Identifying Regions for Asymptotic Expansions in QCD

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The method of regions (MoR)

- ***Statement:*** $\text{entire space} = R_1 \cup R_2 \cup \cdots \cup R_n$

$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

The original integral, $\mathcal{I}$, can be restored by summing over contributions from the regions.

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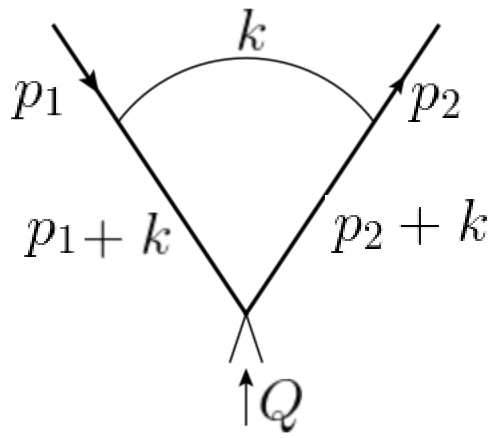
The original integral, $\mathcal{I}$, can be restored by summing over contributions from the regions.

- Proposed by Beneke and Smirnov in 1997, no proof yet.
- The regions are chosen using heuristic methods based on examples and experience.
- The integration measure is the **entire space** for each term.

The method of regions (MoR)

Example: one-loop Sudakov form factor

(Becher, Broggio, Ferroglia 2014)



The on-shell limit kinematics

$$p_1^\mu \sim Q\ (\underset{+}{1},\ \underset{-}{\lambda},\ \underset{\perp}{\lambda^{1/2}}), \qquad p_2^\mu \sim Q\ (\underset{+}{\lambda},\ \underset{-}{1},\ \underset{\perp}{\lambda^{1/2}})$$

$$p_1^2/Q^2 \sim p_2^2/Q^2 \sim \lambda \to 0$$

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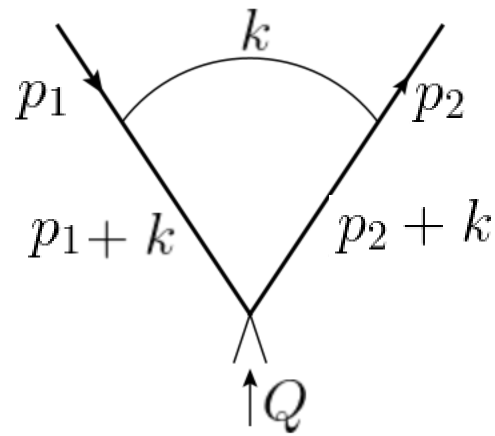

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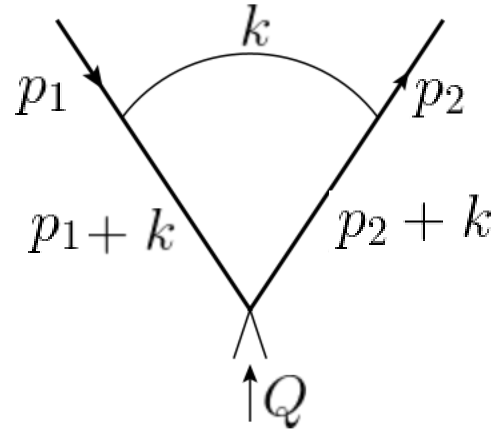
The Feynman integral

$$\mathcal{I} = \mathcal{C} \cdot \int d^D k \frac{1}{(k^2 + i0)\,((p_1+k)^2 + i0)\,((p_2+k)^2 + i0)}$$

can be evaluated directly, or, we can apply the method of regions.

The method of regions (MoR)

Step 1: identify **4** regions in total:



Hard region: $k^\mu \sim Q(1,1,1)$

Collinear-1 region: $k^\mu \sim Q(1,\lambda,\lambda^{1/2})$

Collinear-2 region: $k^\mu \sim Q(\lambda,1,\lambda^{1/2})$

Soft region: $k^\mu \sim Q(\lambda,\lambda,\lambda)$

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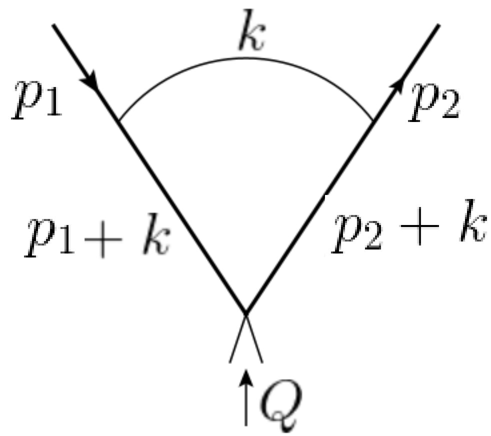

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$\textbf{Collinear-2 region: } k^\mu \sim Q(\lambda,1,\lambda^{1/2})$

$\textbf{Soft region: } k^\mu \sim Q(\lambda,\lambda,\lambda)$

Step 2: perform expansion around each region:

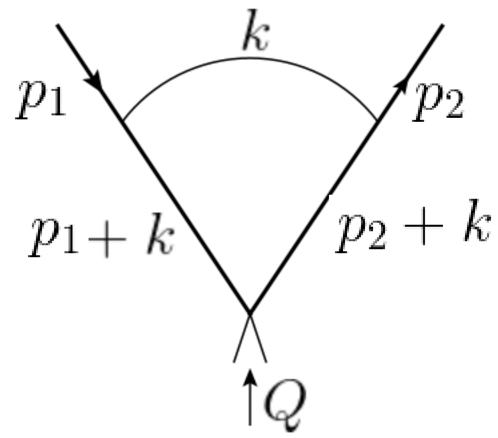
$$\mathcal{I}_H = \mathcal{C}\cdot\int d^Dk \frac{1}{(k^2+i0)\,(k^2+2p_1\cdot k+i0)\,(k^2+2p_2\cdot k+i0)}+\dots$$

$$\mathcal{I}_{C_1} = \mathcal{C}\cdot\int d^Dk \frac{1}{(k^2+i0)\,((p_1+k)^2+i0)\,(2p_2\cdot k+i0)}+\dots$$

$$\mathcal{I}_{C_2} = \mathcal{C}\cdot\int d^Dk \frac{1}{(k^2+i0)\,(2p_1\cdot k+i0)\,((p_2+k)^2+i0)}+\dots$$

$$\mathcal{I}_S = \mathcal{C}\cdot\int d^Dk \frac{1}{(k^2+i0)\,(2p_1\cdot k+p_1^2+i0)\,(2p_2\cdot k+p_2^2+i0)}+\dots$$

The method of regions (MoR)



Step 1: ....

Step 2: ....

Step 3: sum over their contributions, and the original integral is reproduced:

$$\mathcal{I} = \mathcal{I}_H + \mathcal{I}_{C_1} + \mathcal{I}_{C_2} + \mathcal{I}_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{(-p_1^2)}\ln\frac{Q^2}{(-p_2^2)} + \frac{\pi^2}{3} + \ldots\right)$$

This equality holds to **all** orders of λ!

More examples are presented in Smirnov's book "*Applied Asymptotic Expansions in Momenta and Masses*".

Parameter representation

The Lee-Pomeransky representation

(Lee & Pomeransky 2013)

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{C}\int_0^\infty \left(\prod_{e\in G} dx_e x_e^{\nu_e-1}\right)\left(\mathcal{P}(\boldsymbol{x};\boldsymbol{s})\right)^{-D/2}$$

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x},\boldsymbol{s}),$$

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1}\prod_{e\notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x},\boldsymbol{s}) = -\sum_{T^2} s_{T^2}\prod_{e\notin T^2} x_e + \mathcal{U}(\boldsymbol{x})\sum_e m_e^2 x_e\ .$$

spanning trees **spanning 2-trees**

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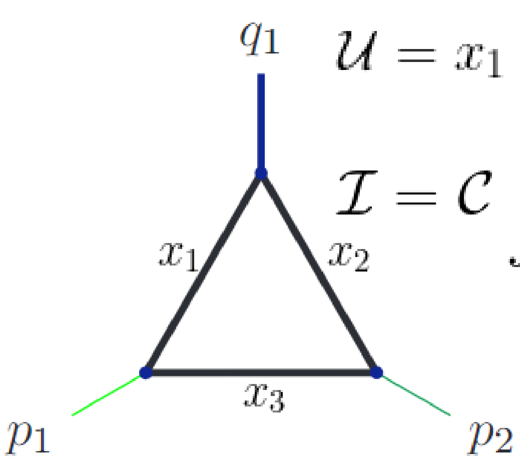

$$\mathcal{U} = x_1 + x_2 + x_3, \quad \mathcal{F} = (-p_1^2)x_1x_3 + (-p_2^2)x_2x_3 + (-q_1^2)x_1x_2$$

$$\mathcal{I} = \mathcal{C}\int_0^\infty dx_1dx_2dx_3x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}$$

$$\cdot\left(x_1+x_2+x_3-p_1^2x_1x_3-p_2^2x_2x_3-q_1^2x_1x_2\right)^{D/2}$$

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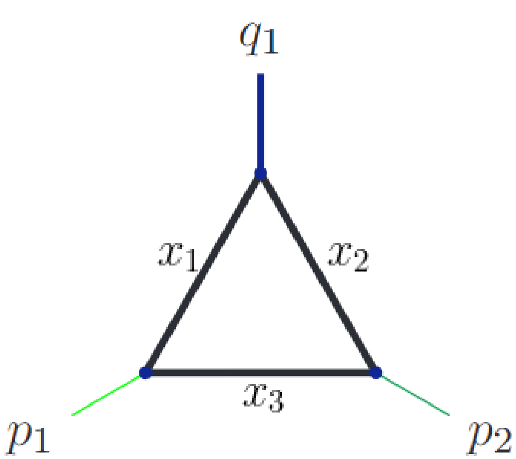


Each region → a certain scaling of the $\boldsymbol{x}$

Hard region : $\boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$
Collinear region to $\boldsymbol{p_1}$ **:** $\boldsymbol{x_1, x_3 \sim \lambda^{-1},\ x_2 \sim \lambda^0}$
Collinear region to $\boldsymbol{p_2}$ **:** $\boldsymbol{x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}}$
Soft region : $\boldsymbol{x_1, x_2 \sim \lambda^{-1},\ x_3 \sim \lambda^{-2}}$

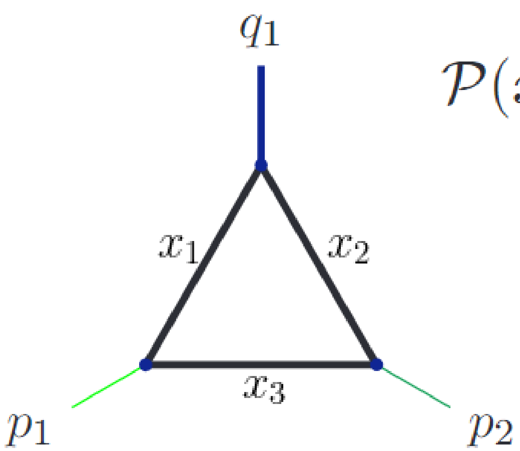
Identifying regions from Newton polytopes

- Given the Lee-Pomeransky polynomial,

$$\mathcal{P}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x};\boldsymbol{s}),$$

take the **exponents** of each term:

$$s x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \to (v_1, v_2, \ldots, v_n; a) \quad \text{if} \quad s \sim \lambda^a Q^2$$



$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0;0) **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

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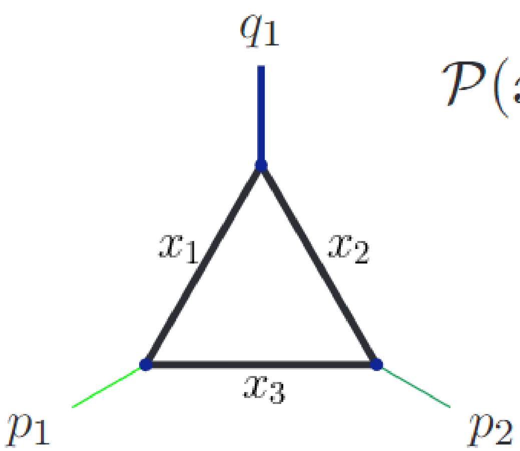


$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

- $x_1 \rightarrow$ **(1,0,0;0)**
- $x_2 \rightarrow$ **(0,1,0;0)**
- $x_3 \rightarrow$ **(0,0,1;0)**
- $p_1^2 x_1 x_3 \rightarrow$ **(1,0,1;1)**
- $p_2^2 x_2 x_3 \rightarrow$ **(0,1,1;1)**
- $q_1^2 x_1 x_2 \rightarrow$ **(1,1,0;0)**

Construct a Newton polytope = the **convex hull** of all these points.

