Reducing Feynman integrals using Blade

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SLAC

Based on work with Xiao Liu, Yan-Qing Ma and Wen-Hao Wu (to appear)

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Outline

- Introduction
  - Collider physics -> scattering amplitude -> perturbative calculation -> IBP reduction
- Block-triangular form
  - Key ideas -> algorithms -> example
- Blade
  - Usage -> benchmarks -> new features
- Summary and outlook

Collider physics

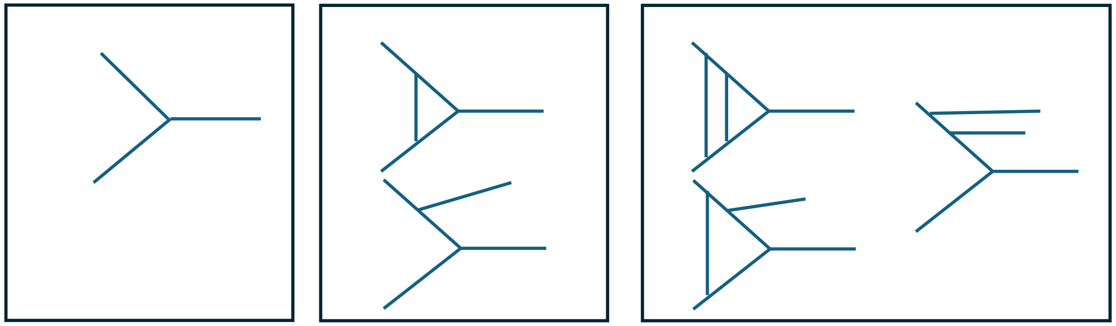
- Main way of exploring elementary particles and forces
  - 1911: Rutherford scattering experiment
  - 2012: Higgs discovery
- Questions
  - New physics (dark matter, neutrino-oscillation, …)
  - Spontaneous symmetry breaking, exotic Higgs potential
- Percent level uncertainty (HL-LHC, lepton-collider)
  - Opportunities to discovery new physics and learn Standard Model
  - Challenges to theoretical predictions

Scattering amplitudes

- Bridge between QFT and experiments
  - Cross-section, differential distributions
- Perturbative calculation

$$\sigma = \sigma_{LO}(1 + \alpha\, \Delta_1 + \alpha^2\, \Delta_2 + \alpha^3 \Delta_3 + \cdots)$$

  - More precise prediction -> higher order calculation
  - Higher loops , higher multiplicity



…

Current status of high order calculation

- One loop
  - Solved problem
- Two loop five (six) point
  - $p\, p \to \gamma\, \gamma\, g$ Agarwal, Buccioni, Manteuffel et al, 2105.04585
  - $p\, p \to H\, b\, \bar{b}$ Badger, Simon, Hartanto et al, 2107.14733
  - $p\, p \to H\, t\, \bar{t}$ Catani, Devoto, Grazzini et al, 2210.07846
  - Six-point massless planar masters Henn, Matijašić, Miczajka et al, 2403.19742
- Three loop four point
  - $q\, \bar{q} \to g\, g$ Caola, Chakraborty, Gambuti et al, 2207.03503
  - $q\, \bar{q} \to Z, \gamma^*, W^\pm + g$ Gehrmann, Jakucik, Mella et al, 2307.15405
- …

Perturbative calculation: generate integrand

- Integrand
  - Feynman diagrams + Feynman rules
  - On-shell techniques
- Operations
  - Tensor decomposition/ Projection/ Squared Amplitudes
  - Color algebra, Lorentz algebra
  - Topology classification (unique sectors)
- Linear combinations of scalar Feynman integrals

$$\mathcal{A} = \sum^{10^4 \sim 10^6} a_i \, I_i$$

$$I(\nu) = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^d l_i}{\mathrm{i}\pi^{d/2}} \right) \frac{D_{k+1}^{-\nu_{k+1}} \dots D_N^{-\nu_N}}{D_1^{\nu_1} \dots D_k^{\nu_k}}$$

$D = q^2 - c$

$D = q \cdot p - c$

q, p are linear combinations of loop momenta and external momenta

Perturbative calculation: integral reduction

- Linear relations among Feynman integrals
  - E.g. Integration-by-part identities

