Fragmentation functions: definitions & sum rules

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Jefferson Lab and Old Dominion University

Based on:

John Collins, TCR Phys.Rev.D 109 (2024) 1, 016006

and other works in progress

Loopfest 2024,

Southern Methodist University, Dallas, Texas May 21, 2024

Fragmentation functions

- Definition: start from $\langle \mathbf{k} | \mathbf{k}' \rangle$

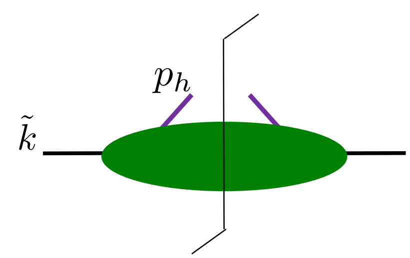
Insert number density operator

$$d_{(0),h/j}(z) \equiv \int \mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}\, d_{(0),h/j}(z,\boldsymbol{p}_\mathrm{T})$$

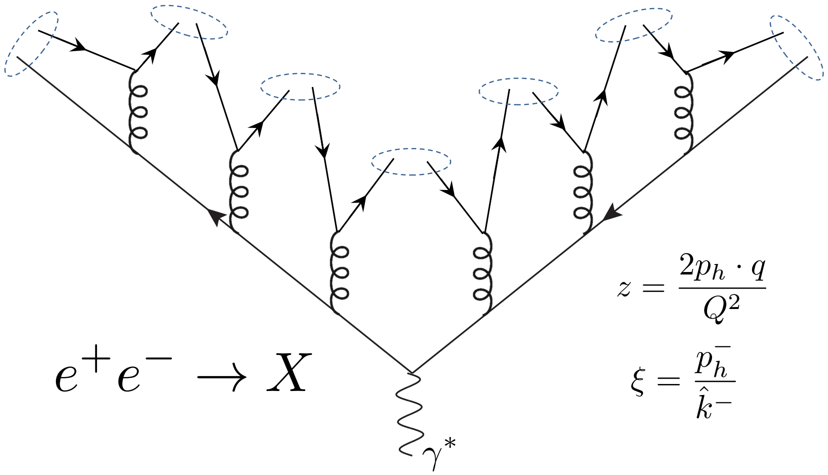
$$= \frac{\mathrm{Tr}_D}{4}\sum_X z^{1-2\epsilon}\int \frac{\mathrm{d}x^+}{2\pi} e^{ik^-x^+}\gamma^- \langle 0|\psi_j^{(0)}(x/2)\,|h,X,\mathrm{out}\rangle\langle h,X,\mathrm{out}|\overline{\psi}_j^{(0)}(-x/2)\,|0\rangle$$

Graphically

$$d(\xi,\boldsymbol{p}_{hp\mathrm{T}}) = \frac{1}{4\xi}\int \frac{\mathrm{d}k_H^-}{(2\pi)^4}\mathrm{Tr}\left[\gamma^- \quad \right]$$



Cross sections

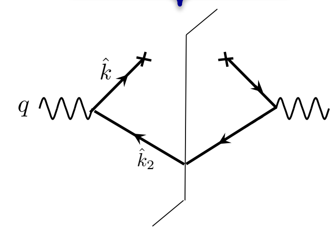


$$e^+e^- \to X$$

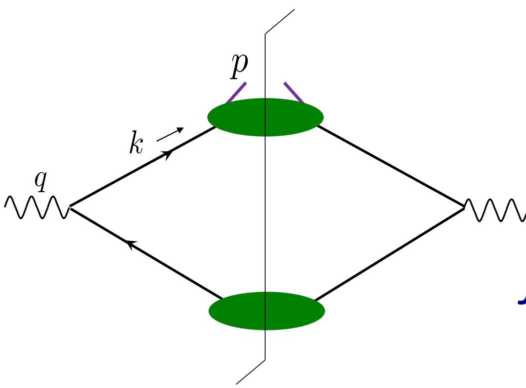
$$z = \frac{2p_h \cdot q}{Q^2}$$

$$\xi = \frac{p_h^-}{\hat{k}^-}$$

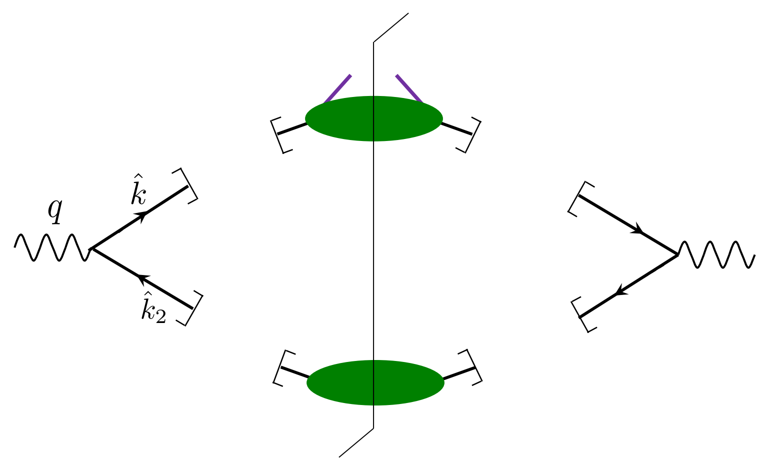
$$\frac{\mathrm{d}^3\boldsymbol{p_h}}{2E_h(2\pi)^3} = \sum \int_z^1 \frac{\mathrm{d}\xi}{\xi^2} \underbrace{\left(2E_{\hat{k}}(2\pi)^3 \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}^3\hat{\boldsymbol{k}}}\right)} d(\xi;\mu) + O\left(\frac{\Lambda^2}{zQ^2}\right)$$



$+ \cdots$



factorization



Fragmentation functions

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- In QCD: Include Wilson lines and color trace.

