration by parts (IBP)

↓

Sum of master integrals

$$A = \sum_{i \in basis} c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE)
or numerical integration

↓

Integrated amplitude

$$A = \sum_{i,j} \epsilon^j d_{ij}(\vec{p})\, h_{ij}(\vec{p})$$

Numerical Amplitude Computation

- Numerical evaluations avoid problems of manipulating multi-variate expressions

  - Numerical algorithms for one-loop amplitudes during `NLO revolution' [Blackhat, GoSam, Recola, OpenLoops, NJet, Recola,…]

- Challenges:

  - Numerical instabilities in integral reduction

  - Dimension dependence

- Solution:

  - Exact rational arithmetic $\mathbb{Q}$ instead of floating point $\mathbb{R}$ (actually: finite-field arithmetic $\mathbb{F}$) [$\mathbb{Q}$: common for checks; $\mathbb{F}$: vManteuffel, Schabinger 15]

  - Focus on simple rational functions

  - Functional reconstruction of analytic expressions [Peraro 16]

$$\vec{p} \to \mathbb{Q}\,, \quad \epsilon \to \mathbb{Q}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

? Laurent expansion in ϵ

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

Numerical Amplitude Computation

- Rationality of integral coefficients in ϵ

[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng 17]

$$c(\epsilon) = \frac{n_0 + n_1\,\epsilon + n_2\,\epsilon^2 + \ldots}{d_0 + d_1\,\epsilon + d_2\,\epsilon^2 + \ldots}$$

similar to [Giele, Kunst, Melnikov 08]

- Rational function reconstructed in finite number of evaluations in ϵ:

  - Linear system for unknowns $\{n_i, d_i\}$:

$$c(\epsilon_1)(d_0 + d_1\,\epsilon_1 + d_2\,\epsilon_1^2 + \ldots) = n_0 + n_1\,\epsilon_1 + n_2\,\epsilon_1^2 + \ldots$$

$$c_i(\epsilon_2)(d_0 + d_1\,\epsilon_2 + d_2\,\epsilon_2^2 + \ldots) = n_0 + n_1\,\epsilon_2 + n_2\,\epsilon_2^2 + \ldots$$

$$\ldots$$

  - Efficiency depends on number of evaluations, which in turn depends on degree of rational function

$$\vec{p} \to \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

✓ Laurent expansion in ϵ

Amplitude Reconstruction

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

- Rationality of integral coefficients in momenta

[Peraro 16; Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng, 17]

$$c(\epsilon, \vec{p}) = \frac{n_0(\vec{p}) + n_1(\vec{p})\,\epsilon + n_2(\vec{p})\,\epsilon^2 + \ldots}{d_0(\vec{p}) + d_1(\vec{p})\,\epsilon + d_2(\vec{p})\,\epsilon^2 + \ldots}$$

$$c(\epsilon, \vec{p})(d_0(\vec{p}) + d_1(\vec{p})\,\epsilon + d_2(\vec{p})\,\epsilon^2 + \ldots) = n_0(\vec{p}) + n_1(\vec{p})\,\epsilon + n_2(\vec{p})\,\epsilon^2 + \ldots$$

$$n_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}}\,(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \ldots)\,, \;\; s_{ij} = (p_i + p_j)^2,$$

similar for $d_i(\vec{p})$

↪ linear systems for numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $c_i(\epsilon, \vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

↪ Efficient and numerically stable analytic forms of amplitudes

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i\, d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

Laurent expansion in ϵ

Caravel Program

- C++ implementation of numerical-unitarity approach [Abreu, Dormans, Febres Cordero, HI, Kraus, Page, Pascual, Ruf, Sotnikov; '20]

  ↪ Numerical values for integral functions in amplitude remainders

- Applications in QCD and gravity

- Public code: https://gitlab.com/caravel-public

- Auxiliary programs for input data:
  - Qgraf, Mathematica
  - Computational algebraic geometry & Singular [Decker, Greuel, Pfister, Schönemann]

Caravel

- Contributors:

  - FSU group with Febres Cordero, Figueiredo, …

  - Mexico: Kraus

  - CERN: Abreu

  - UZH/PSI: HI, Kuschke, Sotnikov

  - Ghent U.: Page

  - Edinburgh: De Laurentis

Structure — `Good' Integral Bases

- Analytic properties yield crucial simplifications in expressions:
  - Good integral bases lead to factorisation

$$c(\epsilon,\vec{p}) = \frac{\mathrm{poly}_1(\epsilon,\vec{p})}{\mathrm{poly}_2(\epsilon,\vec{p})} = \frac{\mathrm{poly}_1(\epsilon,\vec{p})}{\mathrm{poly}_2(\epsilon)\ \mathrm{poly}_3(\vec{p})}$$