Chetyrkin,Tkachov, NPB(1981)

$$\int \prod_{i=1}^{L} \mathrm{d}^D \ell_i \frac{\partial}{\partial \ell_j^\mu} [v^\mu \mathcal{I}(\vec{v})] = 0$$

- Finite dimensional linear space

Smirnov and Petukhov, 1004.4199

$$I_i = c_{ij} M_j$$

$$\mathcal{A} = \sum_{j}^{10^2 \sim 10^4} a_i c_{ij} M_j$$

  - Master integrals
  - Reduce the number of Feynman integrals, as well as the complexity

Perturbative calculation: evaluate master integrals

- Differential equation — Gehrmann and Remiddi, hep-ph/9912329
  - Derivates of master integrals can be reduced to master integrals (integral reduction)

$$\frac{\partial M_i}{\partial s} = B_{ij}\, M_j$$

e.g. $\frac{\partial}{m^2} \frac{1}{((l+p)^2 - m^2) \ldots D_k} \rightarrow \frac{1}{\left((l+p)^2 - m^2\right)^2 \ldots D_k} + \cdots$

  - Canonical form — Henn, 1304.1806
  - Generalized series expansion — **DiffexExp**, Hidding, 2006.05510; **AMFlow**, Liu and Ma, 2201.11669; **Seasyde**, Armadillo, Bonciani, Devoto et al 2205.03345;
    - Auxiliary mass flow — Liu, Ma and Wang, 1711.09572
- Other methods
  - Sector decomposition — Hepp, Commun. Math. Phys 2 (1966) 301-326
  - Mellin barns transformation — Hergere and Lam, Commun. Math. Phys 39 (1974) 1
  - …

Integral reduction is crucial

- Integral reduction and evaluating master integrals are bottlenecks

  e.g. $g\ g \to g\ g\ g$

  - Master integrals [Papadopoulos, Tommasini and Wever, 1511.09404; Gehrmann, Henn and Presti, 1807.09812; Chicherin, Henn and Mitev, 1712.09610; Abreu, Page and Zeng, 1807.11522; Chicherin, Gehrmann, Henn et al, 1809.06240; Abreu, Dixon, Herrmann et al, 1812.08941]
  - Integral reduction
    - Planar [Chawdhry, Czakon, Mitov et al, 1911.00479, Kallweit, Sotnikov, Wiesemann, 2010.04681; Chawdhry, Czakon, Mitov et al, 2105.06940]
    - Non-planar double pentagon XG, Liu and Ma, 1912.09294; Klappert, Lange, Maierhofer et al, 2008.06494; Agarwal, Buccioni, Manteuffel et al, 2105.04585

- Auxiliary mass flow

  - Calculate Feynman integrals with any loop, any multiplicity and any dimension, provided the integral reduction is obtained

  Liu, Ma and Wang, 1711.09572; Liu and Ma, 2107.01864; Liu and Ma, 2201.11637; Liu, Ma, Tao et al, 2009.07987; Liu and Ma, 2201.11636; Liu and Ma, 2201.11669

IBP reduction

- Ubiquitously used in state-of-art calculations
  - Systematic
  - Many packages exist

[Air, C. Anastasiou and A. Lazopoulos, hep-ph/0404258,
Reduze, A. von Manteuffel and C. Studerus, 0912.2546, 1201.4330
LiteRed, R.NLee, 1212.2685, 1310.1145
Fire , A.V. Smirnov, et al, 0807.3243, 1302.5885, 1408.2372,1901.07808
Kira, Maierhöfer, et al, 1705.05610, 1812.01491, 2008.06494

  - Along with other linear relations: Lorentz invariance identity, Symmetry relations

- Laporta algorithm

Laporta, hep-ph/0102033

  - Generate identities for seed integrals
  - Ordering Feynman integrals
  - Gaussian elimination

IBP reduction

- Difficulties
  - Many equations — E.g. millions of equations, Laporta, 1910.01248
  - Intermediate expression swell
  - Memory intensive & Time-consuming — E.g. Hundreds of GB RAM Klappert, et al., 2008.06494; E.g. Months of runtime Baikov, Chetyrkin and Kühn, 1606.08659