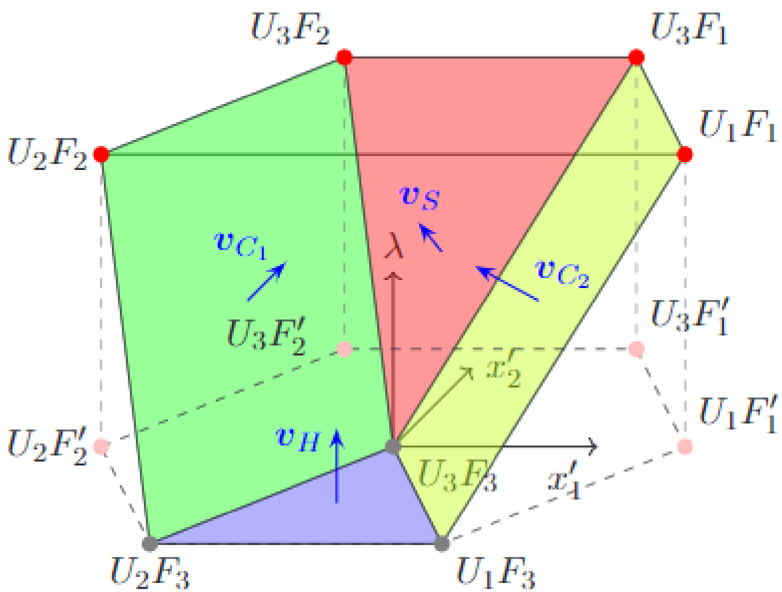
Regions = the lower facets of this Newton polytope.

(Pak & Smirnov 2010; Jantzen, Smirnov, Smirnov, 2012; etc.)

Identifying regions from Newton polytopes

- There have been computer codes based on this approach:

  Asy2, ASPIRE, pySecDec, …



- Timely results may not be available if the graph is not too simple.

  Note that dim(polytope) = #(propagators)+1.

- Also, how to interpret the output in momentum space?

Identifying regions from Newton polytopes

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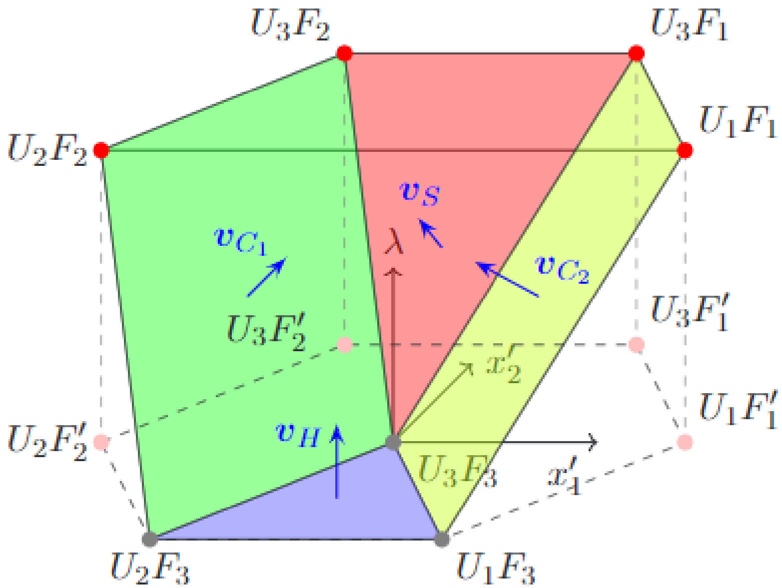


- ***Question: For any expansion of interest, can we establish a general rule, which governs all the regions and specifies all the relevant modes?***

Identifying regions from Newton polytopes

- ***Based on***

  → ***E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk, JHEP07(2023)197,***

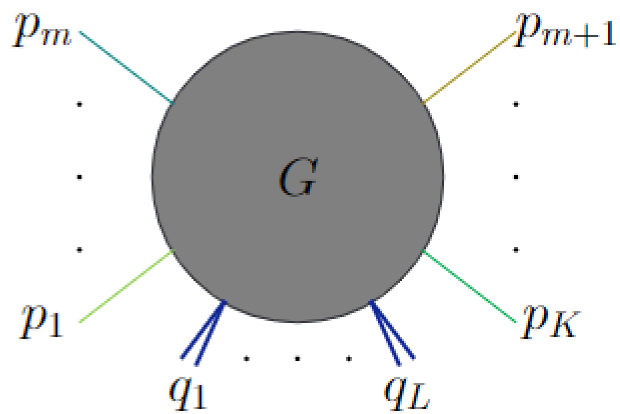
  → ***YM, arXiv:2312.14012,***

  → ***E.Gardi, F.Herzog, S.Jones, YM, to appear.***

This talk will try to answer the question.

The "on–shell expansion"

- We start with the following asymptotic expansion:



massless

$$p_i^2 \sim \lambda Q^2 \quad (i = 1, \ldots, K), \quad q_j^2 \sim Q^2 \quad (j = 1, \ldots, L), \quad p_{i_1} \cdot p_{i_2} \sim Q^2 \quad (i_1 \neq i_2).$$

small virtuality — large virtuality — wide-angle scattering

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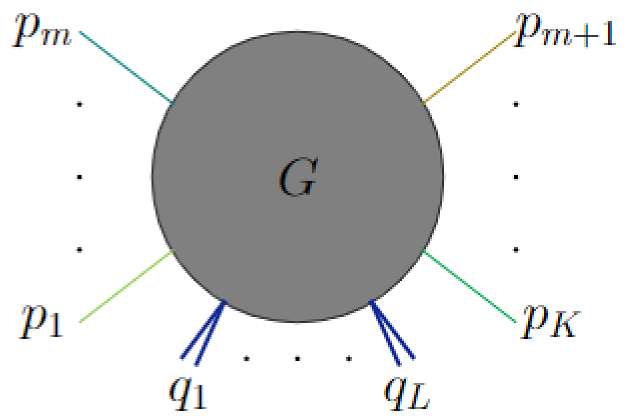

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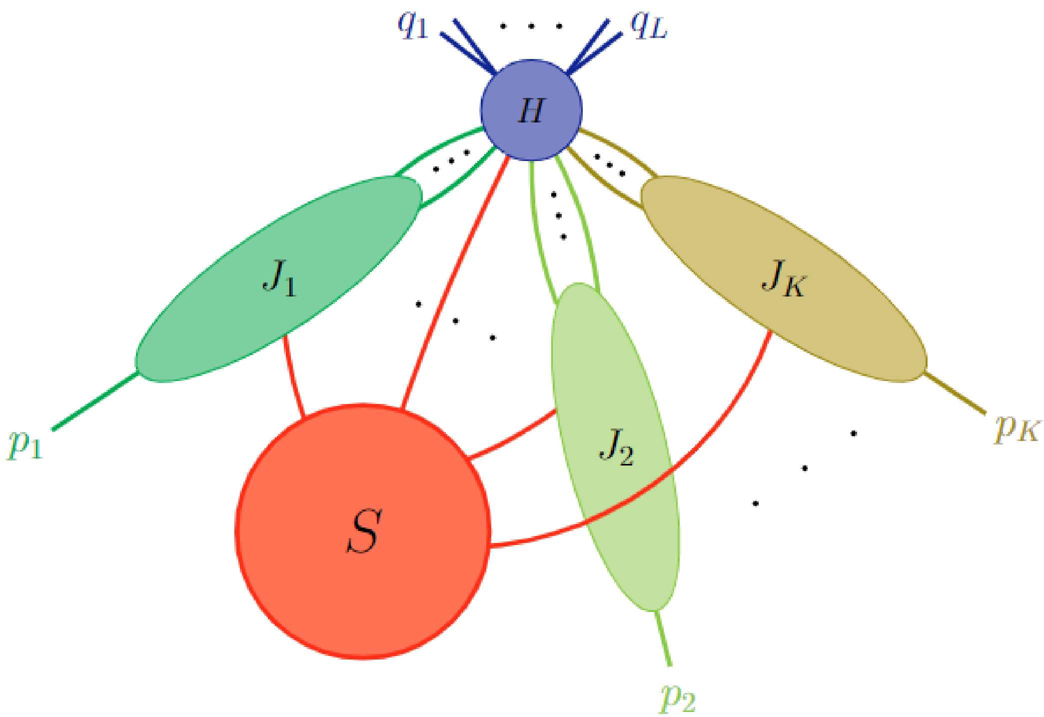
small virtuality **large virtuality** **wide-angle scattering**

- ***Result: the possibly relevant modes are***

$$k_H^\mu \sim Q(1,1,1), \quad k_{C_i}^\mu \sim Q(1,\lambda,\lambda^{1/2}), \quad k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

Regions in the on–shell expansion

- More precisely, the general structure of each region looks like



$$k_H^\mu \sim Q(1,1,1),$$

$$k_{C_i}^\mu \sim Q(1,\lambda,\lambda^{1/2}),$$

$$k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

with additional requirements on the subgraphs H, J, and S.

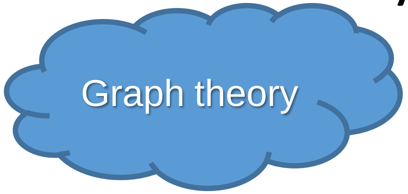
This conclusion was proposed in [Gardi, Herzog, Jones, YM, Schlenk, 2022], and later proved in [YM, arXiv:2312.14012].

Idea of the proof

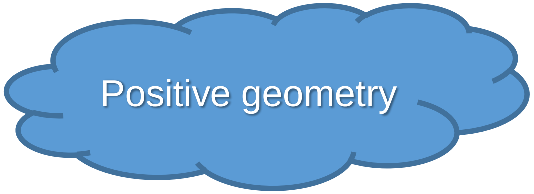
For the Symanzik polynomials,

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1} \prod_{e \notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x};\boldsymbol{s}) = -\sum_{T^2} s_{T^2} \prod_{e \notin T^2} x_e + \mathcal{U}(\boldsymbol{x}) \sum_e m_e^2 x_e \,.$$

- The terms are described by **spanning (2-)trees** of G.
- Furthermore, the terms are described by **weighted spanning (2-) trees** of G for a given scaling of the parameters.
- The **leading** terms are described by the **minimum spanning (2-)trees** of G.