[observed in computations, e.g. Tancredi, Melnikov; recently: Usovitsch 20; Smirnov, Smirnov 20]

↪ universal $\mathrm{poly}_2(\epsilon)$ simplifies reconstruction

  - Canonical kinematic denominators:

$$c(\epsilon,\vec{p}) = \frac{\mathrm{poly}_1(\epsilon,\vec{p})}{\mathrm{poly}_2(\epsilon)\ \prod_i W_i^{m_i}(\vec{p})}$$

[Abreu, Dormans, Febres Cordero, Ita, Page '18]

$W_i(\vec{p})$ … `letters' associated to integral

↪ denominators require to obtain integer exponents m_i

- Factorisation properties simplify reconstruction and improve numerical stability

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

Feynman diagrams

↓ Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i \frac{\tilde{c}_i(\epsilon,\vec{p})}{\hat{c}_i(\epsilon)\prod_j W_j(\vec{p})} I_i(\epsilon,\vec{p})$$

↓ Differential equations (DE) or numerical integration

Integrated amplitude

Laurent expansion in ϵ

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p}) h_{ik}(\vec{p})$$

Structure — Function Bases

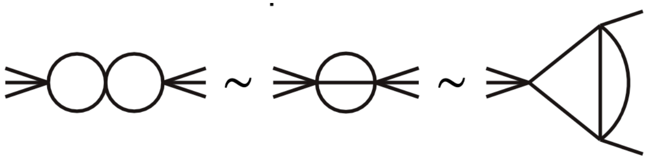
- Integral coefficients very complicated

  - only finite orders in ϵ-expansion needed

  - subtract universal IR/UV poles and reconstruct finite remainders

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_i}(\vec{p})} h_i(\vec{p})$$

- Relations after ϵ expansion



$\sim r_0 + r_1 \epsilon \ln(s) + r_2 \epsilon^2 \ln^2(s) + \dots$

↪ cancellations and simplification

- Reconstruct polynomials $e_i(\vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \dots\} \in \mathbb{Q}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i \frac{\tilde{c}_i(\epsilon, \vec{p})}{\hat{c}_i(\epsilon) \prod_j W_j(\vec{p})} I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

Laurent expansion in ϵ

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_i}(\vec{p})} h_i(\vec{p})$$

Structure — Regularity

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_j}(\vec{p})} h_i(\vec{p})$$

- Simplifications:

  - Regularity of amplitudes/remainders in phase space

  - Many of poles in $W_j(\vec{p}) = 0$ unphysical and cancel $\Longrightarrow$ correlations between numerator polynomials $e_j(\vec{p})$

- Many advanced ideas for reconstruction:

  - Univariate/multivariate partial fractions [Badger, Hartanto, Zoia, 21]

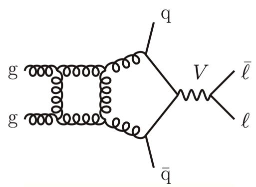
  - Choice of variables, e.g. spinor helicity

  - p-adic numbers [Page, De Laurentis 22; Chawdhry 23] see talk by Chawdhry

  - Reconstruction programs: FireFly [Klappert, Lange 19]

- Example: planar two-loop four-parton + W-boson amplitudes [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 22]

  - Observe factor-50 reduction of needed evaluations



Start

Factor-50 reduction

$\mathcal{R}_g$	$p_5 \parallel p_i$	—K—	Max Ansatz Size: Common Denominator	Max Ansatz Size: Partial Fractioning	Max Non-Zero Terms: Result
	1	58	5500 k	180 k	37 k
$+-\ N_f^0$	2	67	7000 k	480 k	110 k
	3	67	5900 k	380 k	90 k

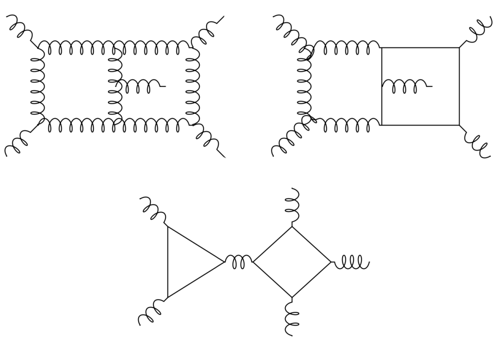
Application: Five-Parton Two-Loop Amplitudes