Sum rules:

• Momentum	$\sum_h \int_0^1 \mathrm{d}z\; z\, d_{h/j}(z) = 1$
• Charge	$\sum_h \mathcal{Q}_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j$

Workman, et al, Particle Data Group, Review of Particle Physics, PTEP 2022, 083C01 (2022) Eq.(19.3)

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• Multiplicity	$\sum_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \langle N\rangle$
• Extensions to multihadron FFs & other correlation functions	$\sum_h \int \mathrm{d}z_1 \mathrm{d}z_2\, d_{h_1h_2/j}(z_1,z_2) = \langle N(N-1)\rangle$ *Pitonyak et al, Phys.Rev.Lett. 132 (2024) 1, 011902*
• TMD FFs, etc…	

General derivations

- Definition (bare)

$$d_{(0),h/j}(z,\boldsymbol{p}_\mathrm{T}) \sim \sum_X \langle \underbrace{\mathrm{quark}}_{\textit{Off-shell quark}} | \overbrace{h, X, \mathrm{out}}^{\textit{Count particles of type h}} \rangle \langle h, X, \mathrm{out} | \mathrm{quark}' \rangle$$

$$\sum_X |h, X, \mathrm{out}\rangle\langle h, X, \mathrm{out}| \equiv \sum_X a^\dagger_{h,p,\mathrm{out}} |X, \mathrm{out}\rangle\langle X, \mathrm{out}| a_{h,p,\mathrm{out}}$$

$$= \underline{a^\dagger_{h,p,\mathrm{out}} a_{h,p,\mathrm{out}}}$$

Operators for on-shell asymptotic states

- Operators for conserved currents (e.g. momentum)

$$\mathcal{P}^\mu = \sum_h \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}}{(2\pi)^{3-2\epsilon}} \, a^\dagger_{h,p,\mathrm{out}} p^\mu a_{h,p,\mathrm{out}}$$

- Sum rules follow from unitarity of asymptotic states

$$\sum_X |X, \mathrm{out}\rangle\langle X, \mathrm{out}| = \widehat{1}$$

- Preserved by standard renormalization

- Straightforward in **nongauge** theories

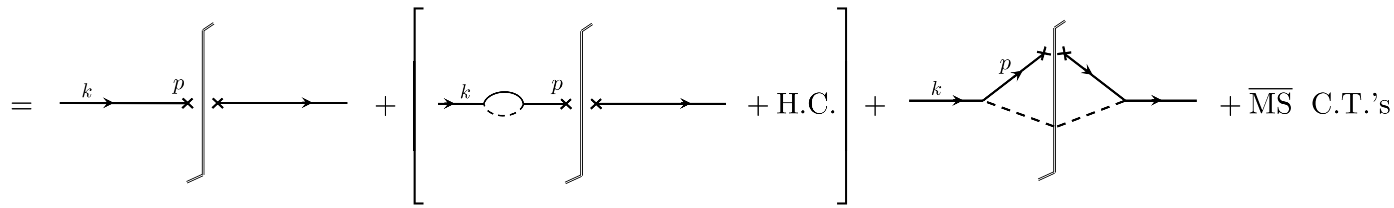
Examples in $\overline{MS}$

Dirac “quark” Scalar “pion”

- Scalar Yukawa theory: $\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial\phi)^2 - m_q\bar{\psi}\psi - \frac{m_\pi^2}{2}\phi^2 - \lambda\,\bar{\psi}\psi\phi$

Quark-in-quark fragmentation function

$d_{q_j/j}(z;\mu)$



$$= \delta(1-z) - a_\lambda(\mu)\delta(1-z)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] + a_\lambda(\mu)(1-z)\left[\ln\left(\frac{1}{(1-z)^2+z}\right) + \ln\frac{\mu^2}{m^2} - 1 + \frac{(1+z)^2}{(1-z)^2+z}\right]$$

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Dirac "quark" *Scalar "pion"*

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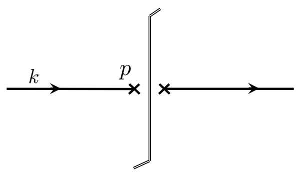
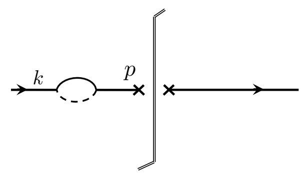
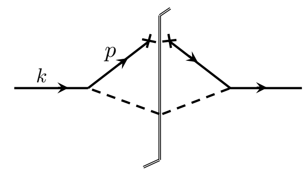
$=$ 

$+ \Big[$ 

$+\,\mathrm{H.C.}\Big] +$ 

$+\ \overline{\mathrm{MS}}\ \ \mathrm{C.T.'s}$

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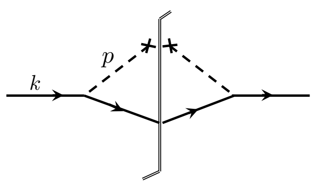
Pion-in-quark fragmentation function

$d_{\pi/j}(z;\mu)\ =$ 

$+\ \overline{\mathrm{MS}}\ \ \mathrm{C.T.}$

$$= a_\lambda(\mu)z\left[\ln\left(\frac{1}{(1-z)+z^2}\right) + \ln\frac{\mu^2}{m^2} - 1 + \frac{(2-z)^2}{(1-z)+z^2}\right]$$