- Selected improvements
  - Finite-field $12345678910 \text{ Mod } 7 = 3$

    Function reconstruction & rational reconstruction

    Manteuffel, Schabinger, 1406.4513
    FiniteFlow, Peraro, 1905.08019
    FireFly, Klappert and Lange, 1904.00009, 2004.01463
    Reconstruction.m, Belitsky, Smirnov and Yakovlev, 2303.02511
  - Syzygy

    $$\frac{\partial}{\ell_j^\mu} \frac{\nu_j}{D_1 \dots D_k} \to \frac{N}{D_1^2 \dots D_k} + \cdots$$

    $$\sum_j \nu_j^\mu \frac{\partial D_i}{\partial \ell_j^\mu} = \beta_i D_i, \quad i = 1, \dots, t,$$

    Gluza, Kajda and Kosower, 1009.0472
    Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
    NeatIBP, Zi-Hao Wu, et al. 2305.08783

    No increased power of denominators
  - Block-triangular form — Minimize the IBP system (require input) (this talk)

    Liu and Ma, 1801.10523, XG, Liu, Ma, 1912.09294
  - Partial fractioned reconstruction — Chawdhry, 2312.03672

Notation of Feynman integrals

$$I[\vec{v}] := \int \dots \int \frac{1}{D_1^{v_1} \dots D_N^{v_N}}$$

$$e.g. I[1,2,1,-2] := \int \dots \int \frac{D_4^2}{D_1 D_2^2 D_3}$$

- t: number of denominators

$$t = \sum_{j|\nu_j>0} 1\,.$$

e.g. $t = 3$

- d: dots

$$d = \sum_{j|\nu_j>0} (\nu_j - 1)\,.$$

e.g. $d = 1$

- r: rank

$$r = - \sum_{j|\nu_j<0} \nu_j\,.$$

e.g. $r = 2$

- Sort integrals: e.g. $t \succ d \succ r$
- Top-sector: integrals with the most denominators, namely t, in the problem.

A simple example of the Block-triangular form

- Reduce top-sector integrals with rank up to 3
- Search 7 relations to reduce 10 integrals to 3 master integrals

Master integral

1st

$$I[1,1,-3] + (s+t)\, I[1,1,-2] + 2s\,t\, I[1,1,-1] + t^3 I[1,1,0] + I[0,1,-2] + I[1,0,-2] + \cdots = 0$$

$$(s-t) I[1,1,-3] + t^2\, I[1,1,-2] + s^2\, t\, I[1,1,-1] + \epsilon\, t^4 I[1,1,0] + (s+t) I[0,1,-2] + t\, I[1,0,-2] + \cdots = 0$$

$$s\, I[1,1,-3] + s^2\, I[1,1,-2] + t^2\, s\, I[1,1,-1] + \epsilon^2 (s^2 t^2 + t^4) I[1,1,0] + t I[0,1,-2] + s\, I[1,0,-2] + \cdots = 0$$

2nd

$$(2-\epsilon) I[0,1,-2] + (2s+t) I[0,1,-1] + (s\,t + t^2) I[0,1,0] = 0$$

$$t\, I[0,1,-2] + \epsilon\, s^2 I[0,1,-1] + s^3\, I[0,1,0] = 0$$

3rd

$$I[1,0,-2] + \epsilon\, t\, I[1,0,-1] + (\epsilon t^2 + s^2) I[1,0,0] = 0$$

$$(s+t)\, I[1,0,-2] + s^2 I[1,0,-1] + t^3 I[1,0,0] = 0$$

Key ideas of the Block-triangular form

1st

$$I[1,1,-3] + (s+t)\, I[1,1,-2] + 2s\,t\, I[1,1,-1] + t^3 I[1,1,0] + I[0,1,-2] + I[1,0,-2] + \cdots = 0$$
$$(s-t) I[1,1,-3] + t^2\, I[1,1,-2] + s^2\, t\, I[1,1,-1] + \epsilon\, t^4 I[1,1,0] + (s+t) I[0,1,-2] + t\, I[1,0,-2] + \cdots = 0$$
$$s\, I[1,1,-3] + s^2\, I[1,1,-2] + t^2\, s\, I[1,1,-1] + \epsilon^2 (s^2 t^2 + t^4) I[1,1,0] + t I[0,1,-2] + s\, I[1,0,-2] + \cdots = 0$$