- Computed and validated by two groups

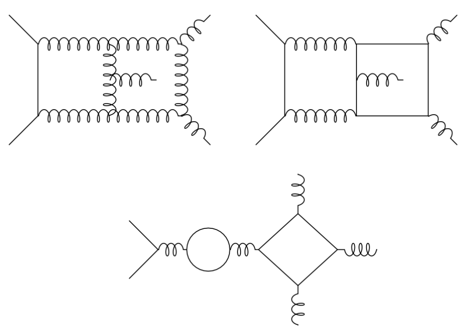
  - Feynman diagrams [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi ‘23]

  - Numerical unitarity [De Laurentis, HI, Klinkert, Sotnikov ’23], methods details to appear [De Laurentis, HI, Page, Sotnikov ’23]. — **discussed here**

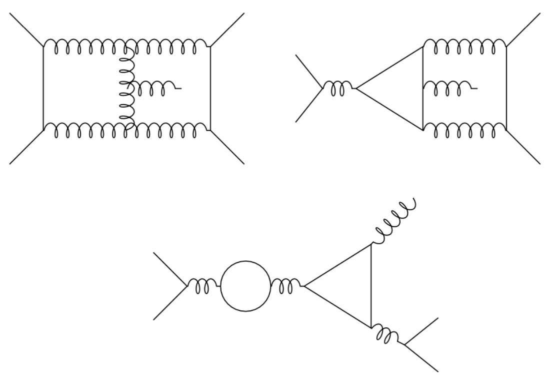
Gluons



2-quark



4-quark



Coefficient Functions

- Basis of functions

$$R = \sum_j r_j h_j = \sum_{i \in \text{basis}, j} r_i M_{ij} h_j \qquad M_{ij} \in \mathbb{Q}$$

with minimal complexity. Complexity measure is mass dimension of coefficient numerators $\mathcal{N}_m$, in common denominator form,

$$r_m(\lambda, \tilde{\lambda}) = \sum_i \frac{\mathcal{N}_m(\lambda, \tilde{\lambda})}{\prod_j W_j^{m_i}(\lambda, \tilde{\lambda})}$$

- Basis change to simplify coefficient: [see also Abreu, Dormans, Febres Cordero, Ita, Page '18]

$$\tilde{r}_i = \sum_{j \in \text{basis}} O_{ij} r_j$$

Helicity remainder	dim(VS($\mathcal{R}$))	LCD ansatz size before basis change	LCD ansatz size after basis change
$R^{(2),(2,0)}_{+++--}$	31	21,910	N/A
$R^{(2),(2,0)}_{++-+-}$	54	54,148	N/A
$R^{(2),(1,0)}_{+++--}$	274	163,635	14,093
$R^{(2),(1,0)}_{+-++-}$	270	241,156	14,552
$R^{(2),(1,0)}_{--+++}$	203	82,180	25,620
$R^{(2),(1,1)}_{+++--}$	31	21,910	N/A
$R^{(2),(1,1)}_{++-+-}$	54	54,148	N/A
$R^{(2),(0,1)}_{+++--}$	226	118,880	4,108
$R^{(2),(0,1)}_{+-++-}$	240	209,018	N/A
$R^{(2),(0,1)}_{--+++}$	157	76,845	8,840
$R^{(2),(-1,1)}_{+++--}$	25	5,320	N/A
$R^{(2),(-1,1)}_{++-+-}$	35	9,384	N/A

Coefficient Functions

- Key structures for constructing basis change:

  - Numerator degree of common denominator form linked to denominator letters

$$r_i = \frac{c_{i1}}{x} + \frac{c_{i2}}{x+y} = \frac{d_i + d_i^x x + d_i^y y}{x(x+y)} \qquad \rightarrow \qquad \tilde{r}_1 = \frac{c_1}{x}, \quad \tilde{r}_2 = \frac{c_2}{x+y}$$

  - Residues for vanishing letters are correlated between coefficients, since spurious nominator poles have to cancel in amplitude

  ↪ basis change constructed to de correlate residues

Correlation of Residues

- Laurent expansion around zeros of letters on univariate slices

$$\lambda_i = \lambda_i(t)\,, \quad \tilde{\lambda}_i = \tilde{\lambda}_i(t)$$

$$r_i = \sum_{m=1}^{q_{ik}} \frac{e^k_{im}}{(t - t_{W_k})^m} + \mathcal{O}(t - t_{W_k})$$

