RECENT PROGRESS IN PLANAR TWO-LOOP SIX-POINT FEYNMAN INTEGRAL COMPUTATION

BASED ON ARXIV:2403.19742 WITH J. HENN, J. MICZAJKA, T. PERARO, Y. XU, Y. ZHANG

ANTONELA MATIJAŠIĆ

LOOPFEST XXVII

DALLAS, TX

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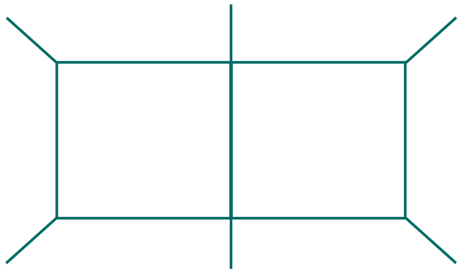
MAX-PLANCK-INSTITUT FÜR PHYSIK

European Research Council
Established by the European Commission

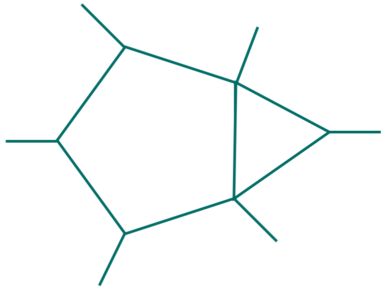
KEY MESSAGE

Can evaluate six-point Feynman integrals at high precision within a few minutes by solving their canonical DEs.

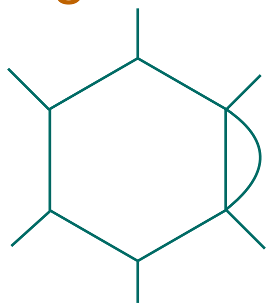
Double-Box



Pentagon-Triangle



Hexagon-Bubble



High energy physics has entered the precision era.

- Measurements of observables for many LHC processes are now **available at 1% precision**
- On theory side, one bottleneck are reliable and fast evaluations of **two-loop Feynman integrals**

TWO-LOOP FEYNMAN INTEGRALS

- State-of-the-art: two-loop five-point Feynman integrals (five on-shell legs, four on-shell + one off-shell leg, one or two massive propagtors)

[Gehrmann, Henn, Lo Presti ’18; Abreu, Dixon, Herrmann, Page, Zeng ’18; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia ’18; Abreu, Ita, Moriello, Page, Tschernow, Zeng ’20; Chicherin, Sotnikov, ’20; Abreu, Ita, Page, Tschernow ’21; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia ’23; Badger, Becchetti, Cahubey, Marzucca ’23; Febres Cordero, Figueiredo, Kraus, Page, Reina ‘23]

[cf. talk by Kraus]

- Very little is known about **two-loop six-point processes** in general theories
- Phenomenologically interesting: 4-jet production @ LHC
- Theoretically interesting:
  - Analytic structure of QCD function space
  - Wilson loops with Lagrangian insertion