2nd

$$(2-\epsilon) I[0,1,-2] + (2s+t) I[0,1,-1] + (s\,t + t^2) I[0,1,0] = 0$$
$$t\, I[0,1,-2] + \epsilon\, s^2 I[0,1,-1] + s^3\, I[0,1,0] = 0$$

3rd

$$I[1,0,-2] + \epsilon\, t\, I[1,0,-1] + (\epsilon t^2 + s^2) I[1,0,0] = 0$$
$$(s+t)\, I[1,0,-2] + s^2 I[1,0,-1] + t^3 I[1,0,0] = 0$$

- Relations among chosen set of FIs
  - No increased denominator power
  - Fewer propagators (t) with lower rank (r) (mass dimension)
- Solved strictly block-by-block
  - The most complicated integrals are reduced to simpler integrals in the first block
  - These simple integrals are reduced to even simpler integrals in other blocks

➡ **Faster numeric evaluation & reduced memory consumption**

How to choose FIs in each block?

- Integral extension
  - Operate on target integrals and return a set of FIs
  - Relations among FIs are simple to search
  - Two build-in schemes

e.g. $I[1,1,-3] \xrightarrow{integral\ extension}$
$\{I[1,1,-3], I[1,1,-2], I[1,1,-1], I[1,1,0],$
$I[1,0,-2], I[1,0,-1], I[1,0,0],$
$I[0,1,-2], I[0,1,-1\ ], I[0,1,0\ ]\}$

- Sector-wise
  - Integrals not belonging to master integrals as G_1
  - Apply integral extension on G_1 to obtain a set of FIs, denoted as G
  - Search enough relations among G to reduce G_1 to other (simpler) integrals
- Improvement
  - The number of G_1 is not too large -> distribute one sector into a few blocks
  - Better integral extension scheme?

How to find relations among Feynman integrals?

- Step 1: numeric IBP

$$I_i(\epsilon, \vec{s}) = \sum_{j=1}^{n} C_{ij}(\epsilon, \vec{s}) M_j(\epsilon, \vec{s})$$

- Step 2: search

$$\sum_{i=1}^{N} Q_i(\epsilon, \vec{s}) I_i(\epsilon, \vec{s}) = 0,$$

s_i : kinematic invariants
ϵ : dimensional regulator
monomials with degree bound

- Make ansatzes:

$$Q_i(\epsilon, \vec{s}) = \sum_{\mu_0=0}^{\epsilon_{max}} \sum_{\vec{\mu} \in \Omega_{d_i}} \tilde{Q}_i^{\mu_0 \mu_1 \ldots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r},$$

- Substitute the numeric IBP and ansatz:

Vanish because master integrals are independent

$$\sum_{\mu_0, \vec{\mu}} \sum_{j=1}^{n} \tilde{Q}_i^{\mu_0 \mu_1 \ldots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\epsilon, \vec{s}) M_j(\epsilon, \vec{s}) = 0.$$

- Constraints of unknowns: $$\sum_{\mu_0, \vec{\mu}} \tilde{Q}_i^{\mu_0 \mu_1 \ldots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\epsilon, \vec{s}) = 0, \; j = 1, \ldots, n.$$

- Repeat at different numeric points -> adequate constraints -> relations obtained (with degree bound)

How to find enough & good relations?

- Search algorithm: from simple to complex

    1. Set degree bound
    2. Search relations among Feynman integrals
    3. If relations facilitate reduction, stop; else, increase degree bound and go to step 2.

low degree

$$I[1,1,-3] + (s+t)\, I[1,1,-2] + 2s\,t\, I[1,1,-1] + t^3 I[1,1,0] + I[0,1,-2] + I[1,0,-2] + \cdots = 0$$