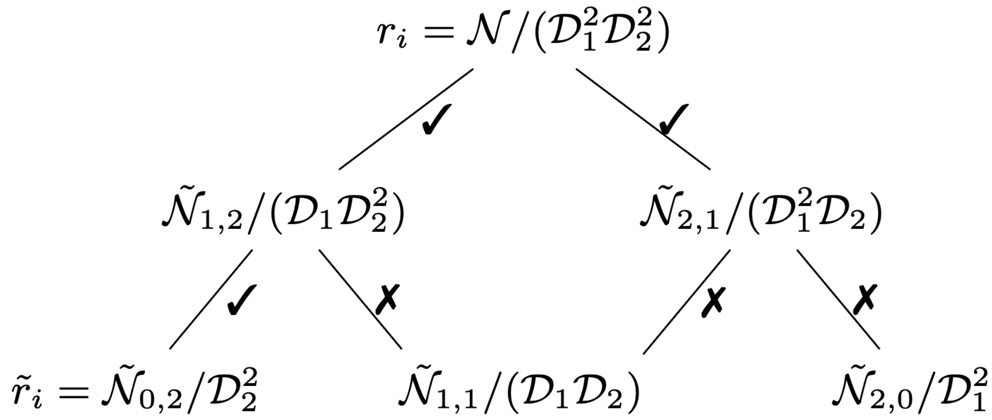
- Transforms to

$$\tilde{r}_i = \sum_{m=1}^{q_{ik}} \frac{\tilde{e}^k_{im}}{(t - t_{W_k})^m} + \mathcal{O}(t - t_{W_k})\,, \quad \tilde{e}_{jm} = O_{ij} e_{jm}$$

- Impose that the leading residues vanishes,

$$\sum_j O_{ij} e_{jq_k} = 0$$

↪ Pole reduced



- Intersection of null spaces to remove maximal number of leading residues $\{e_{jq_k}\}_j$ in coefficient functions

- Numerical analysis: combining numerical data from multiple slices

Quark Amplitudes from Gluon Amplitudes

[De Laurentis, HI, Sotnikov [2311.18752]

- Ansatz for coefficient functions
  - 2q3g and 4q1g amplitudes
  - Similar cancellation mechanisms and pole structure as gluon amplitudes
  - Coefficients differ from gluon functions by phase weight:

$$|i\rangle \to t|i\rangle, \quad |i] \to \frac{1}{t}|i]$$

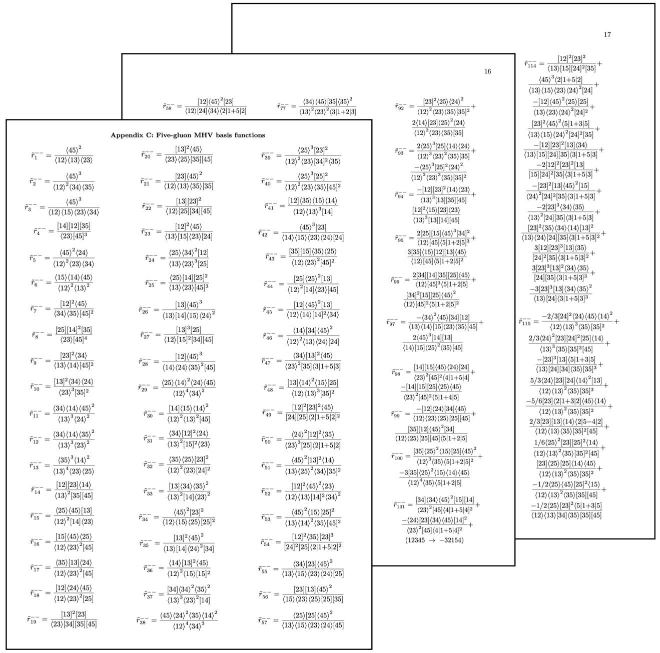
↪ construct Ansatz by multiplying with phase-weight factors similar to supersymmetry Ward Identities

- Example

$$\tilde{r}_{73}^{-}(q^+, q^-, g^+, g^+, g^-) = \frac{\langle 14\rangle}{\langle 24\rangle} \cdot \tilde{r}_{18}^{---}(g^+, g^-, g^+, g^+, g^-) \quad \text{with} \quad \tilde{r}_{18}^{---}(g^+, g^-, g^+, g^+, g^-) = \frac{[1{,}4]\langle 25\rangle\langle 45\rangle}{\langle 24\rangle[24]\langle 34\rangle^2}$$

- Validation with numerical evaluation from Caravel program: obtain 50% of 2q3g functions and 90% of 4q1g functions for free