Examples in $\overline{MS}$

- Moments

$$\int_0^1 \mathrm{d}z\, d_{q_j/j}(z;\mu) = 1 - a_\lambda(\mu)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] + a_\lambda(\mu)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] = 1\,,$$

$$\int_0^1 \mathrm{d}z\, d_{\pi/j}(z;\mu) = a_\lambda(\mu)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right]\,,$$

$$\int_0^1 \mathrm{d}z\, z d_{q_j/j}(z;\mu) = 1 - a_\lambda(\mu)\left[-\frac{13}{9} + \frac{\pi}{\sqrt{3}} + \frac{1}{3}\ln\frac{\mu^2}{m^2}\right]\,,$$

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Examples in $\overline{MS}$

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- Check sum rules:

✓ $$\sum_h \int_0^1 \mathrm{d}z\; z\, d_{h/j}(z) = 1$$

✓ $$\sum_h \mathcal{Q}_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j$$

✓ $$\sum_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \langle N \rangle$$

(Sort of)

Jlab DIS multiplicities ≈ 5

EIC $\approx$ *12 or 13*

A paradox in the definitions?

- Are FFs zero?

$$\langle \mathrm{quark}|\mathrm{quark}'\rangle = \sum_X \langle \mathrm{quark}|\underbrace{X, \mathrm{out}\rangle}_{T\to\infty}\langle X, \mathrm{out}|\mathrm{quark}'\rangle = 0$$

Asymptotic hadronic states

$$\langle \mathrm{quark}|\mathrm{hadron}\rangle = 0$$

A paradox in the definitions?

- Are FFs zero?

$$\langle \mathrm{quark} | \mathrm{quark}' \rangle = \sum_X \langle \mathrm{quark} | \underbrace{X, \mathrm{out} \rangle}_{T \to \infty \ \textit{Asymptotic hadronic states}} \langle X, \mathrm{out} | \mathrm{quark}' \rangle = 0$$

$$\langle \mathrm{quark} | \mathrm{hadron} \rangle = 0$$

- <u>Non-gauge theories</u>: Fields directly correspond to asymptotic physical states

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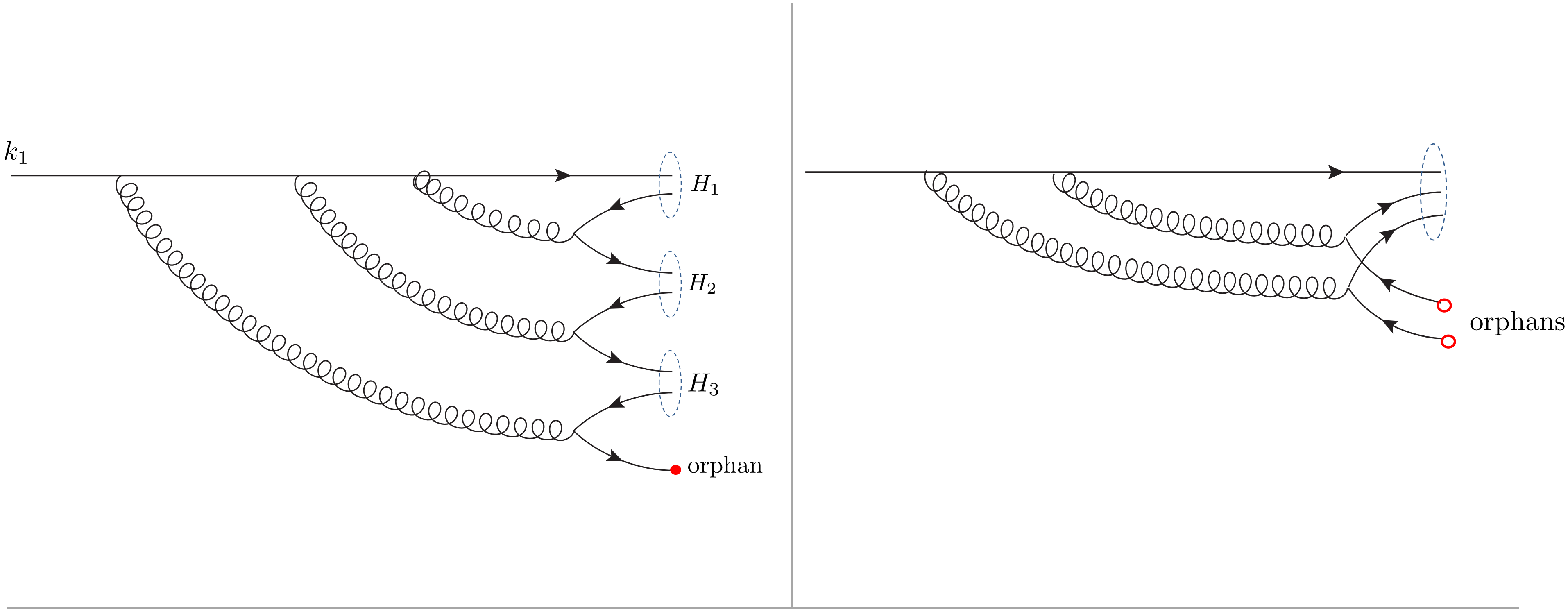
Asymptotic hadronic states

$$\langle \mathrm{quark} | \mathrm{hadron} \rangle = 0$$

- Non-gauge theories: Fields directly correspond to asymptotic physical states
- In QCD/QED: Local fields are not gauge invariant – they do not create unambiguously physical particle states
  - $\overline{\psi}(y)|0\rangle \to \overline{\psi}(y) WL[\infty, y; n]|0\rangle$
  - Wilson line is a source of color charge
  - Asymptotic states must include a kind of "quark/Wilson line bound state"

$$\underbrace{\mathcal{E}}_{\text{Normal hadronic Fock space}} \to \mathcal{E} \otimes \underbrace{\mathcal{B}}_{\text{Space of quark-Wilson bound states}}$$