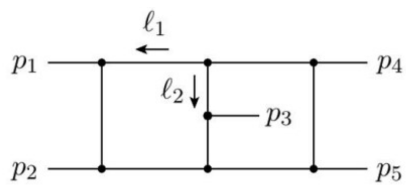
$$(s-t) I[1,1,-3] + t^2\, I[1,1,-2] + s^2\, t\, I[1,1,-1] + \epsilon\, t^4 I[1,1,0] + (s+t) I[0,1,-2] + t\, I[1,0,-2] + \cdots = 0$$

$$s\, I[1,1,-3] + s^2\, I[1,1,-2] + t^2\, s\, I[1,1,-1] + \epsilon^2 (s^2 t^2 + t^4) I[1,1,0] + t I[0,1,-2] + s\, I[1,0,-2] + \cdots = 0$$

high degree

Example of the block-triangular form

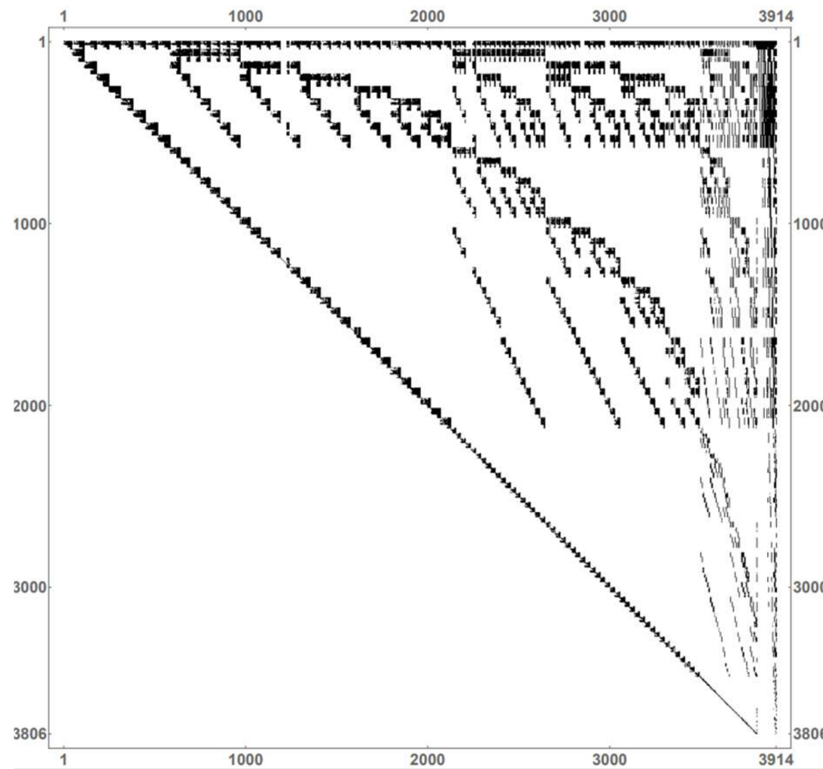
- Massless double pentagon
  - $\epsilon, s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$
  - rank 5 numerators



- Comparison
  - IBP system
    - $\sim 3 * 10^5$ relations
    - Numeric sampling under finite field :

      3.9s per phase point
  - Block-triangular form
    - 3806 relations to reduce 3914 integrals to

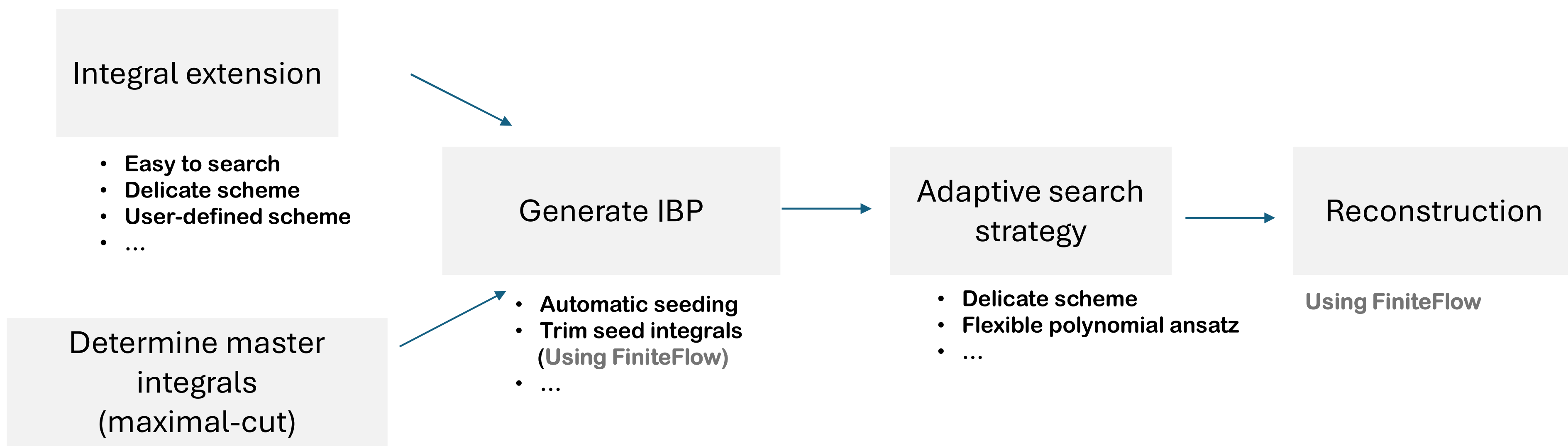
      108 MIs
    - 0.17s per phase point