Spinor-Helicity Results



- Gluon MHV rational functions fit on three pages of paper appendix.
- All rational functions fitted in a single finite field. Matrices M_{ij} and O_{ij} required multiple fields
- Size of full result dominated by matrices
- Can study analytic properties of amplitudes: no tr_5 singularities, no overlapping co-planar poles $[i\,|\,j+k\,|\,i\rangle$
- Use discrete symmetries to obtain generating set of functions

Gluon helicities	Vector-space dimension	Generating set size
$+++\ ++$	24	3
$+++\ +-$	440	33
$+++\ --$	937	115

Particle Helicities	Vector-space dimension	Generating set size
$u^+\bar{u}^- g^+ g^+\ g^+$	424	91
$u^+\bar{u}^- g^+ g^+\ g^-$	844	449
$u^+\bar{u}^- d^+\bar{d}^- g^-$	435	124

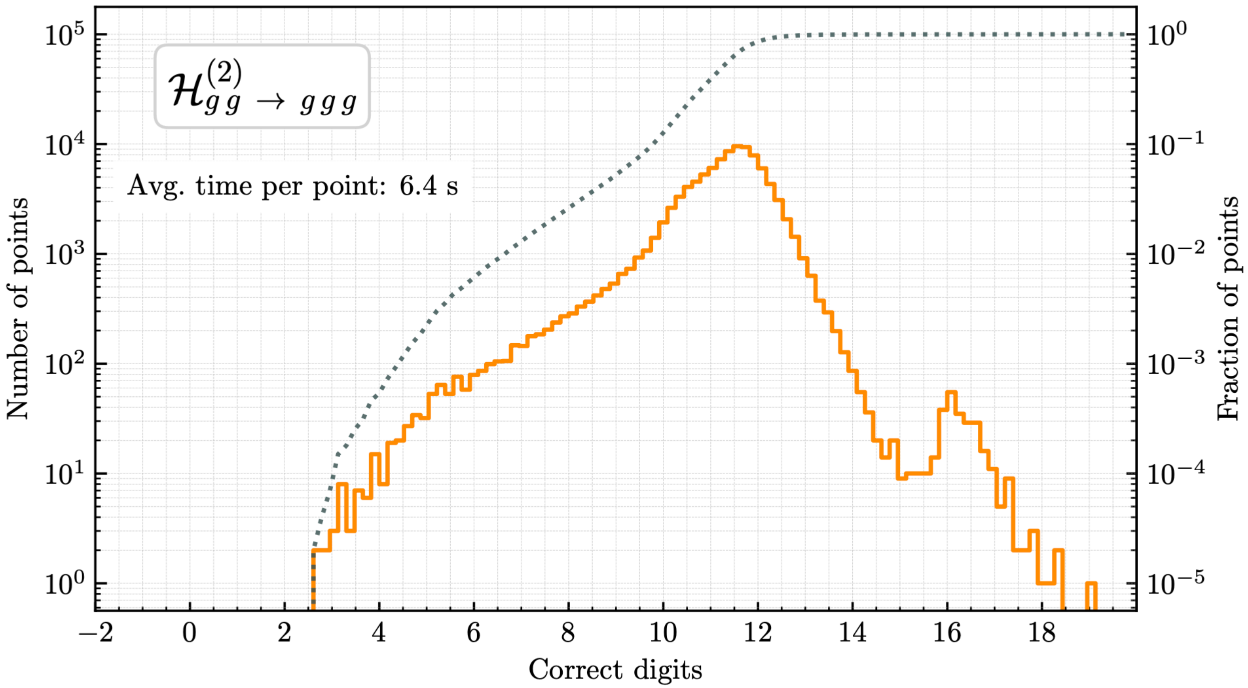
- 35k numerical evaluations: slices and 5k random points

Results

- C++ code for 2-loop remainders
  - gitlab.com/five-point-amplitudes/FivePointAmplitudes-cpp
- Analytic expressions
  - Spinor-helicity functions printed in papers
  - zenodo.org/records/10142295 and zenodo.org/records/10231547
- Stable and fast evaluations for cross sections

[De Laurentis, HI, Klinkert, Sotnikov [2311.100086]

[De Laurentis, HI, Sotnikov [2311.18752]



Conclusions

- Real demand for precision predictions for ongoing and future LHC physics program
- Discussed status of NNLO five-point processes and some key methods
- Progress relies on advancing analytic understanding: differential equation method, integral evaluation, amplitude computation, integral reduction
- Key recent methods: exact numerical evaluations (finite fields), functional reconstruction & understanding of interplay of integral functions and coefficients
- One-mass five point processes in reach given integrals and reconstruction methods
- New amplitudes computations and new formal developments are the way to go for broad availability of NNLO results.