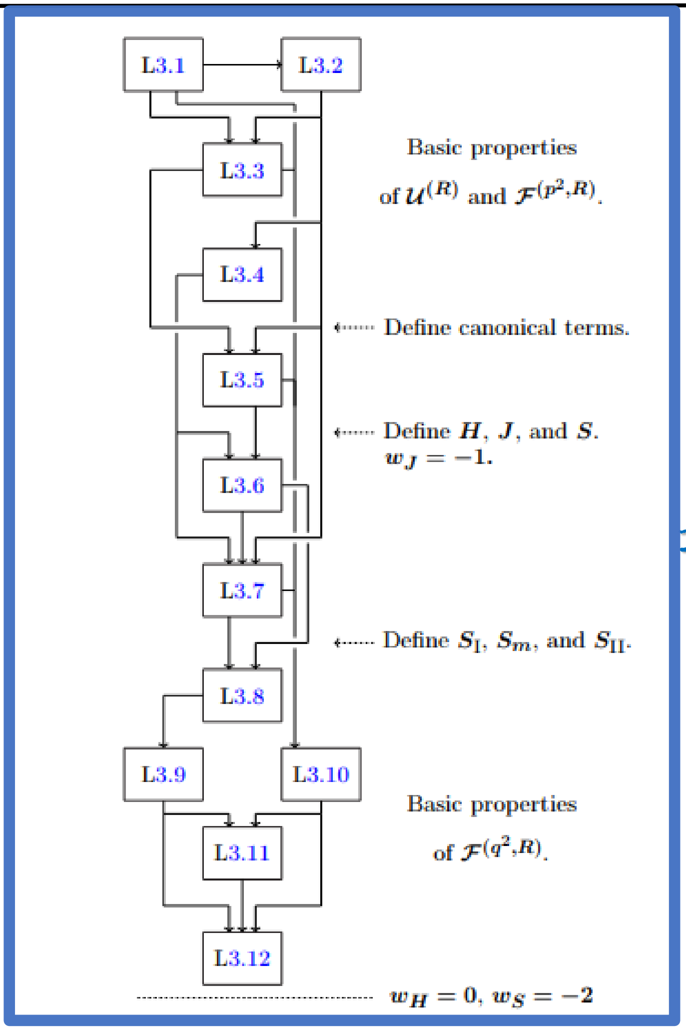


- Meanwhile, regions ←→ lower facets of the Newton polytope.



The proof

- Long and technical.

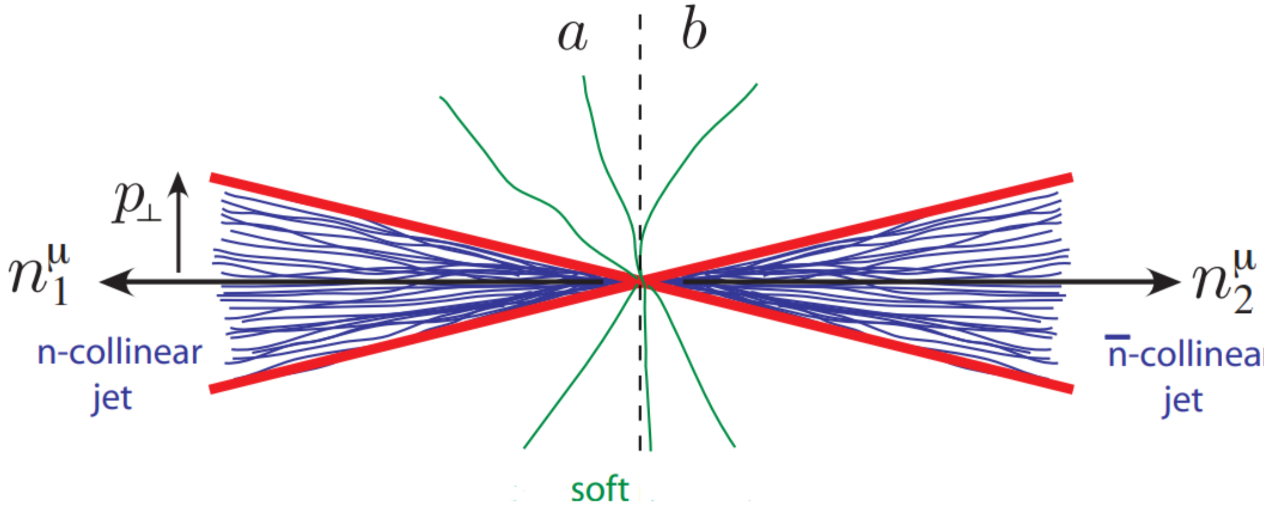


12 lemmas, ~50 pages…

- It works exclusively for the on-shell expansion, but can be slightly modified to apply to some other expansions.

Phenomenology

- Soft-Collinear Effective Theory (SCET): an effective theory describing the interactions of soft and collinear degrees of freedom in the presence of a hard interaction.
- For example, the SCET describing $e^+e^- \to \gamma^* \to \mathbf{dijets}$



involves the hard mode (integrated out), the collinear modes, and the soft mode.

Phenomenology

- Soft-Collinear Effective Theory (SCET): an effective theory describing the interactions of soft and collinear degrees of freedom in the presence of a hard interaction.
- The **SCETI Lagrangian** (leading order):

$$
\begin{aligned}
\mathcal{L} &= \sum_n (\mathcal{L}_{n\xi} + \mathcal{L}_{ng}) + \mathcal{L}_{\text{soft}} \\
&= \sum_n \Bigg(e^{-ix\cdot\mathcal{P}} \overline{\xi}_n \left(in\cdot D + i\not{D}_{n\perp} \frac{1}{i\overline{n}\cdot D_n} \not{D}_{n\perp} \right) \frac{\not{\overline{n}}}{2} \xi_n \\
&\quad + \frac{1}{2g^2} \mathrm{Tr}\{[i\mathcal{D}^\mu, i\mathcal{D}_\mu]^2\} + \tau \mathrm{Tr}\{[i\mathcal{D}_\mathrm{s}^\mu, A_{n\mu}]^2\} + 2\mathrm{Tr}\{b_n [i\mathcal{D}_\mathrm{s}^\mu, [i\mathcal{D}_\mu, c_n]]\} \Bigg) \\
&\quad + \overline{\psi}_\mathrm{s} i\not{D}_\mathrm{s} \psi_\mathrm{s} - \frac{1}{2} \mathrm{Tr}\{G_\mathrm{s}^{\mu\nu} G_{\mathrm{s},\mu\nu}\} + \tau_\mathrm{s} \mathrm{Tr}\{(i\partial_\mu A_\mathrm{s}^\mu)^2\} + 2\mathrm{Tr}\{b_\mathrm{s} i\partial_\mu i\mathcal{D}_\mathrm{s}^\mu c_\mathrm{s}\} \,.
\end{aligned}
$$

- ***We have shown that, in the regime of the on–shell expansion, nothing can go beyond the prediction of SCET, as long as all the regions are predicted by lower facets.***

Subtleties

- So far our analysis is based on

region ←→ lower facet

- But in principle, a region may also come from the inside of the Newton polytope, when terms in the Lee-Pomeransky polynomial cancel.

Can such regions be relevant for the on-shell expansion of (massless) wide-angle scattering?

Subtleties

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  region ←→ lower facet

- But in principle, a region may also come from the inside of the Newton polytope, when terms in the Lee-Pomeransky polynomial cancel.

- For long, we have believed that only "facet regions" are involved in massless wide-angle scattering kinematics, because prior to this work, the only known "non-facet regions" are the threshold region and the Glauber region, which are not relevant here.

- We did test quite many examples, all supporting the statement above…

Subtleties

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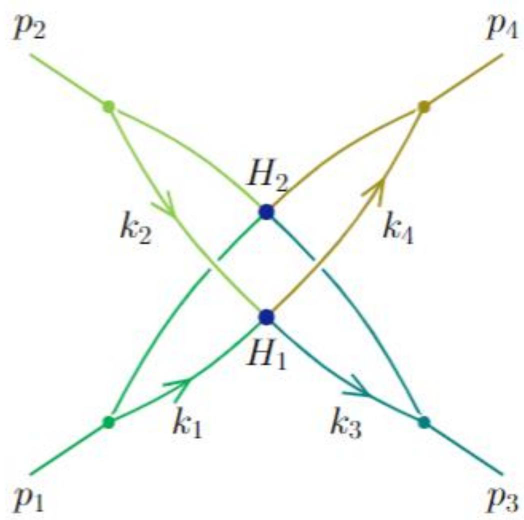
- For long, we have believed that only “facet regions” are involved in massless wide-angle scattering kinematics, because prior to this work, the only known “non-facet regions” are the threshold region and the Glauber region, which are not relevant here.

- ***… until recently we found a counterexample in the framework of wide–angle scattering.***

Subtleties

- The "*Landshoff scattering*":

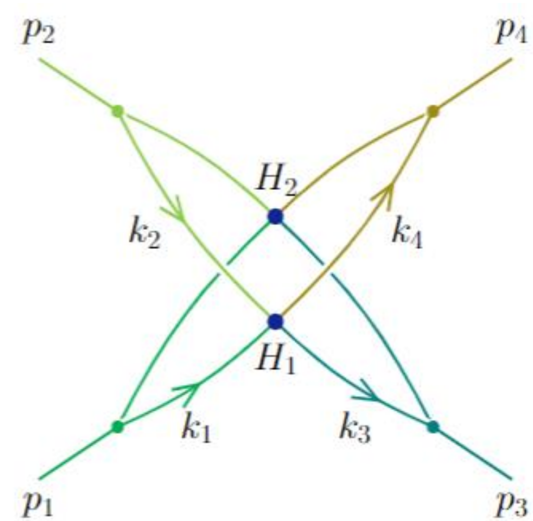


- The cancellation structure is $s_{12} \cdot (x_1 x_4 - x_2 x_3) \cdot (x_5 x_8 - x_6 x_7)$.

(E.Gardi, F.Herzog, S.Jones, YM, to appear)

Subtleties

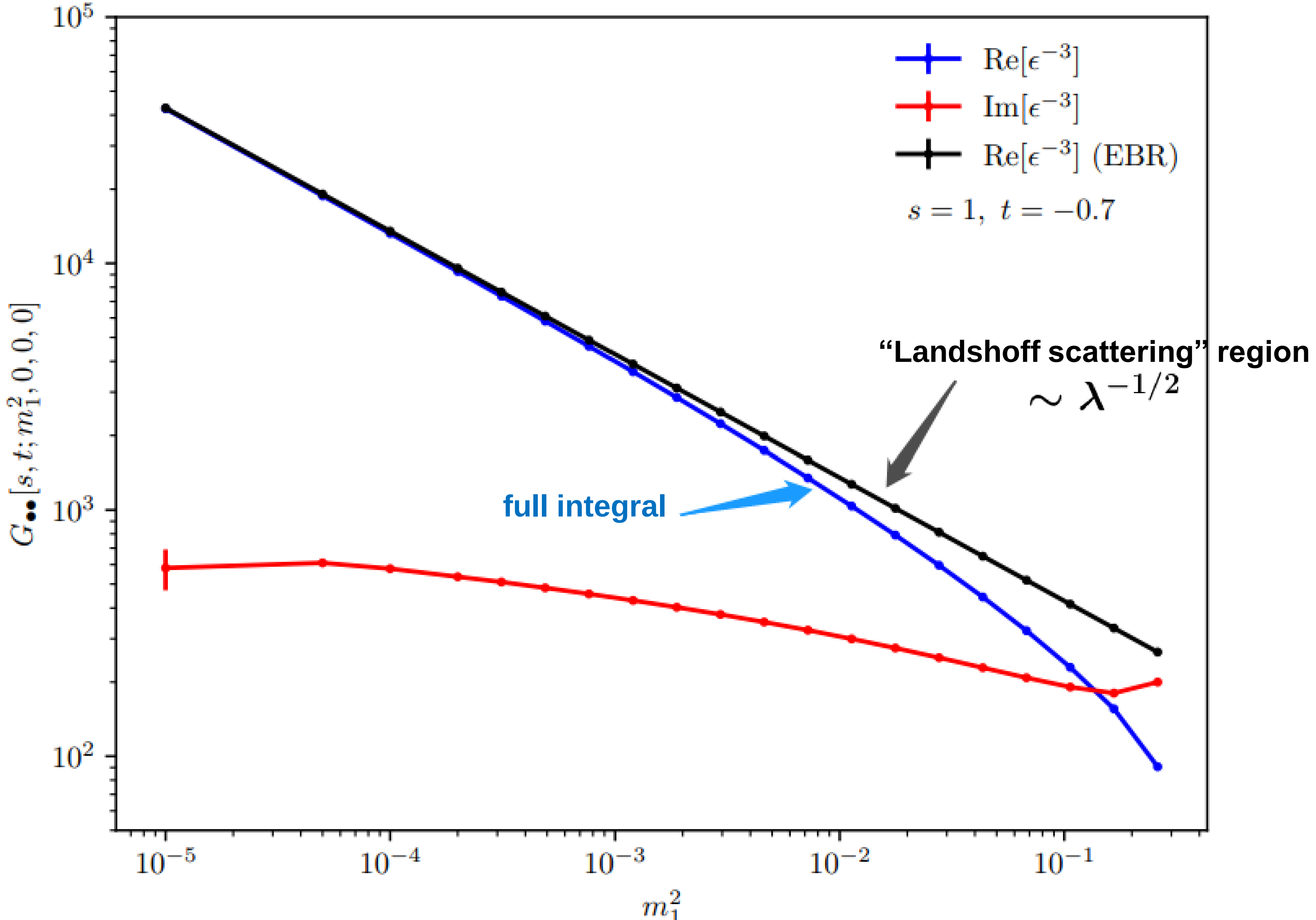
- The "***Landshoff scattering***":



- The cancellation structure is $s_{12}\cdot(x_1x_4 - x_2x_3)\cdot(x_5x_8 - x_6x_7)$.
- In scalar theory, from straightforward power counting, above is **the only** region that contributes to the leading asymptotic behavior (power divergence). So this region must be included.
- This region **cannot** be detected by Asy2.