XG, Liu, Ma, 1912.09294

Blade

- Framework



Usage of Blade

- Download

Blade

Link: https://gitlab.com/multiloop-pku/blade

项目信息

Block-triangular form improved Feynman integral decomposition .

- Install

```
chmod +x auto_install
./auto_install
```

sample_install.md

- Usage

- 1_automatic - introduction to automatic reduction of Feynman loop integrals;
- 1_preferred_masters - introduction to automatic reduction with user-defined master integrals;
- 1_userdefined_target - introduction to automatic reduction of user defined target integrals;

...

Basic usage

```
family = dbox;
dimension = 4-2*eps;
loop = {l1,l2};
leg = {p1,p2,p3,p4};
conservation = {p4->-p1-p2-p3};
replacement = {p1^2 -> 0, p2^2 -> 0, p3^2 -> msq, p1 p2 -> s/2, p1 p3 -> t/2, p2 p3 -> (-s-t)/2};
propagator = {(l1)^2,(l1+p1)^2,(l1+p1+p2)^2,l2^2-msq, (l1+l2)^2-msq, (l2-p1-p2)^2-msq, (l2-p3)^2, (l1-p3)^2,(l2+p1)^2};
topsector = {1,1,1,1,1,1,1,0,0};
numeric = {s -> 1};
```

```
BLFamilyDefine[family,dimension, propagator,loop,leg,conservation, replacement,topsector,numeric]
```

```
target={BL[dbox,{1,1,1,1,1,1,1,0,-3}],BL[dbox,{1,1,1,1,1,1,1,-1,-2}],
BL[dbox,{1,1,1,1,1,1,1,-2,-1}],BL[dbox,{1,1,1,1,1,1,1,-3,0}],
BL[dbox,{2,1,1,1,1,1,0,0,-2}], BL[dbox,{2,1,1,1,1,1,0,-2,0}],
BL[dbox,{0,1,2,2,1,1,0,0,-2}], BL[dbox,{0,1,2,2,1,1,0,-2,0}]};
```

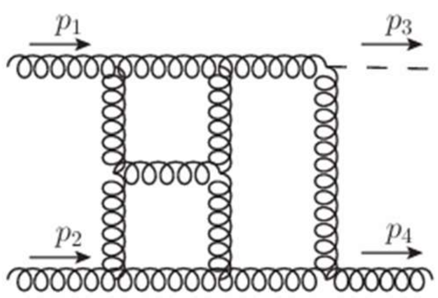
```
res = BLReduce[target];
```

```
de = BLDifferentialEquation[{t}]
```

- Define the integral family using “BLFamilyDefine”
- Reduce target integrals using “BLReduce”
- Construct the differential equations using “BLDifferentialEquation”

Example 1: 3-loop 4-point one massive external line

- $pp \to H + j$ $@N^3LO_{HTL}$ and other processes
- Comparison



121 MIs

Three variables：ϵ , m^2, t . $(s \to 1)$

r_{max}	CPU · h (With block-triangular form)	CPU · h (Without block-triangular form)
3	60	1800
4	180	11000

r_{max}	$\boldsymbol{t_{IBP}}$	$\boldsymbol{t_{BL}}$	N_{search}	N_{fit}	N_{recon}	N_{primes}
3	41s	0.1s	400	150	42000	4