Visualizing the issue



- At least one "orphan" (anti)quark is always left over

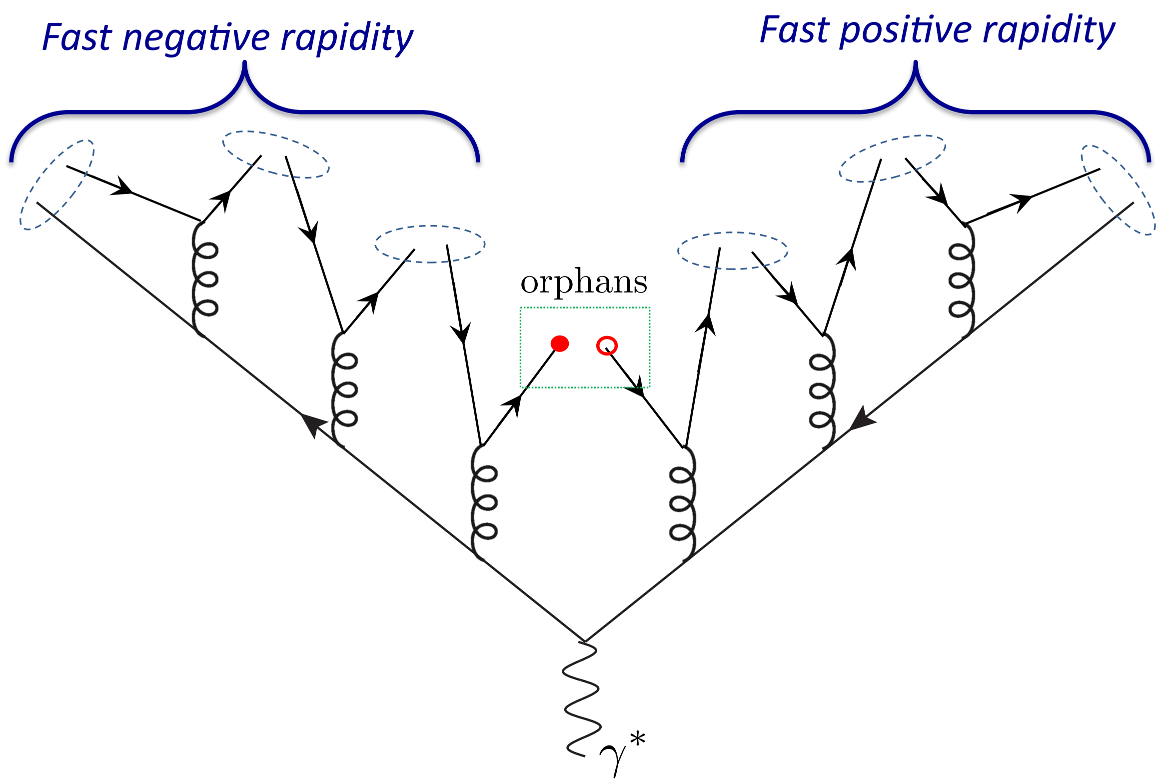
What occurs in a factorization derivation?

- Must match FFs onto full process in region of unclassifiable ≈ 0 rapidity hadrons