Pick an integral family I

Take derivatives ∂_{v_i}

$dI = M \cdot \tilde{I}$

Solve IBP identities: $\tilde{I} = B \cdot I$

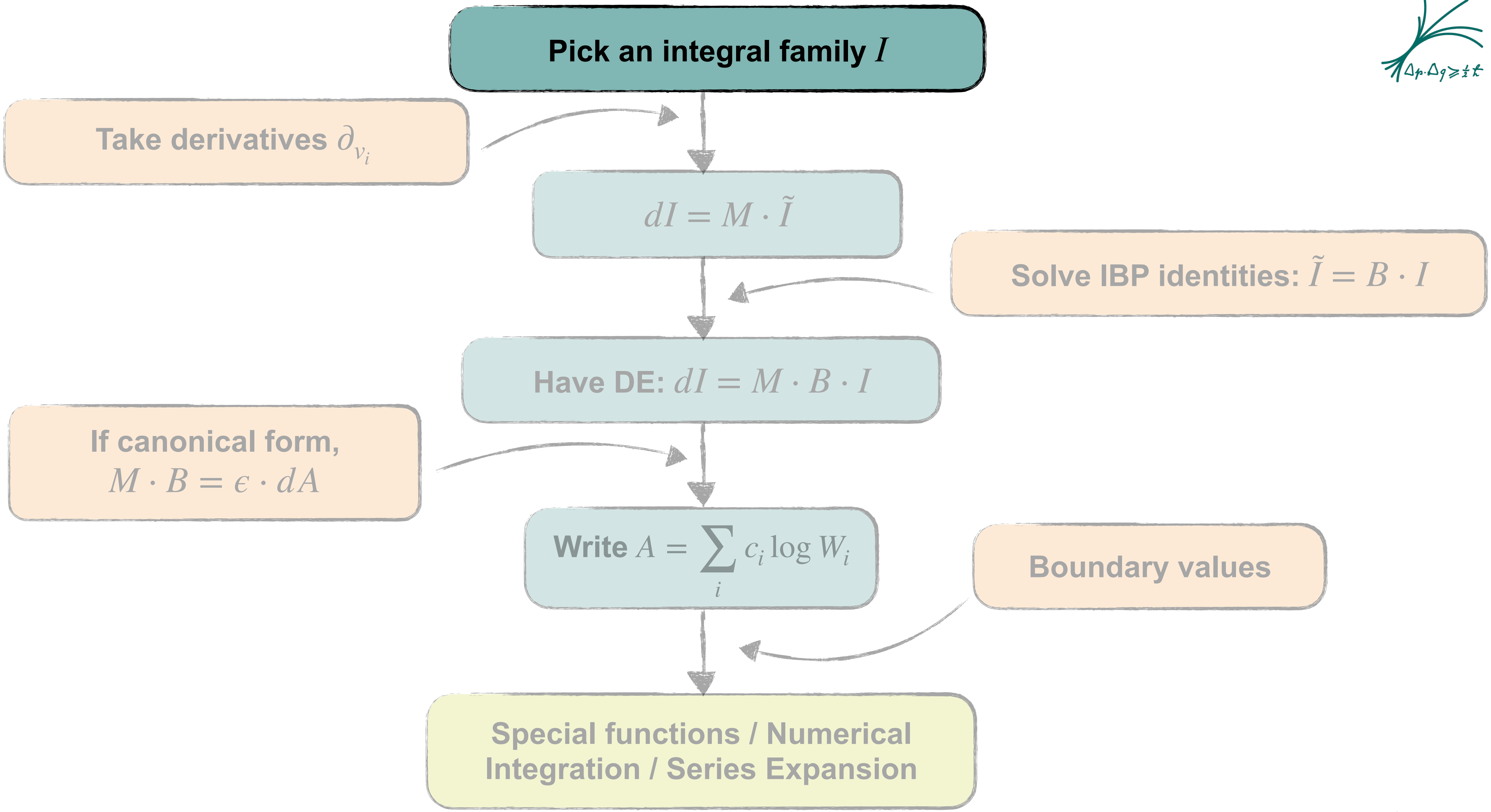
Have DE: $dI = M \cdot B \cdot I$

If canonical form,
$M \cdot B = \epsilon \cdot dA$

Write $A = \sum_i c_i \log W_i$

Boundary values

Special functions / Numerical Integration / Series Expansion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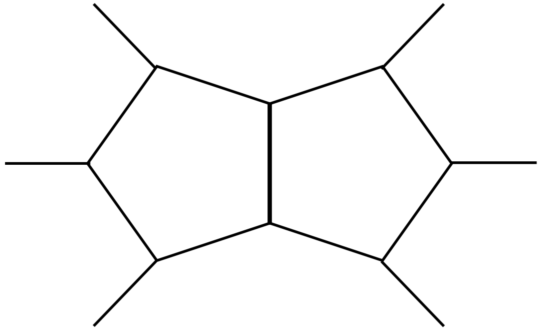
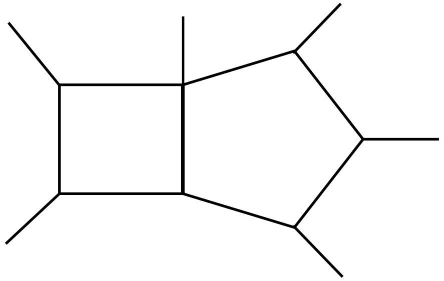
PLANAR TWO-LOOP SIX-POINT INTEGRAL FAMILIES

Double-Pentagon



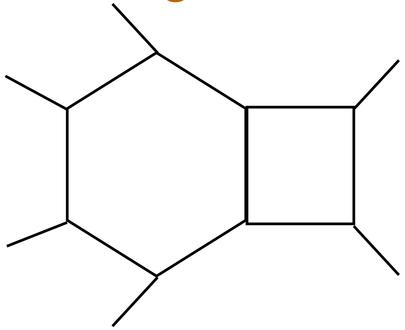
5 (+ 251) MI

Pentagon-Box



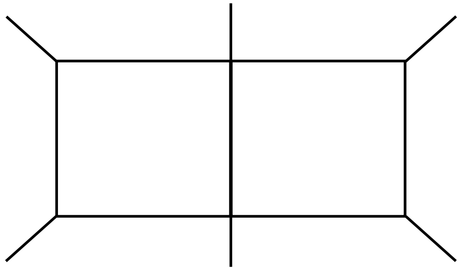
3 (+ 114) MI

Hexagon-Box



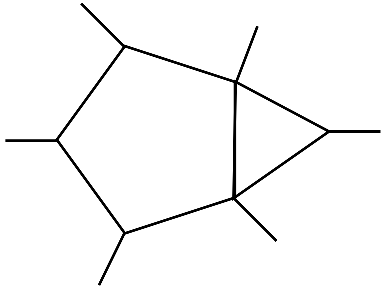
1 (+ 202) MI

Double-Box



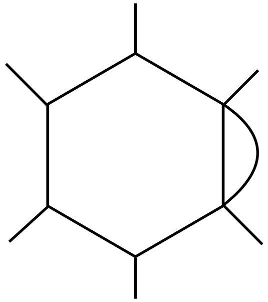
7 (+ 59) MI

Pentagon-Triangle



1 (+ 43) MI

Hexagon-Bubble



1 (+ 31) MI

+ subsector integrals

PLANAR TWO-LOOP SIX-POINT INTEGRAL FAMILIES

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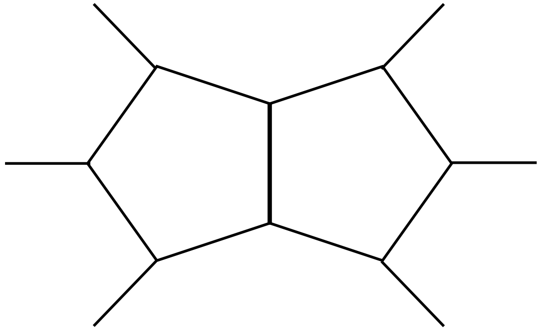

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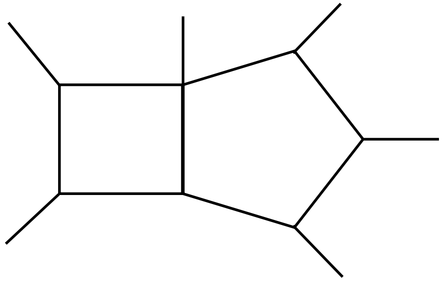

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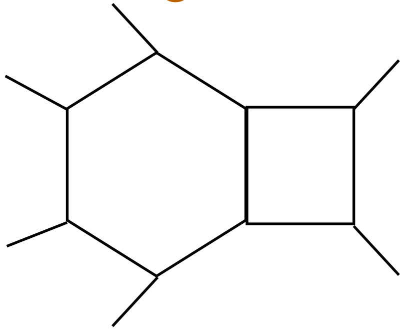

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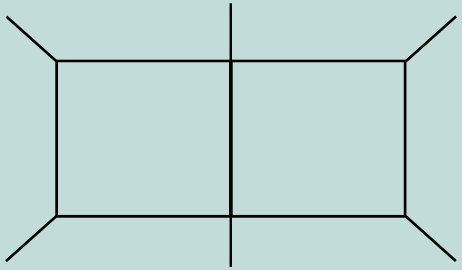

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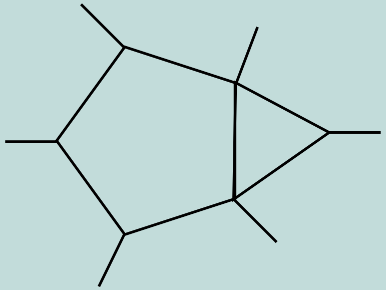

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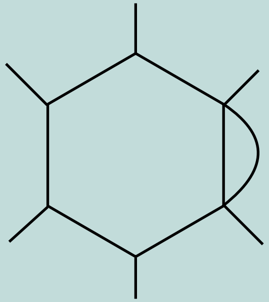


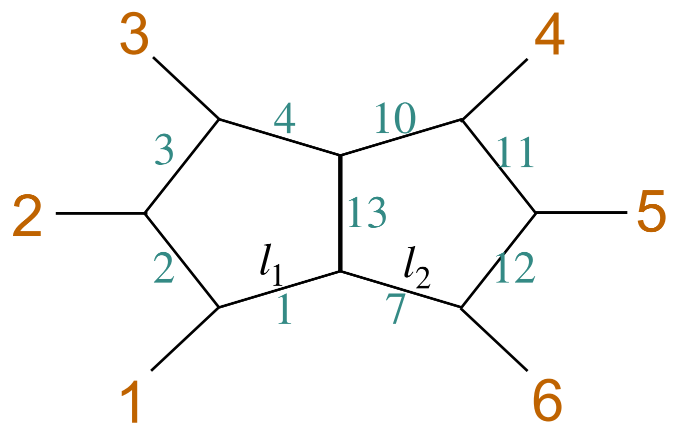
1 (+ 31) MI

In 2403.19742!