(E.Gardi, F.Herzog, S.Jones, YM, to appear)

Numerical evidences



Graph by Stephen Jones

(E.Gardi, F.Herzog, S.Jones, YM, to appear)

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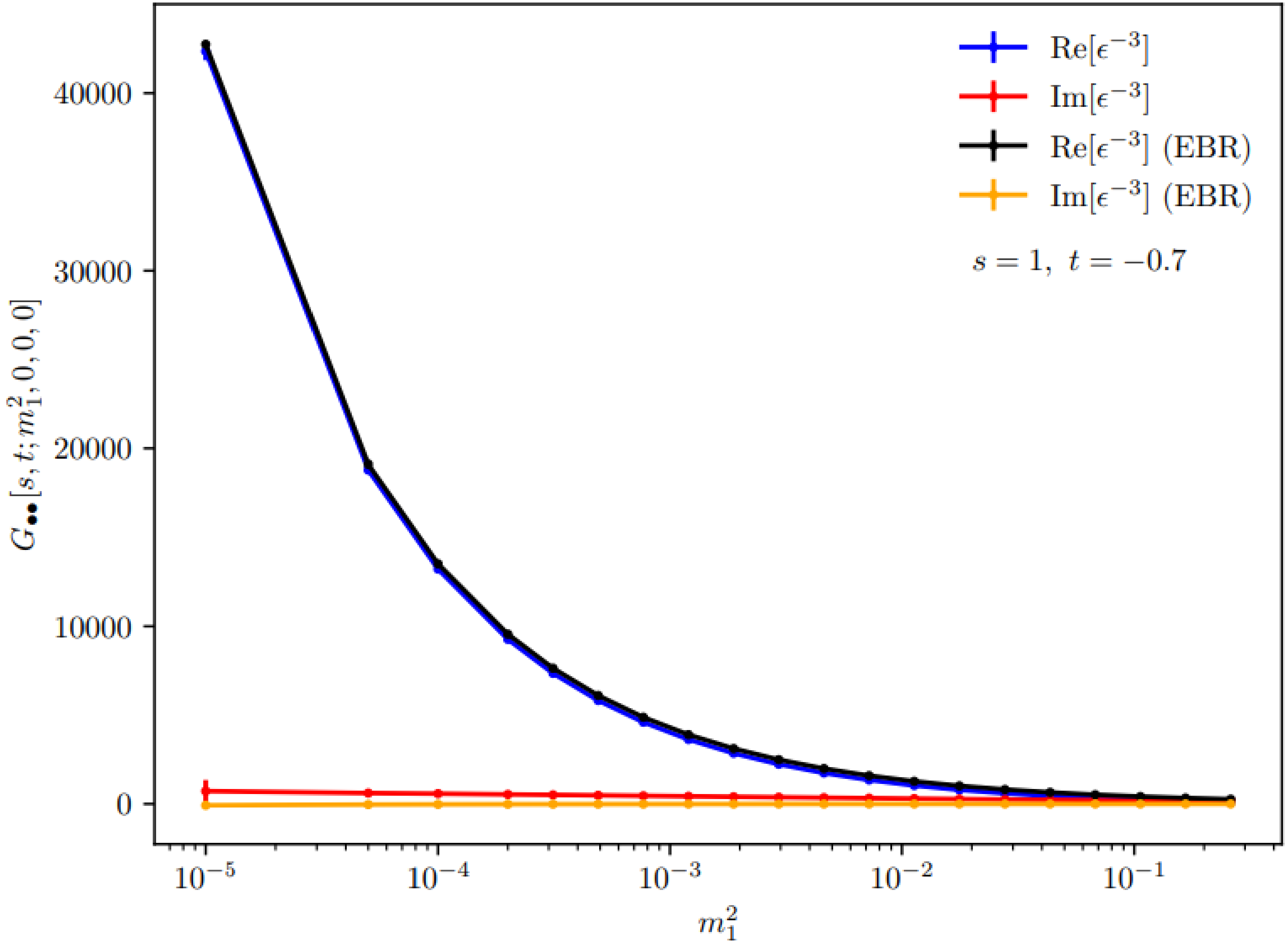

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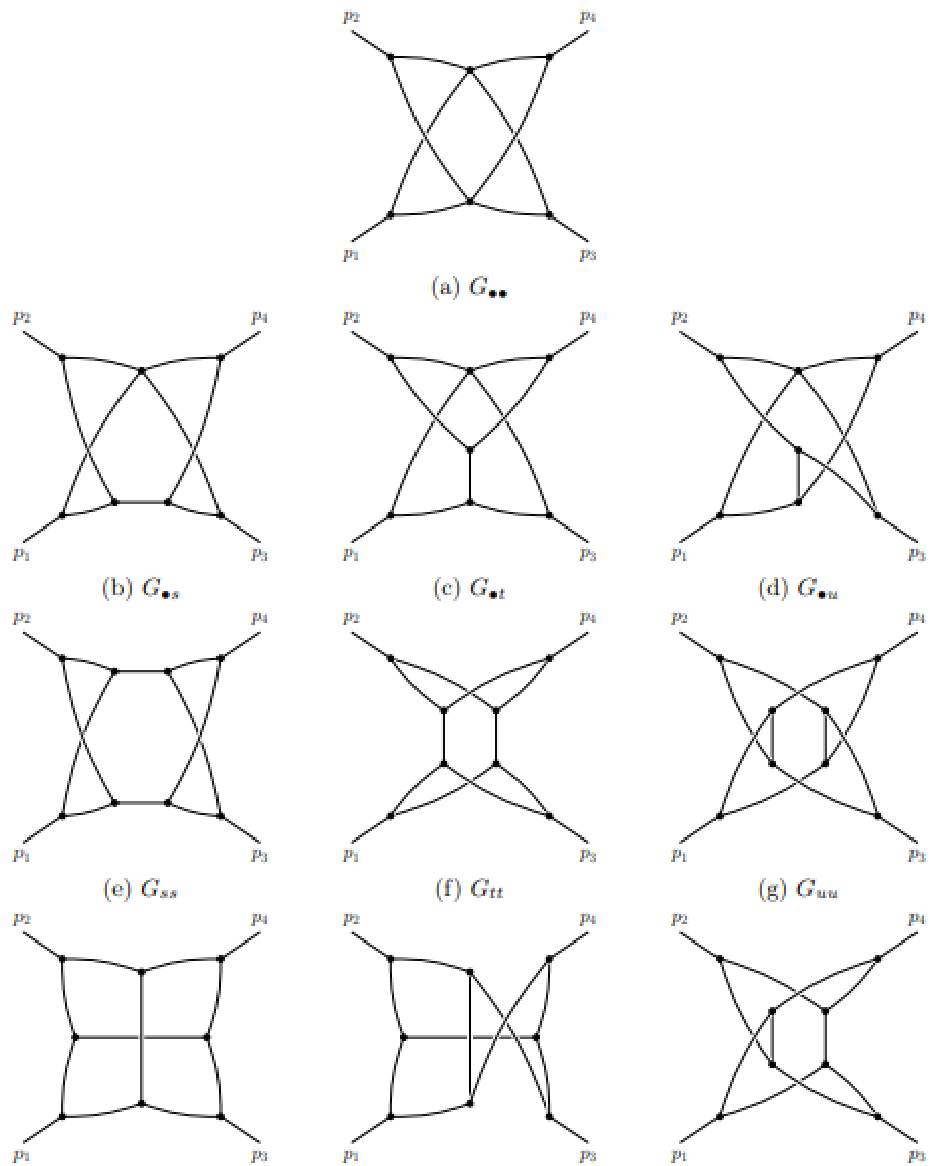
The whole list of subtle graphs

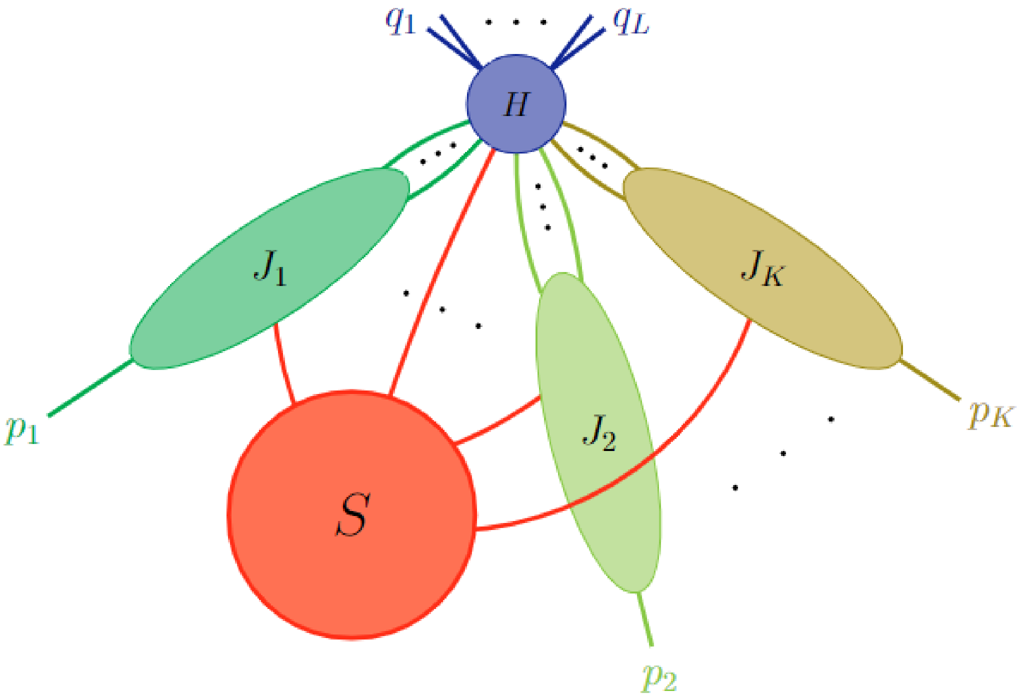
- 8-propagator
- 9-propagator
- 10-propagator