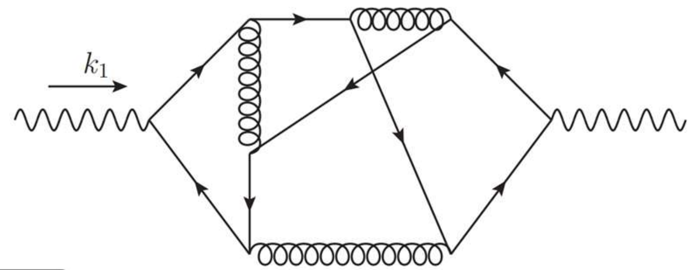
- r_{max} represents the largest rank of top-sector Feynman integrals
- Block-triangular form enhance the IBP reduction efficiency by 1-2 orders
- Improvement on sampling: (4*42000*41)/(4*42000*0.1+ (1*400+3*150)*41) ≈ 133.

Example 2: 4-loop 2-point one massive internal line

- $e^+ e^- \to \gamma^*/Z^* \to t\,\bar{t}$ @ N^3LO_{QCD}

  Chen, XG, He et al, 2209.14259

- Comparison



369 MIs

Two variables: ϵ , m_t^2. $(k_1^2 \to 1)$

r_{max}	CPU · h (With block-triangular form)	CPU · h (Without block-triangular form)
3	120	1200
4	280	8000

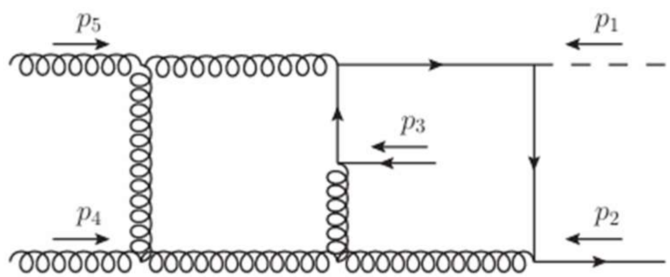
r_{max}	t_{IBP}	t_{BL}	N_{search}	N_{fit}	N_{recon}	N_{primes}
4	440s	1.1s	128	64	6041	8

- The advantages of block-triangular form becomes more pronounced as the task grows in complexity

Example 3: 2-loop 5-point one massive external line

- $pp \to Wjj,\ pp \to Hb\bar{b}, \ldots @N^2LO_{QCD}$

  Abreu, Chicherin, Ita et al. 2306.15431
  Hartanto, Poncelet, Popescu et al. 2205.01687
  Abreu, Cordero, Ita et al, 2110.07541
  Badger, Hartanto, Kryś et al, 2107.14733, 2201.04075
  Badger, Hartanto, Zoia, 2102.02516



142 MIs

six variables: $\epsilon, m^2, s_{23}, s_{34}, s_{45}, s_{51}, (s_{12} \to 1)$

- Comparison
  - **Trick**: $\epsilon = \epsilon_0$

r_{max}	t_{IBP}	t_{BL}	N_{search}	N_{fit}	N_{recon}
5	6s	0.16s	2000	1000	$?10^5$

  - The block-triangular form is about 38 times faster than the numeric IBP
  - Probes for reconstruction are indeterminate due to exhausting 1.5 TB memory

New features: general integrand

- General integrand

(1) $$\frac{\mathcal{F}_i}{\prod_{a=1}^{N} D_a^{\nu_a}}, \quad \text{for } i = 1, \cdots, n,$$

  - Provided that $\frac{\partial \mathcal{F}_i}{\partial D_a}$ can be expressed as linear combination of terms in the form of (1) with coefficients independent of loop momenta.
  - IBP reduction holds

- Applications
  - Symbolic reduction
    - Recurrence relations
  - Generating functions