+ subsector integrals

NOTATION AND KINEMATICS

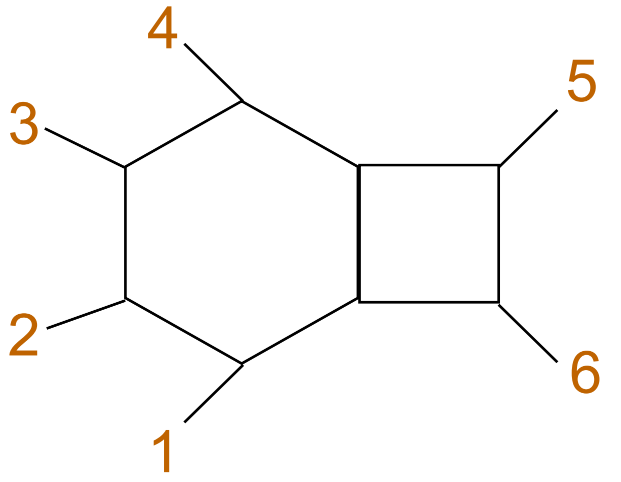
$$I^{d_0}[a_1,\ldots,a_{13}] = e^{2\epsilon\gamma_E}\int \frac{\mathrm{d}^{d_0-2\epsilon}l_1\mathrm{d}^{d_0-2\epsilon}l_2}{i\pi^{(d_0-2\epsilon)}}\frac{1}{D_1^{a_1}\ldots D_{13}^{a_{13}}}$$



$$D_1 = l_1^2,\ D_2 = (l_1+p_1)^2,\ \ldots\ ,\ D_{13} = (l_1+l_2)^2 \qquad a_i \in \mathbb{Z}$$

External momenta:

$$p_i^2 = 0, \quad i = 1,\ldots,6 \qquad \sum_{i=1}^{6} p_i = 0 \qquad p_i \in \mathbb{R}^{D_{\text{ext}}}$$



NOTATION AND KINEMATICS

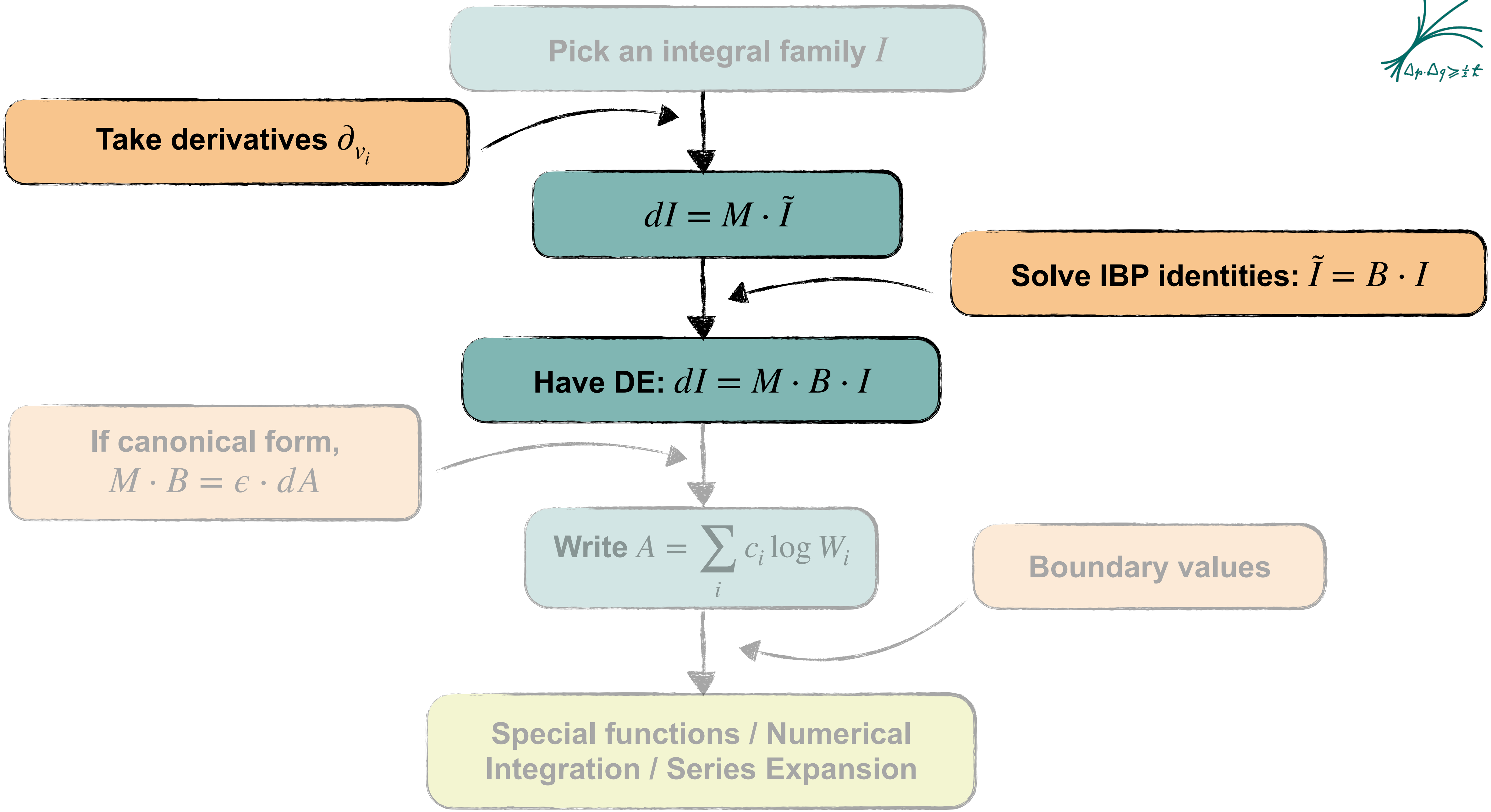
- For $D_{ext} > 4$, nine independent Mandelstam invariants

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}\}$$

$$s_{ij} = (p_i + p_j)^2, \qquad s_{ijk} = (p_i + p_j + p_k)^2$$

- For $D_{ext} = 4$, Gram determinant constraint

$$\begin{aligned} 0 &= G(p_1, p_2, p_3, p_4, p_5) = \det(p_i \cdot p_j), \qquad 1 \leq i, j \leq 5 \\ &= s_{12} s_{23}^2 s_{34} s_{56} + (86 \text{ terms}) \end{aligned}$$



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INTEGRATION VIA DIFFERENTIATION

- Can get value of a function by integrating its first derivatives along some path!

1) Integration by parts (IBP): family is spanned by finite number of basis elements

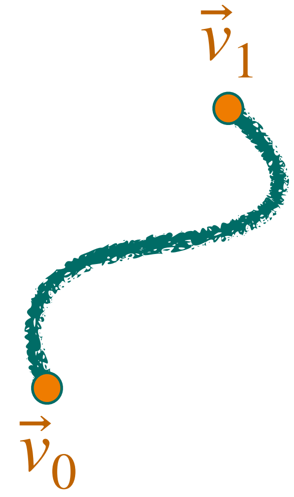
[Chetyrkin, Tkachov '81; Laporta '00]

[Smirnov, Petukhov '10]

$$I^{d_0}[\vec{a}] = \overrightarrow{C} \cdot \vec{I}_{\text{master}}$$

finite basis

2) Derivatives of an integral land in the same family!