- Split unclassifiable hadron(s) & insert zero rapidity Wilson lines

- Slow hadrons lie outside the region relevant to the factorization theorem

$$e^+e^- \to h_1 + h_2 + X$$



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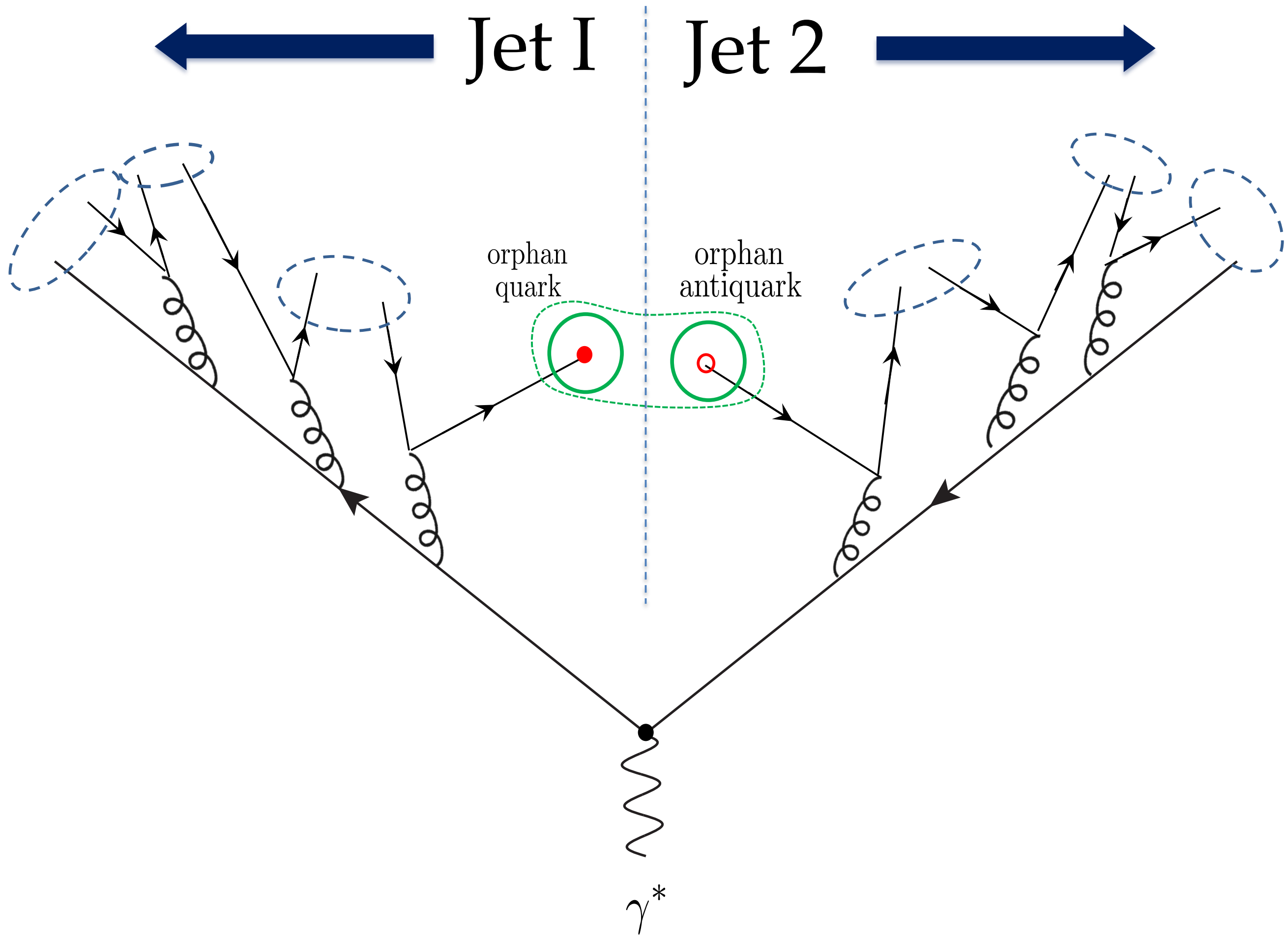
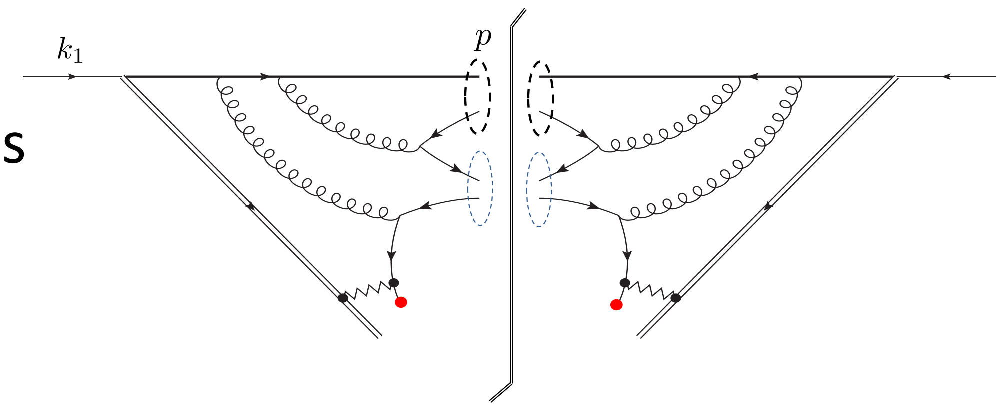


Deficit fragmentation functions

- Proposal: Take the operator definitions of "deficit fragmentation functions" seriously



- Momentum operator becomes

$$\mathcal{P}^\mu = \mathcal{P}^\mu_H + \mathcal{P}^\mu_B$$

$$\mathcal{P}^\mu_H \equiv \sum_{h\in H} \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}}{(2\pi)^{3-2\epsilon}} a^\dagger_{h,p,\mathrm{out}} p^\mu a_{h,p,\mathrm{out}}\,,$$

$$\mathcal{P}^\mu_B \equiv \sum_{b\in B} \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}}{(2\pi)^{3-2\epsilon}} a^\dagger_{b,p,\mathrm{out}} p^\mu a_{b,p,\mathrm{out}}\,.$$

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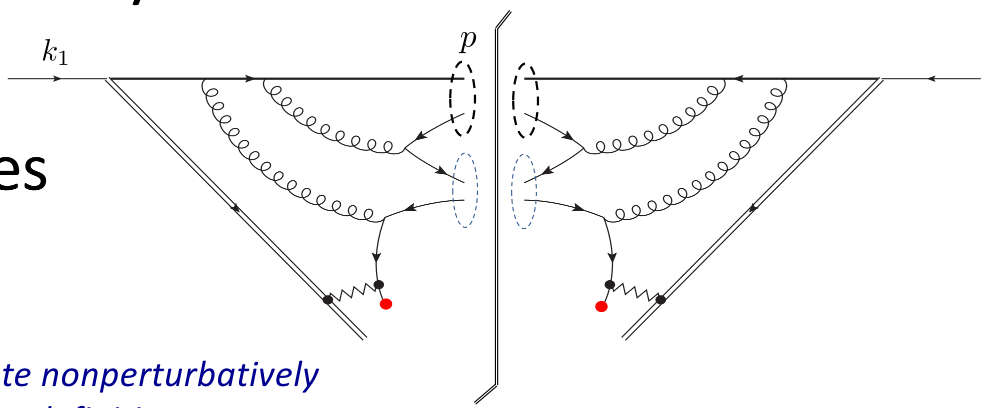

- Momentum sum rule becomes

$$\sum_{h\in H}\int_0^1 \mathrm{d}z\; z\,d_{h/j}(z) = 1 - \underbrace{\sum_{b\in B}\int_0^1 \mathrm{d}z\; z\,d_{b/j}(z)}$$