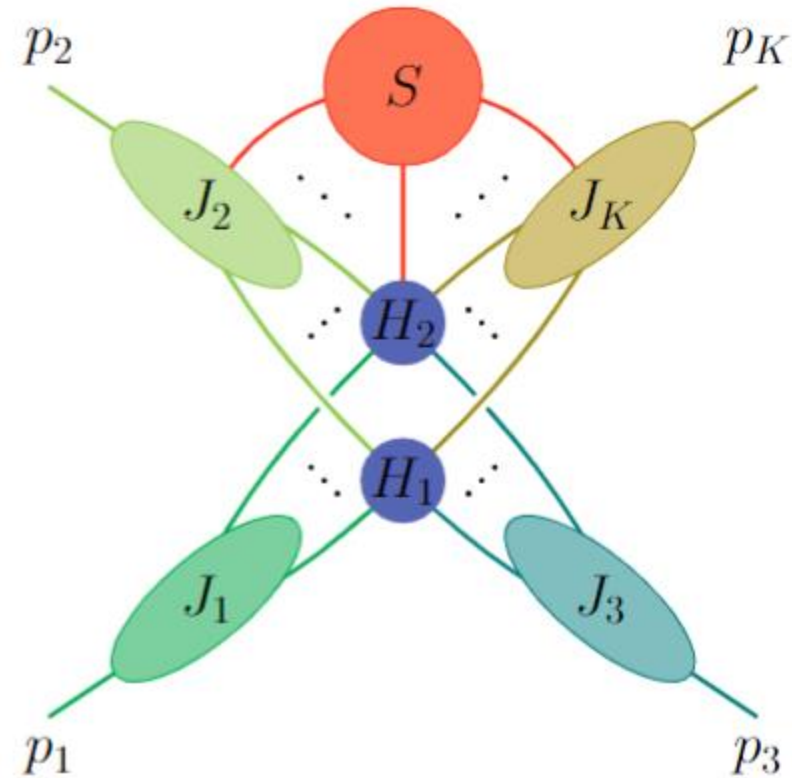
(a) $G_{\bullet\bullet}$
(b) $G_{\bullet s}$
(c) $G_{\bullet t}$
(d) $G_{\bullet u}$
(e) G_{ss}
(f) G_{tt}
(g) G_{uu}
(h) G_{st}
(i) G_{su}
(j) G_{tu}

Regions in the on–shell expansion

Conjecture:



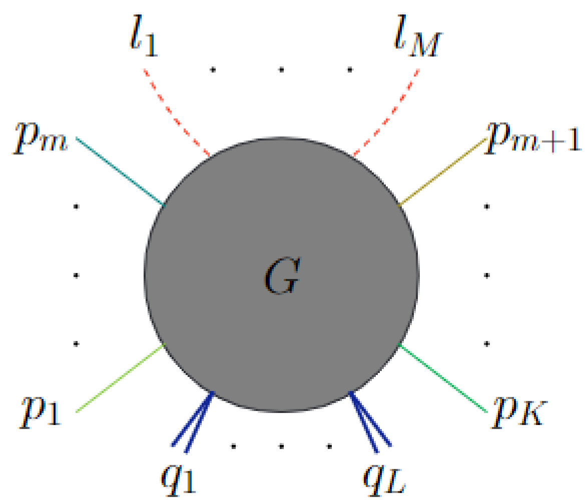
facet



inside

The "soft expansion"

- Including some soft external momenta



exactly on-shell **large virtuality** **exactly on-shell**

$$p_i^2 = 0 \quad (i = 1, \ldots, K), \quad q_j^2 \sim Q^2 \quad (j = 1, \ldots, L), \quad l_k^2 = 0 \quad (k = 1, \ldots, M),$$

$$p_{i_1} \cdot p_{i_2} \sim Q^2 \quad (i_1 \neq i_2), \quad p_i \cdot l_k \sim q_j \cdot l_k \sim \lambda Q^2, \quad l_{k_1} \cdot l_{k_2} \sim \lambda^2 Q^2 \quad (k_1 \neq k_2).$$

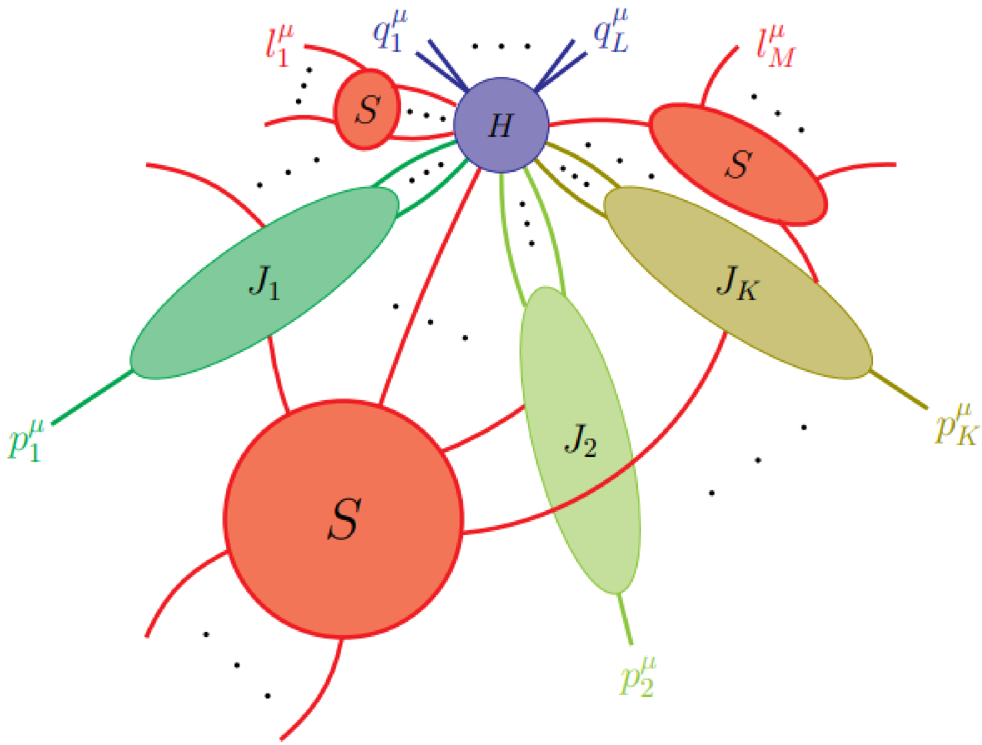
wide-angle scattering **soft momenta**

(YM, arXiv:2312.14012)

Regions in the soft expansion

- ***Result: the possibly relevant modes are:***

$$k_H^\mu \sim Q(1,1,1), \quad k_{C_i}^\mu \sim Q(1,\lambda,\lambda^{1/2}), \quad k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

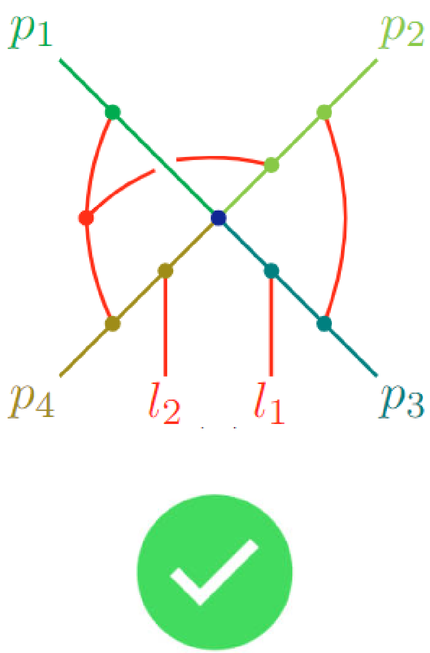


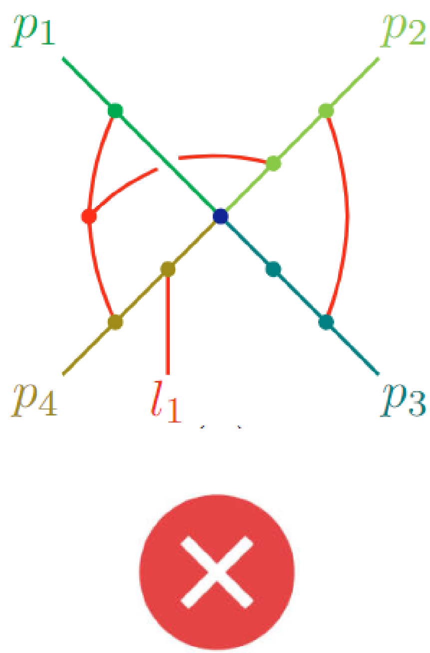
- Interesting feature: additional requirements for the subgraphs.

(YM, arXiv:2312.14012)

Regions in the soft expansion

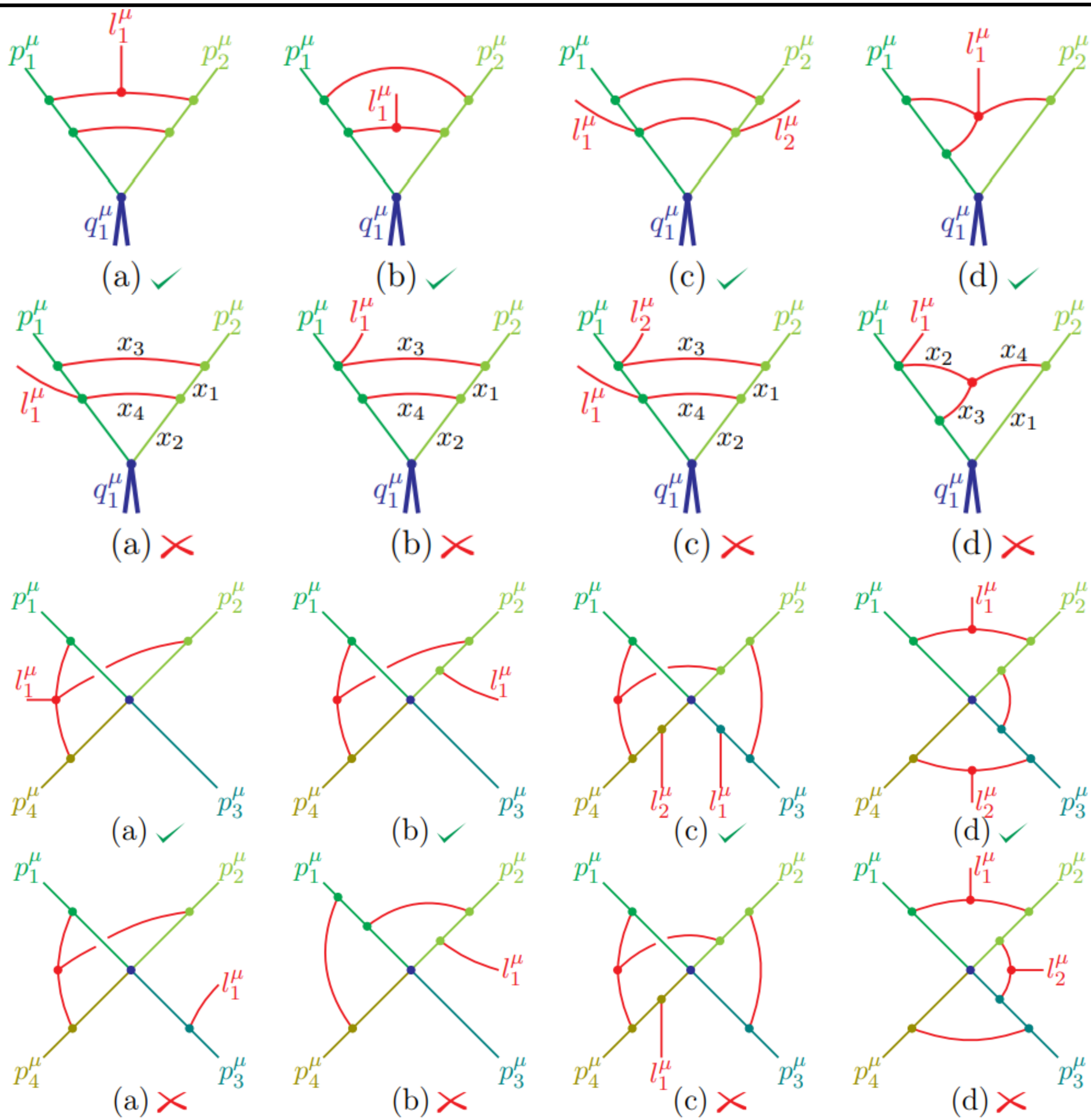
- The interactions between the soft subgraph and the jets follow the "infection-spreading" picture.
- Each jet must be "infected" by some soft external momenta.
- Any soft component adjacent to ≥3 jets can "spread the infection".
- Example:





(YM, arXiv:2312.14012)

Regions in the soft expansion



(YM, arXiv:2312.14012)

Regions in the soft expansion

- This study may also go beyond QCD.
- For example, some rules for the "Soft-Collinear Gravity" coincide with what we have found:

graviton attached to a purely soft vertex.