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$$I^{d_0}[\vec{a}] = \overrightarrow{C} \cdot \vec{I}_{\text{master}}$$

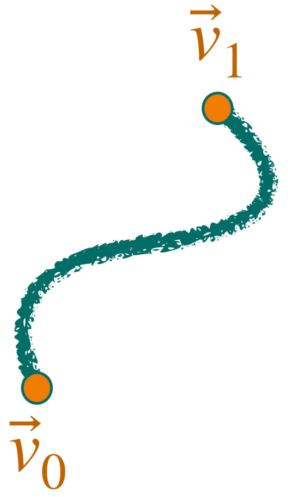
finite basis

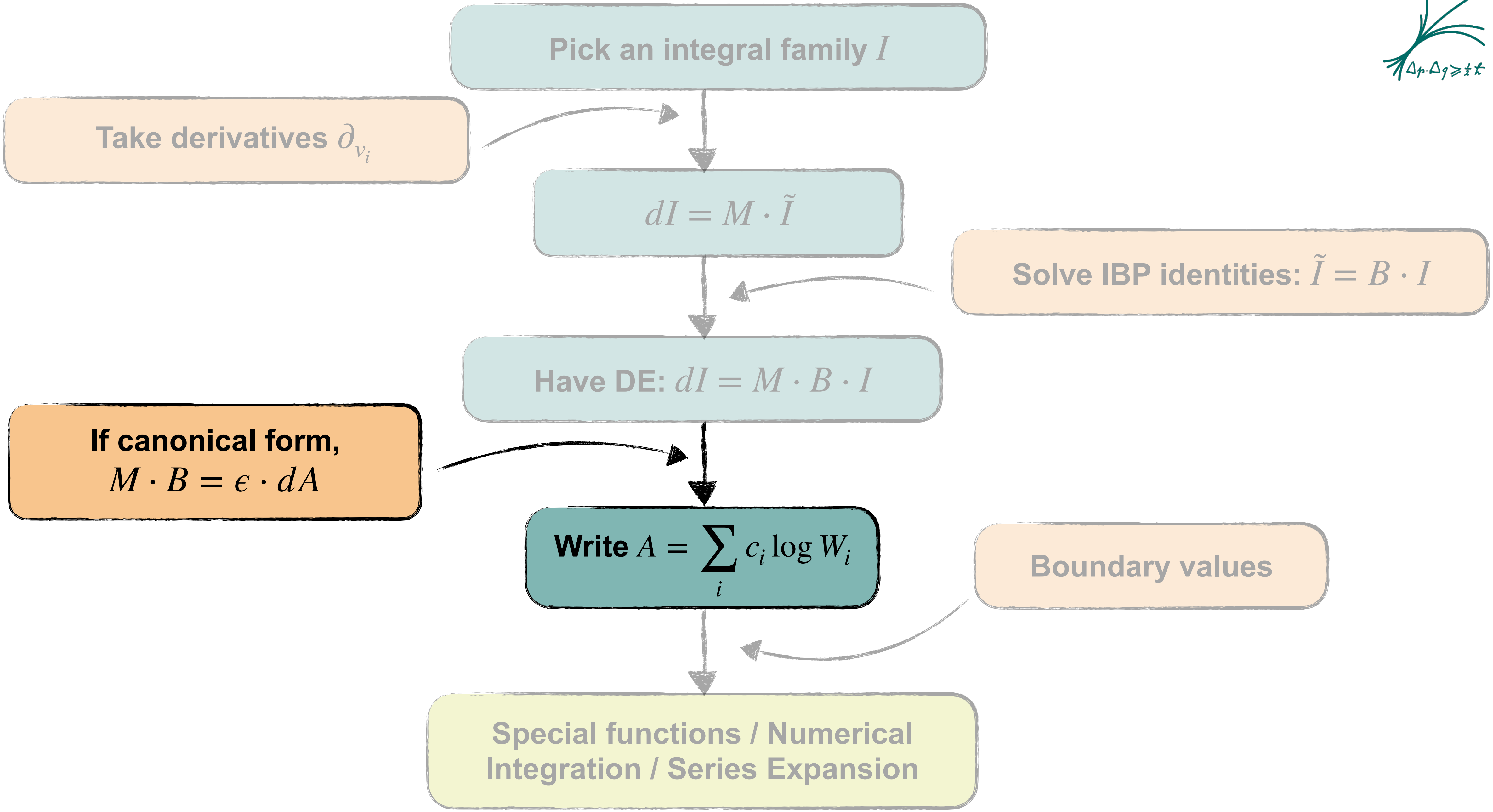
2) Derivatives of an integral land in the same family!

$$\Rightarrow \quad \mathrm{d}\vec{I}_{\text{master}} = \mathrm{d}M(\epsilon, \vec{v}) \cdot \vec{I}_{\text{master}}$$

$$\mathrm{d} = \sum \mathrm{d}v_k \partial_{v_k}$$

[Kotikov '91, Remiddi '97]





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A CANONICAL BASIS

We are looking for a **canonical basis**: [Henn '13]

$$\mathrm{d}\vec{I}_{\text{can}} = \epsilon \, \mathrm{d}A(\vec{v}) \cdot \vec{I}_{\text{can}}$$

Form of A:

$$\mathrm{d}A = \sum_k c_k \, \mathrm{d}\log(W_k)$$

letters

rational constant matrices

Alphabet: $\mathbb{A} = \{W_1, W_2, \ldots, W_N\}$

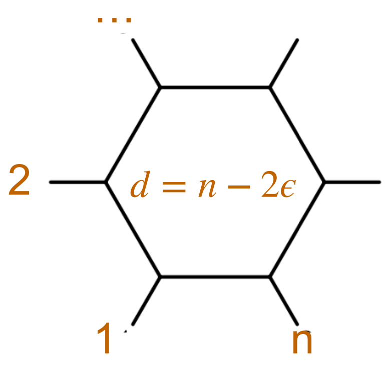
HOW TO CONSTRUCT UT INTEGRALS?

There is no (efficient) algorithmic way at two loops.