A simple example: $\mathcal{F} = \mathcal{D}_4^{a_4}$

$I(1,1,1,-1+a_4) = (s+t-a_4)I(1,1,1,a_4) + a_4 I(1,0,1,a_4)$

$I(1,0,1,-1+a_4) = (a_4 - D)\, I(1,0,1,a_4)$

Application: generating functions

- $\mathcal{F} = e^{\sum_{i=K+1}^{N} x_i \mathcal{D}_i}$

$$G(\{\eta_i\},\{x_i\}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\exp\{\sum_{i=K+1}^{N} x_i \mathcal{D}_i\}}{(\mathcal{D}_1 - \eta_1)\cdots(\mathcal{D}_K - \eta_K)}$$

- Input: differential equations for generating functions

$$\frac{\partial}{\partial \eta_i} \vec{G} = A_{\eta_i} \vec{G} \qquad \frac{\partial}{\partial x_j} \vec{G} = A_{x_j} \vec{G}$$

reasonable computational complexity

- Output: arbitrary high rank/dots reductions
  - e.g. suppose that only $I[1,1,1,0]$ and $G[1,1,1,0]$ are master integrals

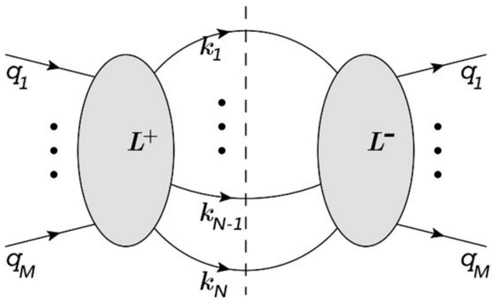
$$I[1,1,1,-20] = \lim_{x_i \to 0, \eta_i \to 0} G[1,1,1,-20] \propto \lim_{x_i \to 0, \eta_i \to 0} \frac{\partial^{20} G[1,1,1,0]}{\partial x_4^{20}}$$

$$\xrightarrow{\text{Differential Equation}} \lim_{x_i \to 0, \eta_i \to 0} f * G[1,1,1,0]$$

$$= g * I[1,1,1,0]$$

New features: symmetry

- Symmetry detection
    - Canonicalize and find the standard representation
    - Symmetry is identified among denominators that have identical prescription. $(D_j + \mathrm{i}0, D_j, D_j - \mathrm{i}0)$



- Simplify symmetry relations
    - Shorten the equation

$$D_a \to D'_a = \sum_b A_{ab} D_b$$

$$\int \mathrm{d}\mu_L \frac{(D^{(l)}_{k+1})^{-\nu_{k+1}} \dots (D^{(l)}_N)^{-\nu_N}}{D_1^{\nu_1} \dots D_k^{\nu_k}} = \int \mathrm{d}\mu_L \frac{(D^{(r)}_{k+1})^{-\nu_{k+1}} \dots (D^{(r)}_N)^{-\nu_N}}{D^{\nu_{\sigma(1)}}_{\sigma(1)} \dots D^{\nu_{\sigma(k)}}_{\sigma(k)}}$$

Choose $\mathcal{D}_a^{(l)}$ and $\mathcal{D}_a^{(r)}$ to find an equation with the minimal number of terms

    - Trim the system

Other new features

- Divide sub-families
  - Group target integrals based on distinct features, such as high rank and high dots e.g. $I[1,1,1,-5]$ and $I[2,2,2,-1]$ -> efficient seeding (see “DivideLevel”)
- Trim seed integrals
  - The rank of seed integrals of sub-sectors can be smaller than that of top-sector -> small system (see “FilterLevel”)
- Spanning-sector reduction
  - Inspired by the generalized-cut and “master-wise” reduction
  - Reduce memory usage significantly
- Complex mode
  - $m_1^2 = 1 + \mathrm{i}\,, m_2^2 = 2 + 2\mathrm{i}$

Summary and outlook

- Summary
  - Block-triangular form is a way to improve the efficiency of IBP reduction
  - Blade is a fully automated integral reduction package, armed with the block-triangular form algorithm
  - Blade has many new features, making it applicable in more general cases
- Outlook
  - User-defined IBP systems (e.g. syzygy equations provided by NeatIBP)
  - Open-source implementation
  - Partial-fractioned reconstruction

Chawdhry, 2312.03672

NeatIBP, Zi-Hao Wu, et al. 2305.08783

Thank you!