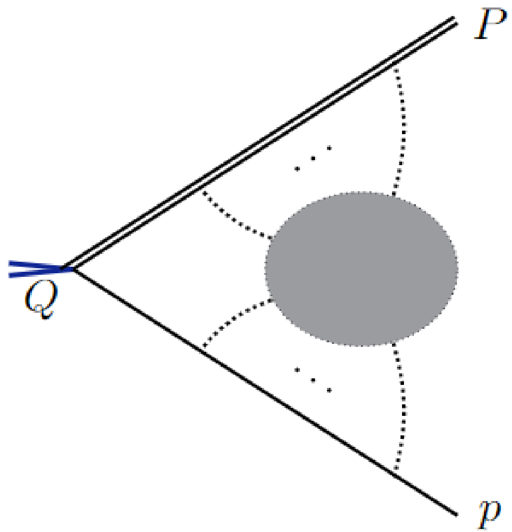
The above argument generalises to the following all-order statement: *In soft loop-corrections to the soft theorem, contrary to the tree-level case, the emitted soft graviton must always attach to a purely-soft vertex, and never directly to any of the energetic particle lines.* The reason is that soft-collinear interactions involve the soft field at the multipole-expanded point x_-^μ to any order in the λ-expansion. Hence, if the emitted graviton couples directly to an energetic line, one can always route its momentum such that the entire loop integral will depend only on $n_{i-}k\, n_{i+}^\mu/2$ of a single collinear direction, i, and no soft invariant can be formed to provide a scale to the loop diagram.

Continuing with two soft loops, whenever the diagram contains a second purely

(Beneke, Hager, Szafron, "*Soft-Collinear Gravity and Soft Theorems*")
See also
(Beneke, Hager, Szafron, 2021)
(Beneke, Hager, Schwienbacher, 2022)
(Beneke, Hager, Sanfilippo, 2023) et al.

The "mass expansion"

- The heavy-to-light decay process:



$$P^2 = M^2 \sim Q^2, \quad p^2 = m^2 \sim \lambda Q^2, \quad P \cdot p \sim Q^2.$$

large mass **small mass**

- In addition to the hard, collinear, and soft modes, more complicated modes can be present.

(YM, arXiv:2312.14012)

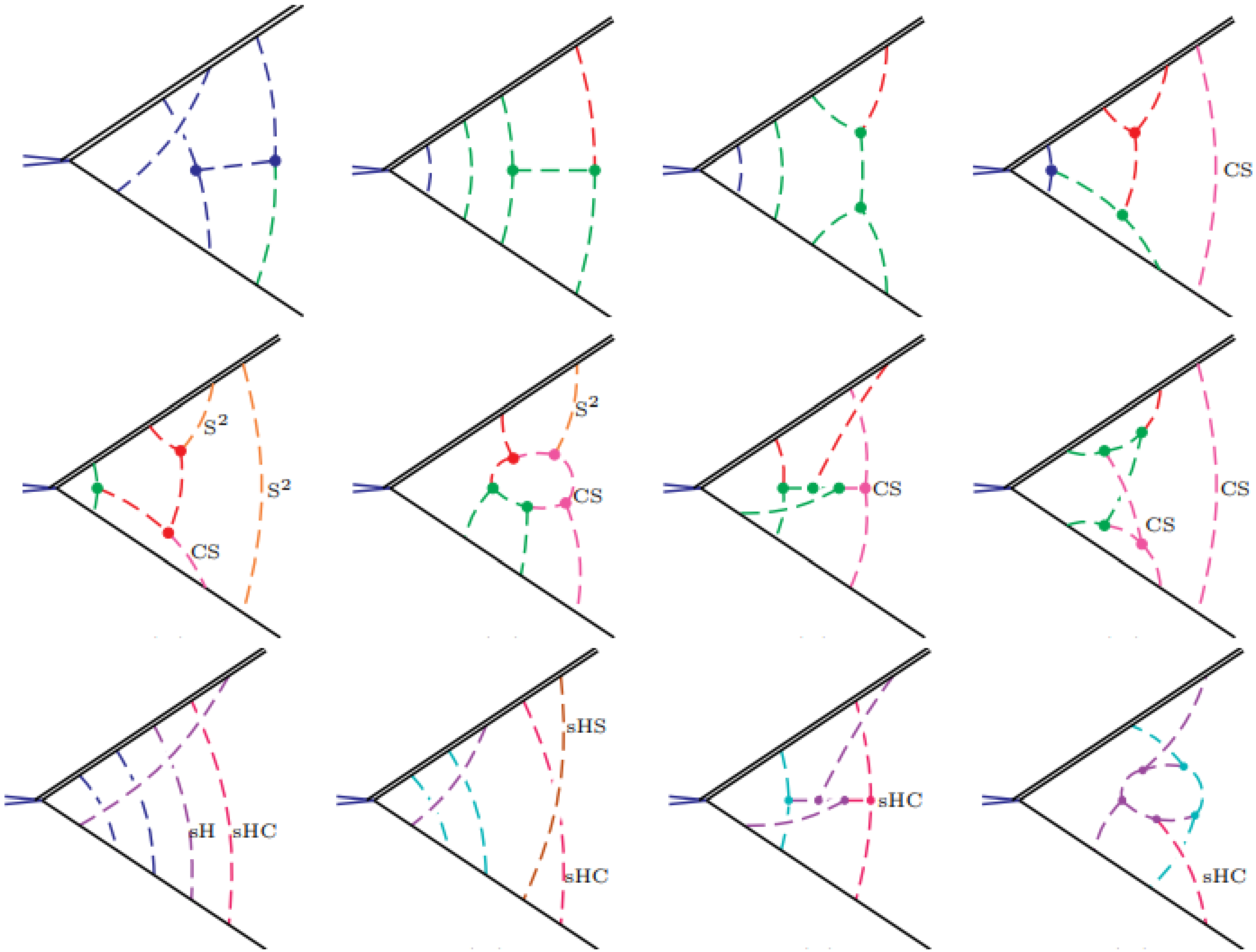
Regions in the mass expansion

• More modes are included:	Starting from
hard mode $Q(1,1,1)$,	1 loop
collinear mode $Q(1,\lambda,\lambda^{½})$,	1 loop
soft mode $Q(\lambda,\lambda,\lambda)$,	2 loops
soft·collinear mode $Q(\lambda,\lambda^2,\lambda^{3/2})$,	3 loops
soft² mode $Q(\lambda^2,\lambda^2,\lambda^2)$, ….	4 loops
semihard mode $Q(\lambda^{½},\lambda^{½},\lambda^{½})$,	2 loops
semihard·collinear, semihard·soft, ….,	3 loops, nonplanar
semicollinear mode $Q(1,\lambda^{1/2},\lambda^{1/4})$,	3 loops, nonplanar
semihard·semicollinear, ….	4 loops, nonplanar

(YM, arXiv:2312.14012)

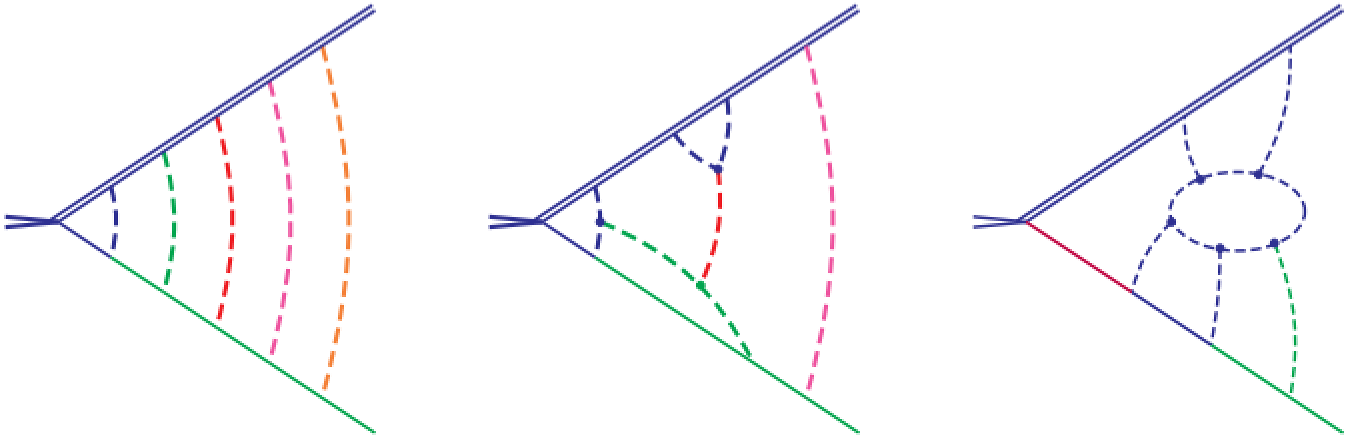
Regions in the mass expansion

- Examples



A formalism for planar graphs

- For planar graphs, each region can be depicted as a "***terrace***".

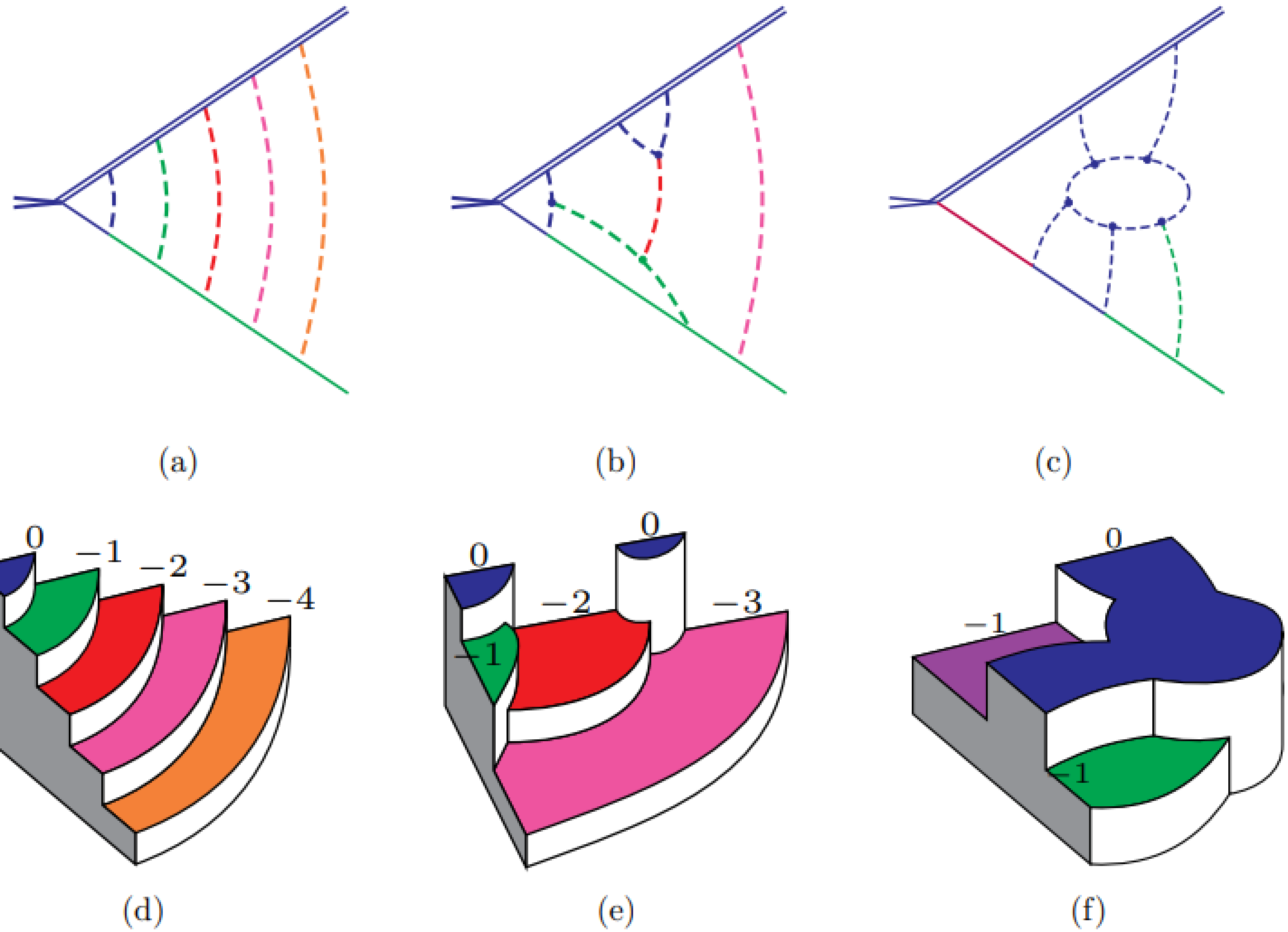


(a) (b) (c)

(YM, arXiv:2312.14012)

A formalism for planar graphs

- For planar graphs, each region can be depicted as a "*terrace*".