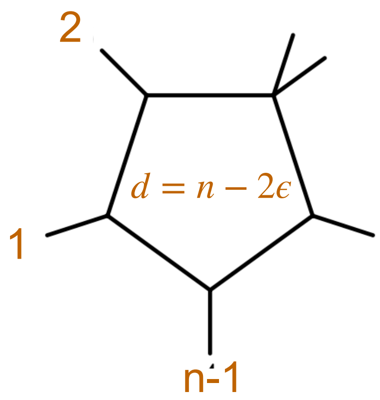
However, we can recycle one-loop insights to some extent…

INGREDIENT 1: N-GONS IN N DIMENSIONS

- At one loop, know how to construct UT integrals for arbitrary # of points:



$= \mathrm{LS}_n \cdot (\text{UT function}),$



$= \overline{\mathrm{LS}}_{n-1} \cdot (\text{UT function})$

[Schlaefli 1867]
[Spradlin, Volovich ’11]
[Flieger, Torres Bobadilla ’22]

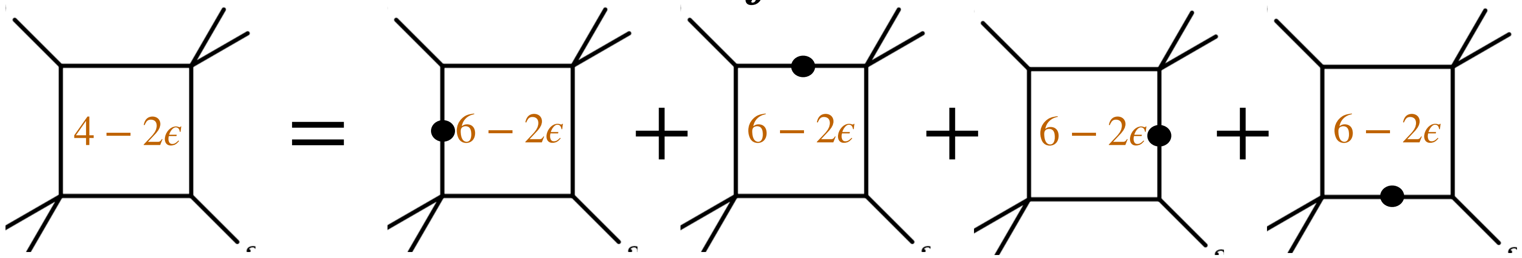
INGREDIENT 2: DIMENSION-SHIFT IDENTITIES

- Obvious from Schwinger Parametrization:

[Tarasov ’96]

$$I \propto \int \mathrm{d}x_1 \ldots \mathrm{d}x_k x^{\nu-1} \mathcal{U}^{-D/2} \exp(-\mathcal{F}/\mathcal{U}) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_k x^{\nu-1} \mathcal{U}\,\mathcal{U}^{-(D+2)/2} \exp(-\mathcal{F}/\mathcal{U})$$

- E.g.



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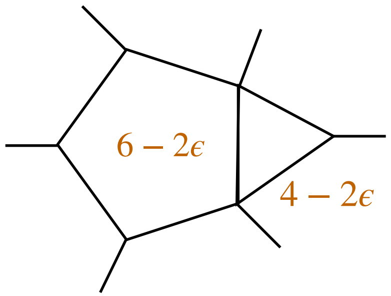

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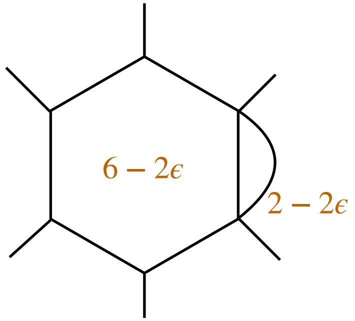

Are UT integrals!

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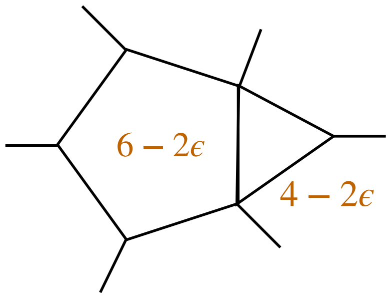

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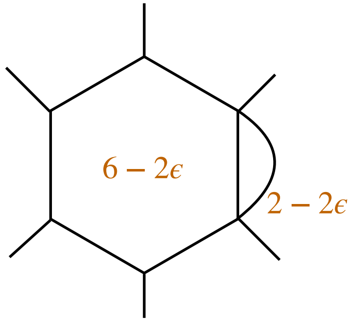

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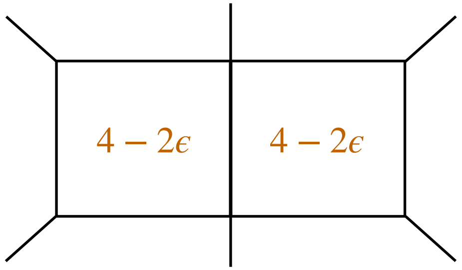
Double-Box



But need 6 more…

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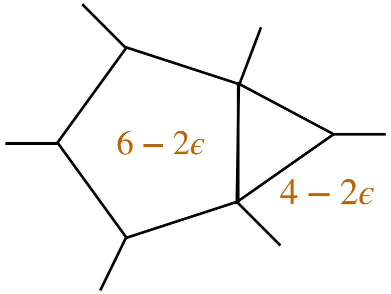

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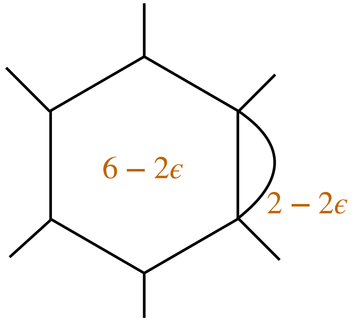

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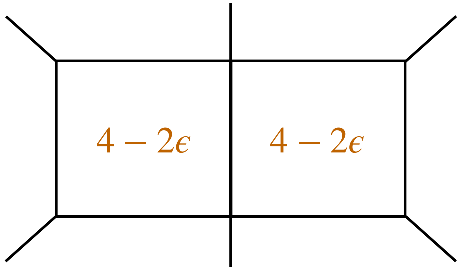
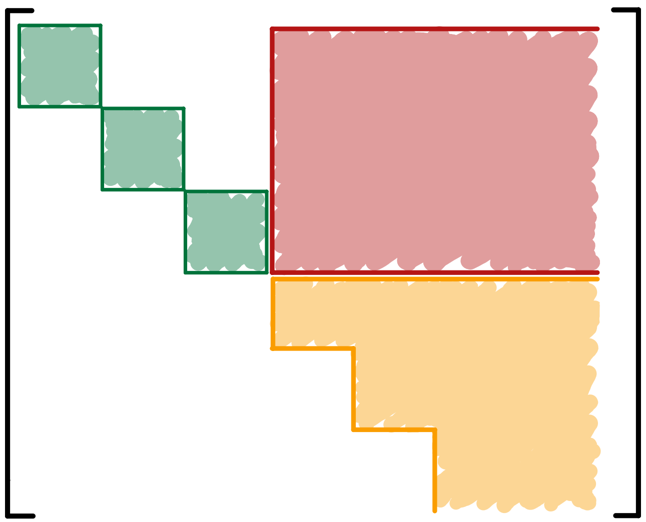


Found 6 more by using Gram determinants with loop momenta as numerators and insisting on definite parity behavior.