Calculate nonperturbatively from the definition

-A. Kerbizi, et al Phys. Rev. D 97, 074010 (2018)

-T. Ito, et al Phys. Rev. D 80, 074008 (2009)

- Other sum rules

$$\sum_{h\in H}\mathcal{Q}_h\int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j - \sum_{b\in B}\mathcal{Q}_b\int_0^1 \underline{\mathrm{d}z\; d_{b/j}(z)}$$

$$\sum_{h\in H}\int_0^1 \mathrm{d}z\; d_{h/j}(z) = \langle N\rangle - \sum_{b\in B}\int_0^1 \mathrm{d}z\; d_{b/j}(z)$$

Model deficit ffs as δ functions

- Momentum sum rule is preserved if deficit ff is $\propto \delta(z)$.

$$\sum_{h\in H}\int_0^1 \mathrm{d}z\; z\, d_{h/j}(z) = 1 - \cancel{\sum_{b\in B}\int_0^1 \mathrm{d}z\; z\, d_{b/j}(z)}$$

- Other sum rules are not

$$\sum_{h\in H}\mathcal{Q}_h\int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j - \sum_{b\in B}\underbrace{\mathcal{Q}_b\int_0^1 \mathrm{d}z\; d_{b/j}(z)}_{\text{Constant}}$$

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Relationship to factorization

- Factorization theorem applies to fixed z and $Q/\Lambda \to \infty$

$$\frac{\mathrm{d}^3\boldsymbol{p_h}}{2E_h(2\pi)^3} = \sum \int_z^1 \frac{\mathrm{d}\xi}{\xi^2} \left(2E_{\hat{k}}(2\pi)^3 \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}^3\hat{\boldsymbol{k}}}\right) d(\xi;\mu) + O\left(\frac{\Lambda^2}{zQ^2}\right)$$

$$\Downarrow$$

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}z} = d(z;Q) + O\left(\frac{\Lambda^2}{zQ^2}\right) + O\left(\alpha_s\right) .$$

$$\Downarrow$$

$$\sum_{\text{hadron types}} \int_0^1 \mathrm{d}z\, \frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}z} \approx \sum_{\text{hadrons}} \int_0^1 \mathrm{d}z\, d(z;Q) = \langle N \rangle$$

Questions

- Are deficit ffs *exactly* localized at z = 0?

- Is there an impact on extractions?
  - Sum rules are preserved by DGLAP, but only if all ffs are included

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} d_{h/j}(z;\mu) = \sum_{j'} \int_z^1 \frac{\mathrm{d}z'}{z'} d_{h/j'}(z/z';\mu) P_{j'j}(z')$$

- Can calculating deficit fragmentation functions nonperturbatively lead to insights about hadronization?

Extractions

- Difficulties with combining evolution and sum rules

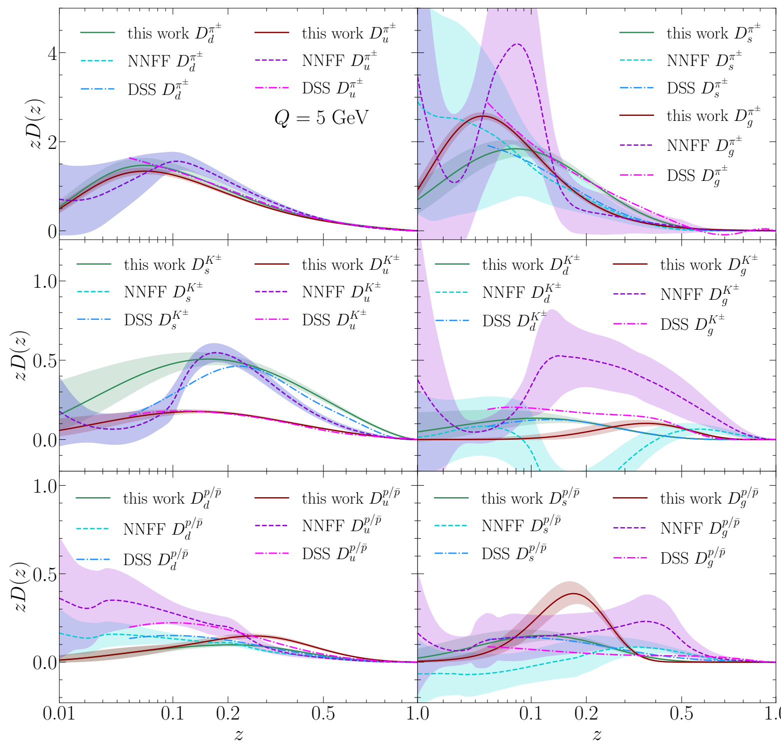
  "This observation renders the energy sum rule (8) a delicate concept for perturbative QCD FFs and we believe it should not be considered within this theoretical framework unless the z → 0 behaviour of FFs is under better control."