(YM, arXiv:2312.14012)

A formalism for planar graphs

- For planar graphs, each region can be depicted as a "***terrace***".

(d) (e) (f)

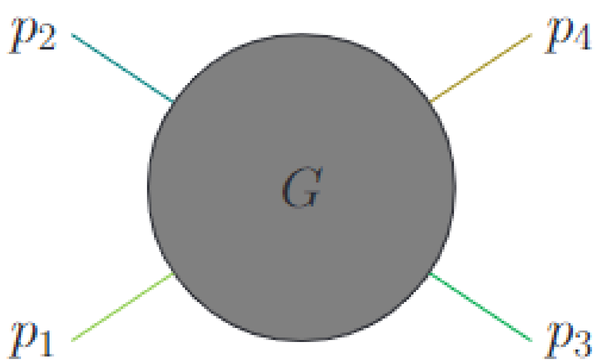
(YM, arXiv:2312.14012)

High–energy expansion of forward scattering

Consider the Regge limit of the 2-to-2 forward scattering.

Regions include:

hard, collinear, soft, *Glauber*, soft·collinear, collinear[3], …



$$(p_1+p_2)^2 = s,$$
$$(p_1+p_3)^2 = t,$$
$$(p_1+p_4)^2 = u,$$

kinematic limit:

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
$$|t| \ll s \sim |u|,$$

(E.Gardi, F.Herzog, S.Jones, YM, to appear)

High–energy expansion of forward scattering

Consider the Regge limit of the 2-to-2 forward scattering.

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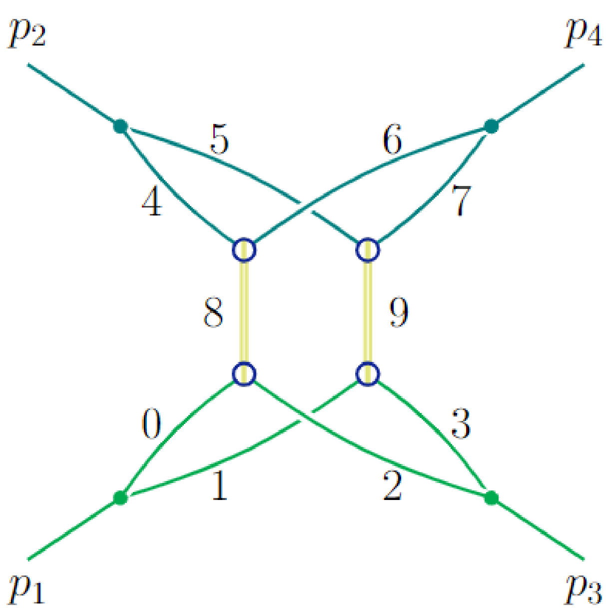


From 3 loops.
Not facets of the Newton polytope.

Due to the cancellation of the following terms

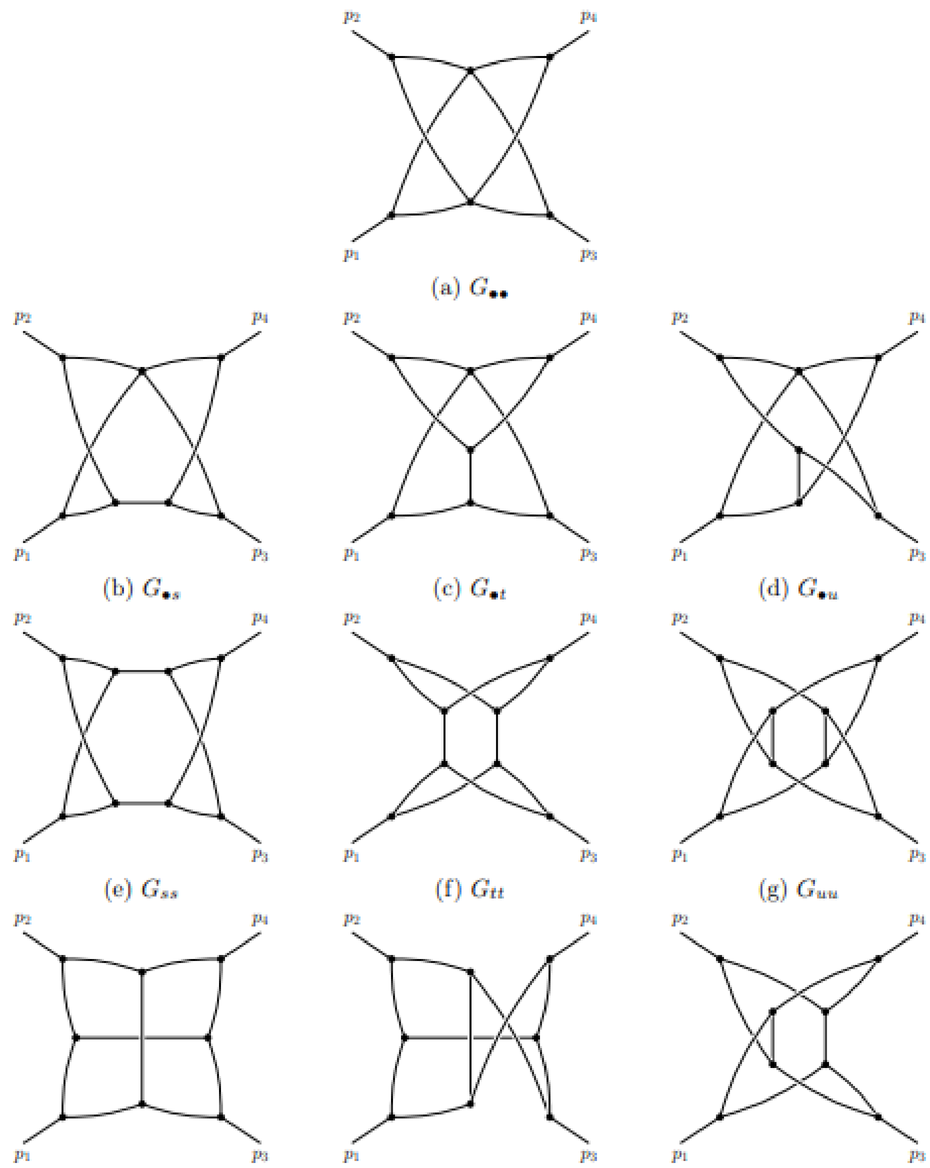
$$s_{12}\cdot(x_1x_4 - x_2x_3)\cdot(x_5x_8 - x_6x_7).$$

Cannot be detected by Asy2 either.

Much more to explore!

(E.Gardi, F.Herzog, S.Jones, YM, to appear)

High–energy expansion of forward scattering



(a) $G_{\bullet\bullet}$

(b) $G_{\bullet s}$ (c) $G_{\bullet t}$ (d) $G_{\bullet u}$

(e) G_{ss} (f) G_{tt} (g) G_{uu}

(h) G_{st} (i) G_{su} (j) G_{tu}

Main conclusion

The regions corresponding to a given graph can be predicted from the infrared picture!

– on–shell expansion: hard, collinear, soft.

– soft expansion: hard, collinear, soft.

– mass expansion: hard, collinear, soft, semihard, soft·collinear, soft^2·collinear, semicollinear, ...

– high–energy expansion: hard, collinear, soft, Glauber, soft ·collinear, ...

The mode interactions follow certain pictures.

Outlook

Hopefully, this work can be helpful to the following aspects.

1. SCET, Glauber-SCET, SCET gravity, etc.
2. Phase space integrals.
3. Local infrared subtractions.
4. Can one even justify the method of regions with the help of our results?
5. Landau analysis of singularities.
6. Mathematical studies of convex/tropical geometry, etc.

…

THANK YOU!

Backup slides

Asymptotic expansion of Feynman integrals

- Evaluating multi-loop Feynman integrals poses significant challenges.

- For Feynman integrals with multiple scales in the external kinematics, a natural idea is to consider the asymptotic expansion.

$$\mathcal{A} \sim \mathcal{A}_0 + \left(\frac{\Lambda_{\text{small}}}{\Lambda_{\text{large}}}\right) \mathcal{A}_1 + \left(\frac{\Lambda_{\text{small}}}{\Lambda_{\text{large}}}\right)^2 \mathcal{A}_2 + \dots$$

- Moreover, asymptotic expansion offers insights into the intricate infrared structure of gauge theory.

- There are various techniques of doing asymptotic expansions.

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- Moreover, asymptotic expansion offers insights into the intricate infrared structure of gauge theory.

- There are various techniques of doing asymptotic expansions.

This talk: the method of regions

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

$x_1 \to$ **(1,0,0;0)**

$x_2 \to$ **(0,1,0;0)**

$x_3 \to$ **(0,0,1;0)**

$p_1^2 x_1 x_3 \to$ **(1,0,1;1)**

$p_2^2 x_2 x_3 \to$ **(0,1,1;1)**

$q_1^2 x_1 x_2 \to$ **(1,1,0;0)**

These points are in the **hard facet**.

Hard region : $\boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$

$$\mathcal{I}_h = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \cdot (x_1 + x_2 + x_3 - q_1^2 x_1 x_2)^{-D/2},$$

$$+\cdots$$

The Newton polytope approach

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x_1 → **(1,0,0;0)**

x_3 → **(0,0,1;0)**

$-p_1^2 x_1 x_3$ → **(1,0,1;1)**

$-q_1^2 x_1 x_2$ → **(1,1,0;0)**

These points are in the **collinear-1 facet**.

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(0,0,1;0)

(1,1,0;0)

(0,1,0;0)

(0,1,1;1)

These points are on the **collinear-2 facet**:

The Newton polytope approach

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

x_3 → **(0,0,1;0)**

$p_1^2 x_1 x_3$ → **(1,0,1;1)**

$p_2^2 x_2 x_3$ → **(0,1,1;1)**

$q_1^2 x_1 x_2$ → **(1,1,0;0)**

These points are on the **soft facet**.