DIFFERENTIAL EQUATION BLOCKS IN CANONICAL FORM

$$\mathrm{d}I = \epsilon\,\mathrm{d}A\,I$$



$$\mathrm{d}\begin{bmatrix} I_{\mathrm{db}} \\ I_{\mathrm{pt}} \\ I_{\mathrm{hb}} \\ I_{\mathrm{sub}} \\ \vdots \end{bmatrix} = \epsilon\,\mathrm{d}\,[\ \cdots\ ]\begin{bmatrix} I_{\mathrm{db}} \\ I_{\mathrm{pt}} \\ I_{\mathrm{hb}} \\ I_{\mathrm{sub}} \\ \vdots \end{bmatrix}$$

Six-particle blocks on maximal cuts [Henn, Peraro, Xu, Zhang '21]

Five-particle integrals with one off-shell leg [Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

Off-diagonal blocks [Henn, AM, Miczajka, Peraro, Xu, Zhang '24]

SPEED UP CALCULATION BY WORKING OVER FINITE FIELD

Pick a UT basis of Integrals I

↓

$dI = M \cdot \tilde{I}$

↓

Have PDE: $\partial_i I = \epsilon \partial_i A \cdot I$

↓

Write $A = \sum_i c_i \log W_i$

Analytical: High cost of time and memory!

↓

Evaluate over finite field and reconstruct rational functions only at the end!

$\mathcal{O}(???) \rightarrow \mathcal{O}(\text{40h per entry})$

[von Manteuffel, Schabinger '15]

We use `FiniteFlow` [Peraro '19]

SPEED UP CALCULATION BY WORKING OVER FINITE FIELD

Pick a UT basis of Integrals I

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$dI = M \cdot \tilde{I}$

↓

Have DE: $dI = \epsilon \; dA \cdot I$

↓

Write $A = \sum_i c_i \log W_i$

If we know alphabet $\{\log W_j\}$ a priori:

↓

Only have to fit the rational entries of the matrices c_i

$\mathcal{O}(\text{years}) \rightarrow \mathcal{O}(\text{hours})$

[Abreu, Page, Zeng ’19]

TWO-LOOP ALPHABET LETTERS

How to find the alphabet $\mathbb{A} = \{W_i\}$?

- Study of singularities of Feynman integrals: **Landau analysis** [Bjorken; Landau; Nakanishi ’59]
- Predicts the components of the Landau variety, i.e. the algebraic variety in kinematic space on which singularities may lie

recent progress: [Fevola, Mizera, Telen ’23,’24]
[Helmer, Papathanasiou, Tellander ’24]
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Output of this analysis:

$$\mathbb{A}_{\text{even}} = \{s_{12}, s_{23}, \ldots, s_{12} - s_{123}, \ldots\}$$ 93 polynomial letters

$$\{\sqrt{Q_i}\} = \{\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \ldots\}$$ 21 square root letters

TWO-LOOP ALPHABET LETTERS

- But we know from experience, that there are also “odd” letters:

$$W_{odd} = \frac{P - \sqrt{Q}}{P + \sqrt{Q}}$$

where $Q \in \{Q_i\} \cup \{Q_i Q_j\}$

P is polynomial in v_i

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$$W_{odd} = \frac{P - \sqrt{Q}}{P + \sqrt{Q}}$$

where $Q \in \{Q_i\} \cup \{Q_i Q_j\}$
P is polynomial in v_i

- Consistent with Landau analysis, if: [Heller, von Manteuffel, Schabinger ’19]

$$\left(P - \sqrt{Q}\right)\left(P + \sqrt{Q}\right) = c \prod_i W_i^{e_i}, \quad W_i \in \mathbb{A}_{even}$$

1) ANSATZ FOR THE POLYNOMIAL

$$P(\vec{v})^2 = Q(\vec{v}) + c\prod_i W_i^{e_i}$$

- e.g. for $\sqrt{Q} = r_{27}$, i.e. a degree 3 ansatz, assume P has coefficients in $\{-2,\ldots,2\}$:

$$\#\ P(\vec{v}) \approx 5^{\binom{9+3-1}{3}} \approx 10^{115}$$

2) ANSATZ FOR THE PRODUCT

$$P(\vec{v})^2 = Q(\vec{v}) + c\prod_i W_i^{e_i}$$

Number of degree one polynomials

$$\#\ \prod_i W_i^{e_i} = \binom{48}{6} + \binom{48}{4}\binom{42}{1} + \ldots \approx 2 \cdot 10^7$$

REFINE THE PRODUCT ANSATZ!

Unpublished [AM, Miczajka '24]

Assume that: $P(\vec{v})^2 = Q(\vec{v}) + W_i(\vec{v}) \cdot (\ldots)$ for some particular letter W_i.

Then, for every $\vec{v}_0 \in \mathbb{Q}^9$ such that $W_i(\vec{v}_0) = 0$, it follows

$$\sqrt{Q(\vec{v}_0)} \in \mathbb{Q}.$$

Hence, we can filter through the even letters to find a reduced set for any given Q.

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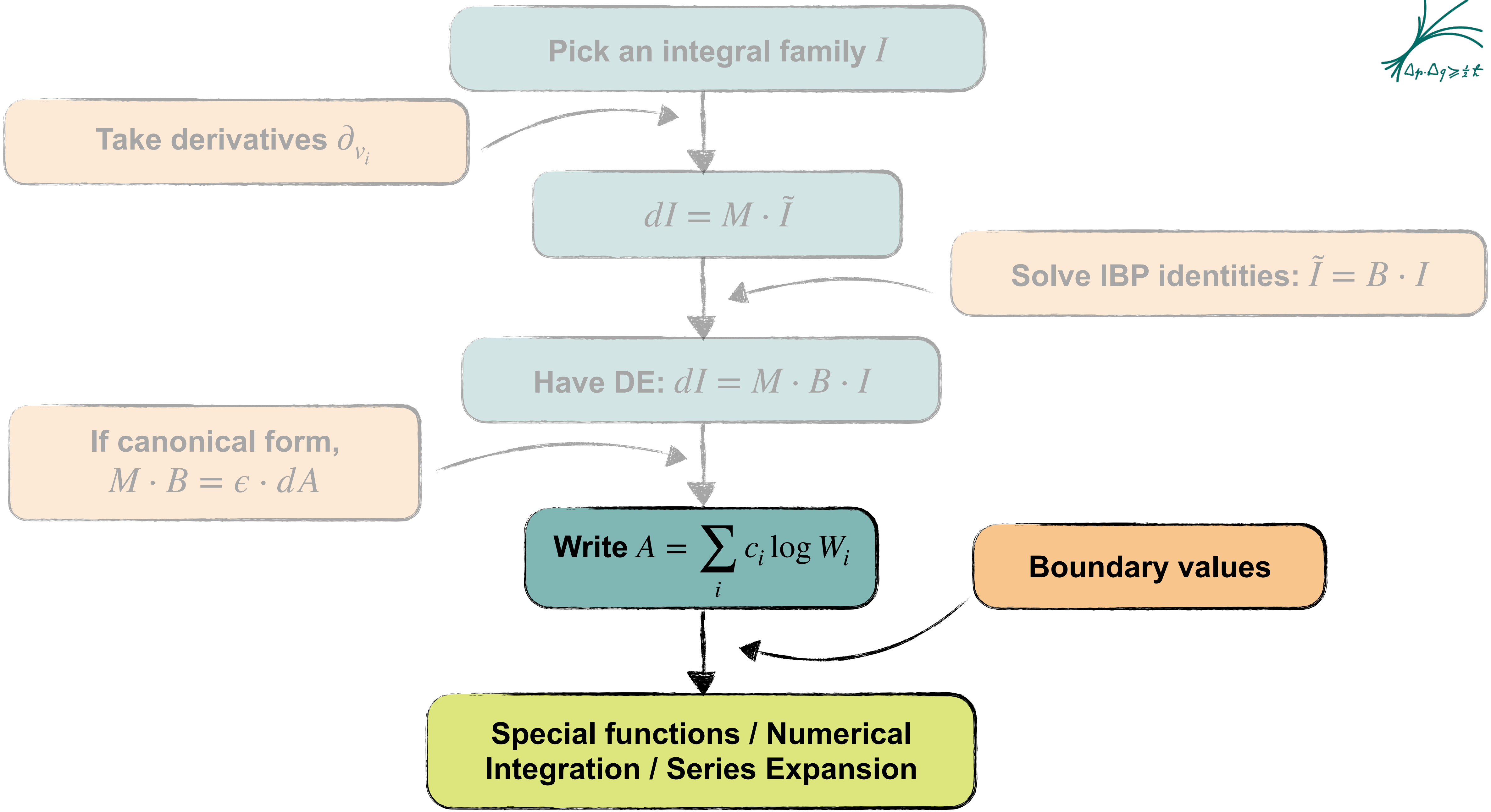
Example:

$$\sqrt{s_{12}^2 + s_{34}^2 + s_{56}^2 - 2s_{12}s_{34} - 2s_{12}s_{56} - 2s_{34}s_{56}} \overset{s_{12}=0}{=} \sqrt{(s_{34} - s_{56})^2} \in \mathbb{Q}.$$

CONJECTURED 2-LOOP HEXAGON ALPHABET

- Started with **93** even letters from subsectors and maximal cut
- Using the algorithm, we construct a total of **109** odd letters
  - out of these, **94** of them have a single square root, **15** have two square roots
  - **89** odd letters can be matched to known data from two-loop five-point and one-loop six-point alphabets; **20** letters are completely new

Together with the **21** square roots, we find an alphabet with 93+21+109=**223** letters.



Pick an integral family I
Take derivatives ∂_{v_i}
$dI = M \cdot \tilde{I}$
Solve IBP identities: $\tilde{I} = B \cdot I$
Have DE: $dI = M \cdot B \cdot I$
If canonical form, $M \cdot B = \epsilon \cdot dA$
Write $A = \sum_i c_i \log W_i$
Boundary values
Special functions / Numerical Integration / Series Expansion

WHAT IT MEANS "TO CALCULATE AN INTEGRAL"

- Usually: Express it in terms of known functions
- Here: it might be possible to express the result in terms of Li_4, $\mathrm{Li}_{2,2}$, but most likely inefficient for evaluations!

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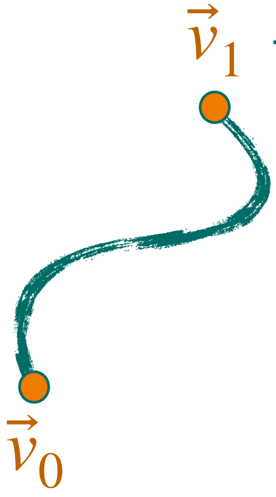

[Caron-Huot, Henn ’14]

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- Instead, use one-fold integral representation along some path!

Step 1: Analytically solve the DE up to weight 2.

Step 2: Write weight 3, 4 solution as integrals over weight 2:

$$\vec{I}^{(3)}(\vec{v}_1) = \vec{I}^{(3)}(\vec{v}_0) + \int_0^1 \mathrm{d}t \, \frac{\partial A}{\partial t} \cdot \vec{I}^{(2)}(t)$$

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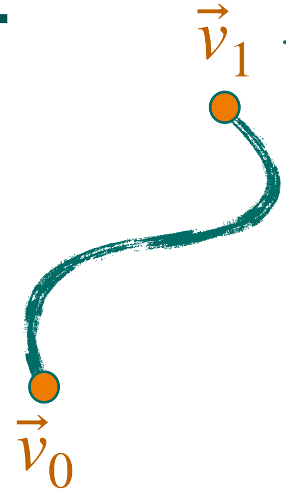

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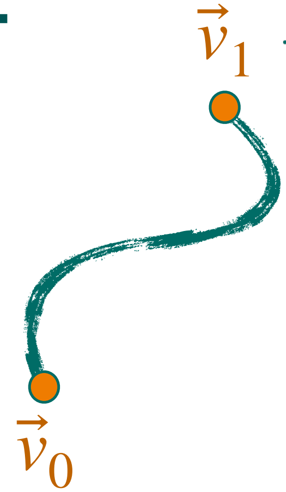
WHAT IT MEANS “TO CALCULATE AN INTEGRAL”



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Improves evaluation time from $\mathcal{O}(3\text{ h}) \rightarrow \mathcal{O}(7\text{ min})$ @ 20 digits precision

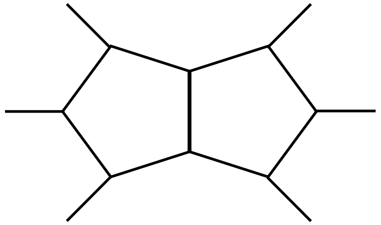
cf. AMFlow [Liu, Ma ’23]

$$\vec{I}^{(4)}(\vec{v}_1) = \vec{I}^{(4)}(\vec{v}_0) + \int_0^1 \mathrm{d}t \; \left(\frac{\partial A}{\partial t} \cdot \vec{I}^{(3)}(\vec{v}_0) + (A(1) - A(t)) \cdot \frac{\partial A}{\partial t} \cdot \vec{I}^{(2)}(t) \right)$$

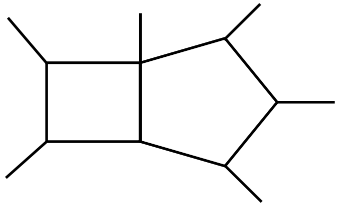
CONCLUSION & OUTLOOK

- Most efficient technique: predict the alphabet & bootstrap the canonical DE
- Supplement with boundary conditions to set up one-fold integral representations at weight 4
- Next steps: proceed with the remaining families

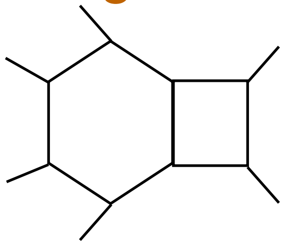
Double-Pentagon



Pentagon-Box



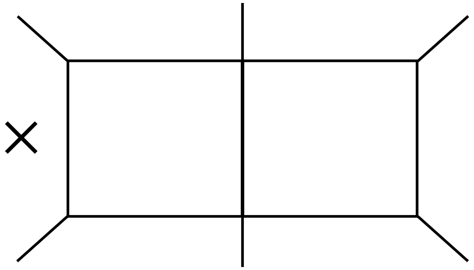
Hexagon-Box

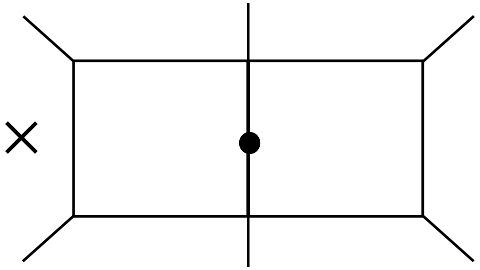


- Analytic continuation to physical scattering region?
- What is the general structure of n-point 2-loop alphabets?