  -S. Kretzer (2000), Phys.Rev.D 62 (2000) 054001

- Recent tests & extractions:

  J. Gao, et al 2401.02781 [hep-ph]

Simultaneous Determination of Fragmentation Functions and Test on Momentum Sum Rule



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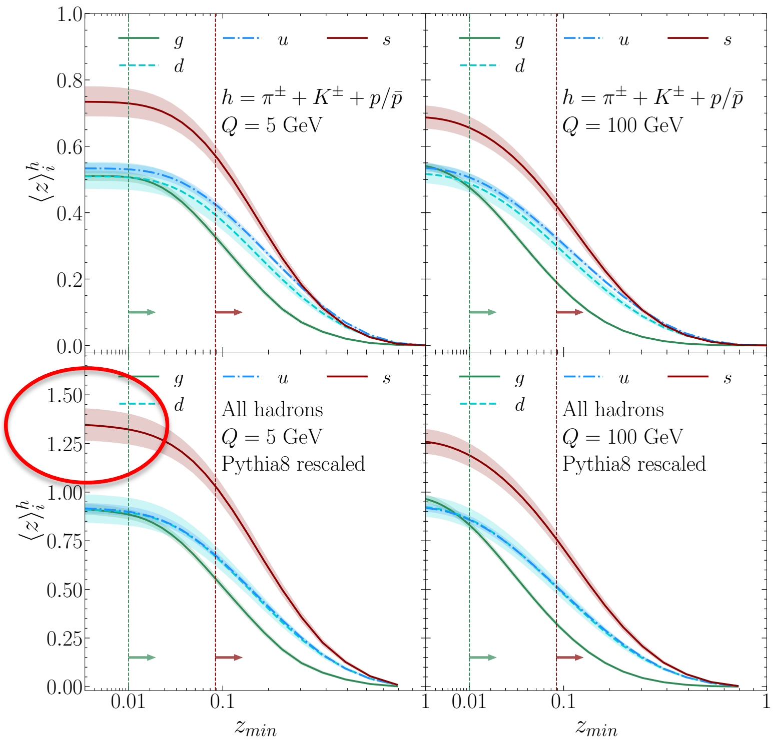
Simultaneous Determination of Fragmentation Functions and Test on Momentum Sum Rule

Fill in neutral hadrons using PYTHIA8



Refining the definition (preliminary)

- Proposal: Start with gauge invariant $q\bar{q}$ wave packet states
  - Basis states: $|k,s,\underline{w}\rangle \equiv \mathrm{Tr}_C \int \mathrm{d}y^+\,\mathrm{d}^2\boldsymbol{y}_\mathrm{T}\, e^{-ik\cdot y}\underline{\bar{u}^s_k\gamma^-\psi(w+L)\mathrm{WL}[w+L,y]}\bar{\psi}(y)\gamma^- u^s_k|0\rangle$

$$L = (l, -e^{-y}, \mathbf{0})$$

$$w = (0,0,\boldsymbol{w}_\mathrm{T})$$

  - Wave packets: $|F,g,l\rangle \equiv g(w,l)\int \frac{\mathrm{d}k^-\,\mathrm{d}^2\boldsymbol{k}_\mathrm{T}}{2k^-(2\pi)^3} F(k,l)|k,w\rangle$

$$\int \frac{\mathrm{d}k^-\,\mathrm{d}^2\boldsymbol{k}_\mathrm{T}}{(2\pi)^3}|F(k,l)|^2 = 1 \qquad \int \mathrm{d}w^+\,\mathrm{d}^2\boldsymbol{w}_\mathrm{T}\,|g(w,l)|^2 = 1 \qquad \lim_{l\to\infty}\int \mathrm{d}w^+\,\mathrm{d}^2\boldsymbol{w}_\mathrm{T}\,\langle F,g,l|F,g,l\rangle = 1$$

  - Work with: $\sum_X \lim_{l\to\infty}\int \mathrm{d}w^+\,\mathrm{d}^2\boldsymbol{w}_\mathrm{T}\,\langle F,g,l\underline{|X\rangle\langle X|}F,g,l\rangle = 1$

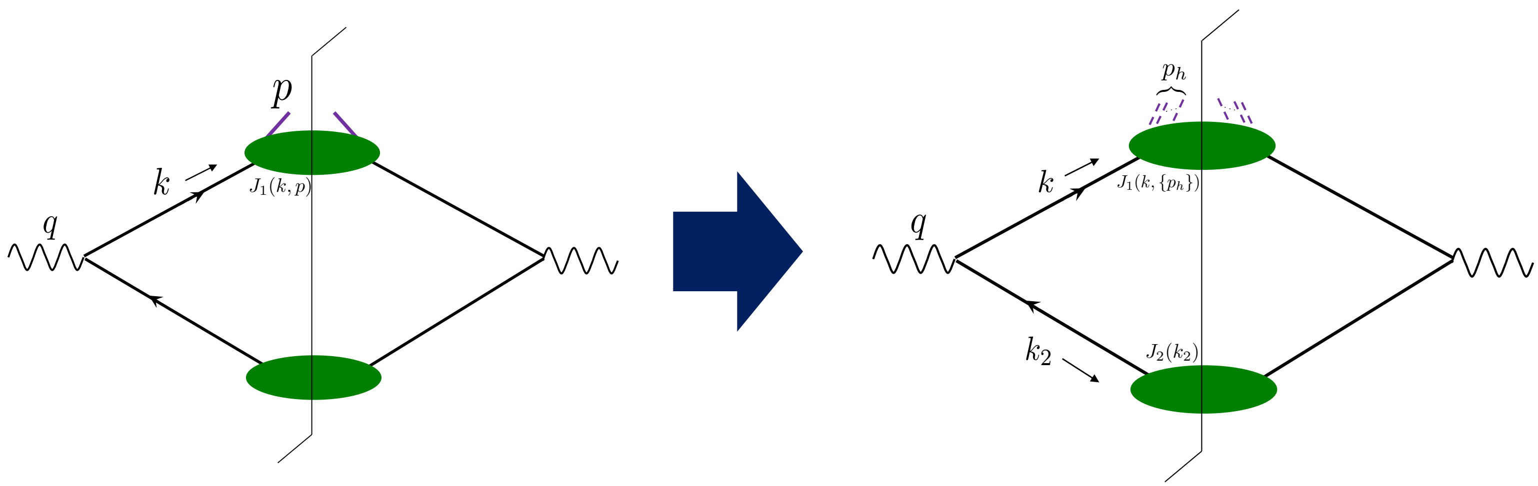
*Asymptotic **Hadron** states*

  - Arrive at standard definition but with

$$|X\rangle\langle X| \to \bar{\psi}(w+L)\gamma^- u^s_k|X\rangle\langle X|\bar{u}^s_k\gamma^-\psi(w+L)$$

Other fragmentation functions

- Dihadron fragmentation or n-hadron fragmentation



- Access to transversity, tensor charge, etc

-A. Bianconi, et al Phys. Rev. D 62 (2000) 034008
-Bacchetta and Radici, Phys. Rev. D67 (2003) 094002

Relationship to factorization

- Factorization theorem applies to fixed z and $Q/\Lambda \to \infty$

$$\frac{\mathrm{d}^3\boldsymbol{p_h}}{2E_h(2\pi)^3} = \sum \int_z^1 \frac{\mathrm{d}\xi}{\xi^2} \left(2E_{\hat{k}}(2\pi)^3 \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}^3\hat{\boldsymbol{k}}}\right) d(\xi;\mu) + O\left(\frac{\Lambda^2}{zQ^2}\right)$$

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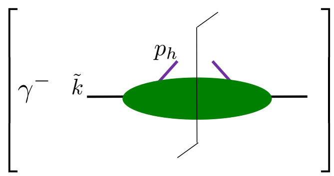
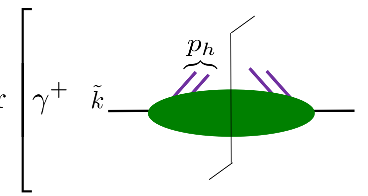
Other fragmentation functions

- $d(\xi, \boldsymbol{p}_{hp\mathrm{T}}) = \frac{1}{4\xi}\int \frac{\mathrm{d}k_H^-}{(2\pi)^4}\mathrm{Tr}$ 

$d(\xi, \boldsymbol{p}_{hp\mathrm{T}}, \{p_h\}) = \mathcal{C}\int \frac{\mathrm{d}k_H^-}{(2\pi)^4}\mathrm{Tr}$ 

- "Parton Model" + # sum rule suggests a definition

$$(2\pi)^3 2E_1 (2\pi)^3 2E_2 \frac{\mathrm{d}\sigma}{\mathrm{d}^3\mathbf{p_1}\mathrm{d}^3\mathbf{p_2}} = \sigma_0 \frac{\Delta}{2} \quad \Rightarrow \quad \mathcal{C} = \frac{1}{64\xi_1\xi_2(2\pi)^3} \sim \frac{1}{\xi^2}$$

- Paradox: Factorization & renormalization give $\mathcal{C} = \frac{1}{4\xi}$

- Problem lies with application of # sum rule

Conclusion

- Status of sum rules is more complicated but more interesting for fragmentation functions than for pdfs
- Relevant for precision tests and phenomenological extractions of fragmentation functions
- Refining definitions: Possible avenues for understanding hadronization?
- Overly literal application of sum rules leads to conflicting results

Backup

Other considerations

- Parton model derivation of momentum sum rule
  - Definition of inclusive cross section

$$\sum_h \int \mathrm{d}^3\mathbf{p}_h \, \frac{\mathrm{d}\sigma^h}{\mathrm{d}x\,\mathrm{d}Q^2\,\mathrm{d}^3\mathbf{p}_h} = \langle N\rangle \frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z\, F_{1,h}(x,z,Q^2) = \langle N\rangle F_1(x,Q^2)$$

$$\Longrightarrow \quad \sum_h \int \mathrm{d}z\, zF_{1,h}(x,z,Q^2) = F_1(x,Q^2)$$

  - Parton model $F_{1,h}(x,z,Q^2) = H_1 f(x) d_h(z)\,,\; F_1(x,Q^2) = H_1 f(x)$

$$\sum_h \int \mathrm{d}z\, zF_{1,h}(x,z,Q^2) = H_1 f(x)\left(\sum_h \int \mathrm{d}z\, z d_h(z)\right) = H_1 f(x) \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z\, z d_h(z) = 1$$

Other considerations

- Experimentalists and theorists means something different by "inclusive!"

- What does $1 = \sum_X |X\rangle\langle X|$ really mean?

  – Not included in (many) experimental SIDIS measurements: $eN \to e + N + \pi$?? $eN \to e + \rho + X$

  – But included in DIS measurements

Other considerations

- What if elastic pions are subtracted?

$$\cancel{eN \to e + N + \pi}$$

Other considerations

- Parton model derivation of momentum sum rule
  - Definition of inclusive cross section

$$\sum_h \int \mathrm{d}^3\mathbf{p}_h \, \frac{\mathrm{d}\sigma^h}{\mathrm{d}x\,\mathrm{d}Q^2\,\mathrm{d}^3\mathbf{p}_h} = \langle N \rangle \frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z \, F_{1,h}(x,z,Q^2) = \langle N \rangle F_1(x,Q^2)$$

$$\Longrightarrow \sum_h \int \mathrm{d}z \, zF_{1,h}(x,z,Q^2) \neq F_1(x,Q^2)$$

No elastic pions

Elastic pions

  - Parton model $F_{1,h}(x,z,Q^2) = H_1 f(x) d_h(z)\,,\; F_1(x,Q^2) = H_1 f(x)$

$$\sum_h \int \mathrm{d}z \, zF_{1,h}(x,z,Q^2) = H_1 f(x) \left(\sum_h \int \mathrm{d}z \, z d_h(z) \right) \neq H_1 f(x) \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z \, z d_h(z) = 1$$