BACK-UP SLIDES

UT INTEGRALS FOR THE DOUBLE BOX FAMILY

$I_i = \epsilon^4 N_i \times$ 

$I_7 = \epsilon^4 N_7 \times$ 

Numerators:

$$N_1 = -\, s_{12} s_{45} s_{234}$$

$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix}$$

$$N_2 = -\, s_{12} s_{45} (l_1 + p_5 + p_6)^2$$

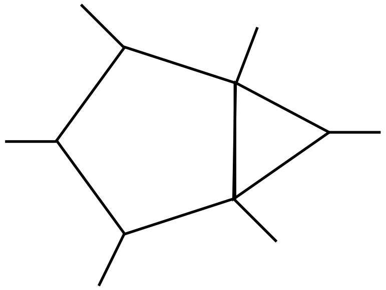
$$N_5 = -\frac{1}{4} \frac{\epsilon_{1245}}{G(p_1, p_2, p_5, p_6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix}$$

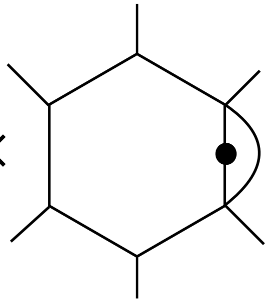
$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix}$$

$$N_6 = \frac{1}{8} \left[G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + (l_1 - p_1)^2 (l_2 - p_6 - p_5)^2 (s_{123} + s_{345}) \right]$$

$$N_7 = \frac{\Delta_6}{G(p_1, p_2, p_4, p_5)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}$$

UT INTEGRALS FOR THE PENTATRI & HEXABUB

$I_{\text{pt}} = \epsilon^4 N_{pt} \times$ 

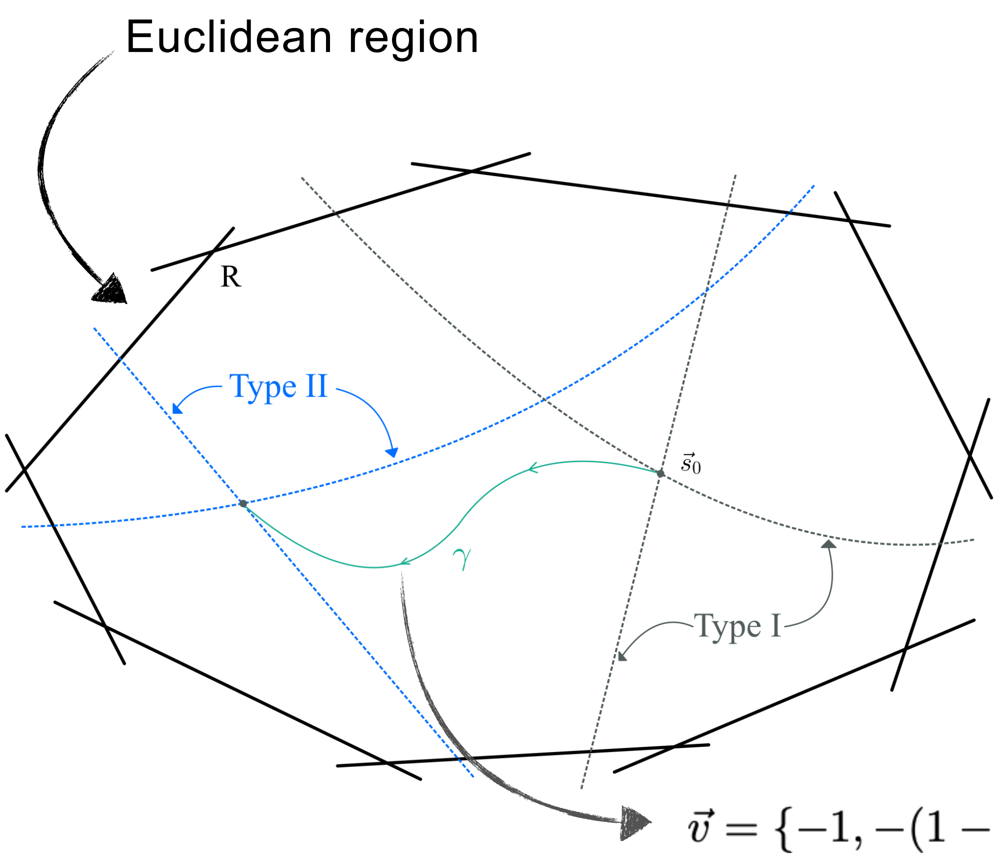
$I_{\text{hb}} = \epsilon^3 N_{hb} \times$ 

Numerators:

$$N_{\text{pt}} = \frac{1}{32\epsilon_{1235}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_5 \\ l_1 & p_1 & p_2 & p_3 & p_5 \end{pmatrix}$$

$$N_{\text{hb}} = \frac{(l_1 + p_6)^2}{32\epsilon_{1234}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_4 \\ l_1 & p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$

FIXING THE BOUNDARY VALUES



- Analytic value of basis integrals at a single point

$$\vec{v}_0 = \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$$

- Absence of certain spurious singularities and by matching to a single sunrise integral

$$\vec{v} = \{-1,-(1-x)^2,-1,-(1-x)^2,-1,-(1-x)^2,-1+x,-1+x,-1+x\}$$

$$\mathbb{A} \to \mathbb{A}_{\text{line}} = \{x, 1-x, x-\rho, x-\bar{\rho}\} \qquad \rho = \frac{1}{2}(1+i\sqrt{3})$$

FIXING THE BOUNDARY VALUES

- Boundary values at $\vec{v}_0 = \{-1, -1, \ldots, -1\}$:

$$I_{\text{db},1} = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\text{Im}[\text{Li}_2(\rho)]\right)\epsilon^4$$

$$I_{\text{db},2} = 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \text{Im}[\text{Li}_2(\rho)]\right)\epsilon^4$$

$$I_{\text{db},6} = -\left(\frac{\pi^4}{540} + \frac{4}{3}\text{Im}[\text{Li}_2(\rho)]^2\right)\epsilon^4$$

$$I_{\text{db},3} = I_{\text{db},4} = I_{\text{db},5} = I_{\text{db},7} = I_{\text{pt}} = I_{\text{hb}} = 0$$