Regions in different representations

- Momentum space:

$$\text{Hard region: } k^\mu \sim Q(1,1,1)$$
$$\text{Collinear-1 region: } k^\mu \sim Q(1,\lambda,\lambda^{1/2})$$
$$\text{Collinear-2 region: } k^\mu \sim Q(\lambda,1,\lambda^{1/2})$$
$$\text{Soft region: } k^\mu \sim Q(\lambda,\lambda,\lambda)$$

- Parameter space:

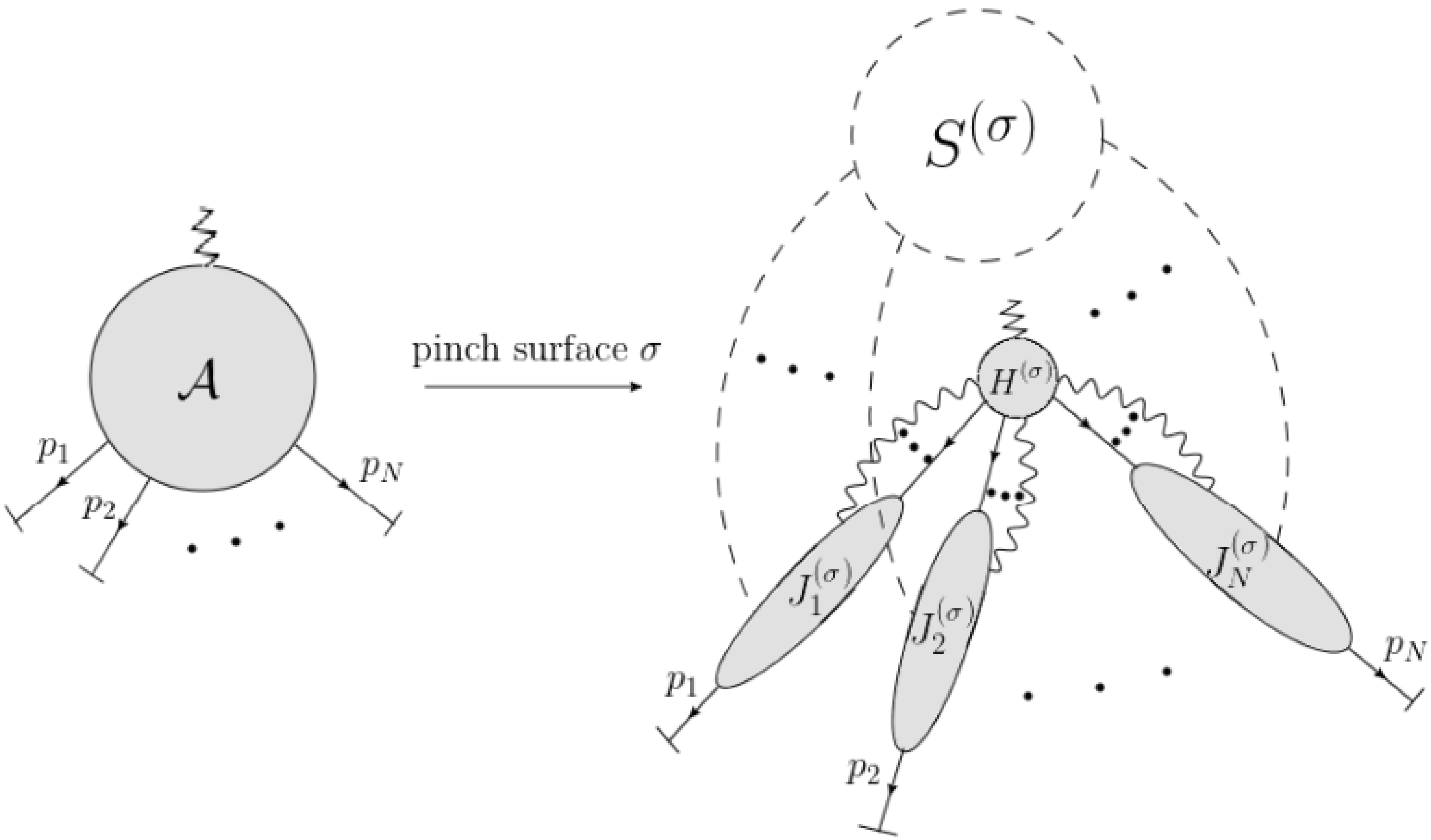
$$\text{Hard region}: \ x_1, x_2, x_3 \sim \lambda^0$$
$$\text{Collinear region to } p_1: x_1, x_3 \sim \lambda^{-1}, \ x_2 \sim \lambda^0$$
$$\text{Collinear region to } p_2: x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}$$
$$\text{Soft region}: x_1, x_2 \sim \lambda^{-1}, \ x_3 \sim \lambda^{-2}$$

- Relation between the scalings:

$$x_e \sim (D_e)^{-1}$$

Infrared structures of wide-angle scattering

- **Generic infrared divergences (pinch surfaces):**



This picture can be obtained from the Landau equations.

Infrared structures of wide-angle scattering

- **The Landau equations**

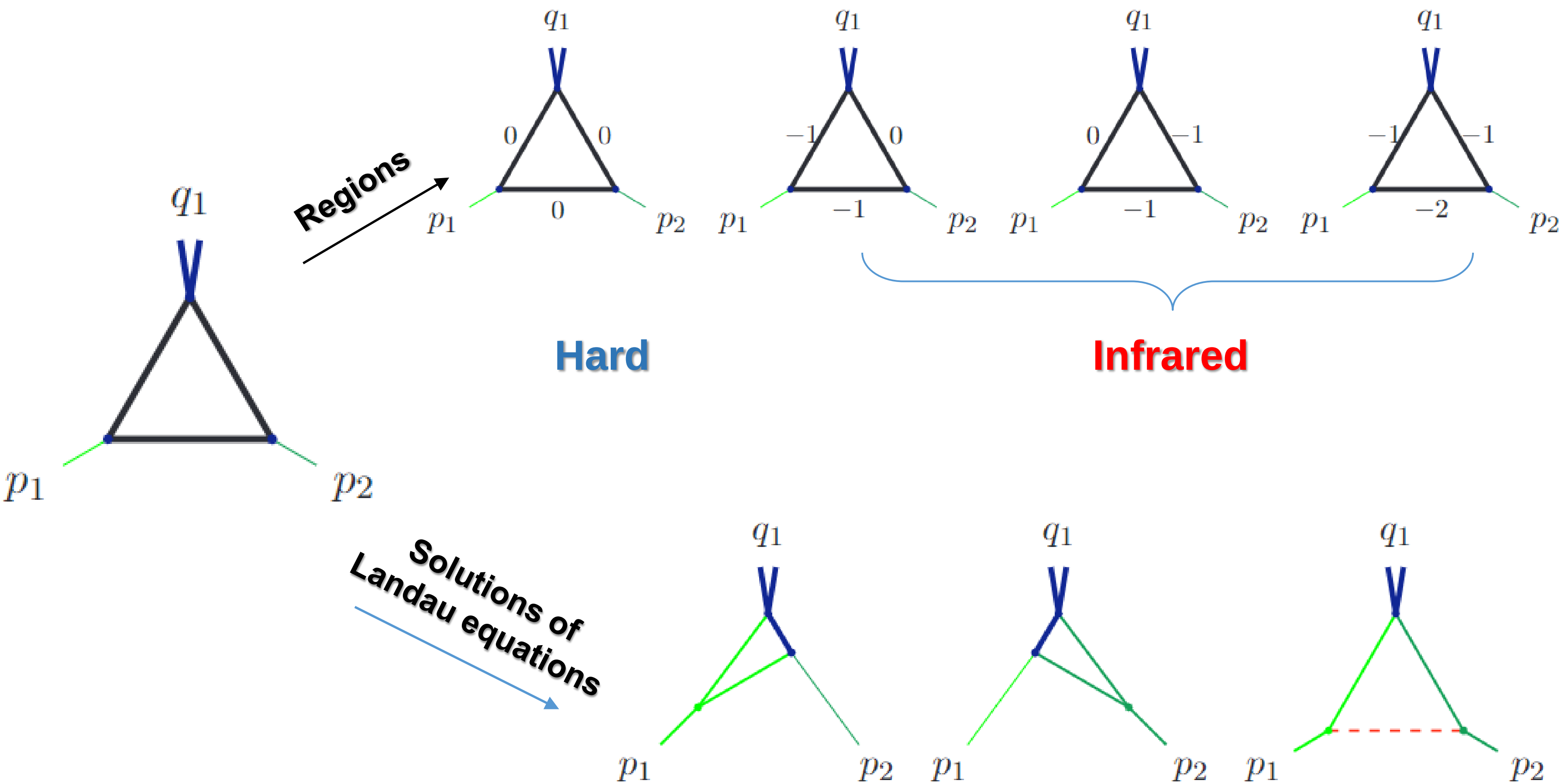
$$\alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$
$$\frac{\partial}{\partial k_a}\mathcal{D}\left(k,p,q;\alpha\right) = 0 \qquad \forall a \in \{1,\ldots,L\}\,.$$

are necessary conditions for infrared singularity. The solutions of the Landau equations are called pinch surfaces.

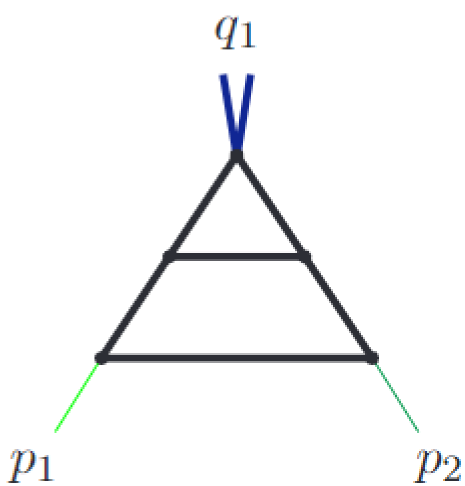
- **The pinch surfaces of hard processes has been studied in detail in the past decades.**

- **Motivation: it looks that the infrared regions are in one-to-one correspondence with the pinch surfaces!**

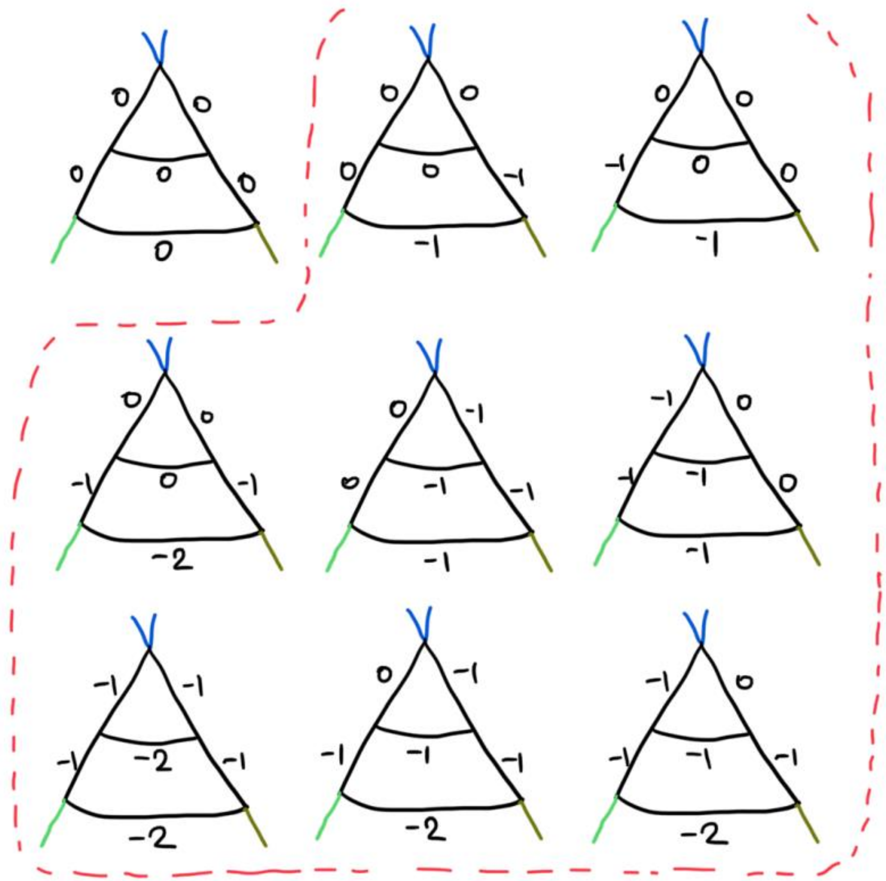
Relating regions to Landau equations



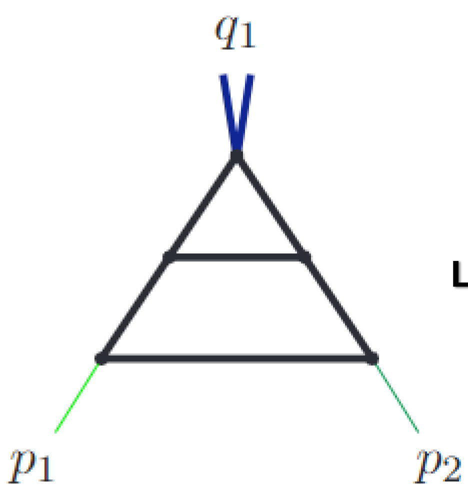
Relating regions to Landau equations



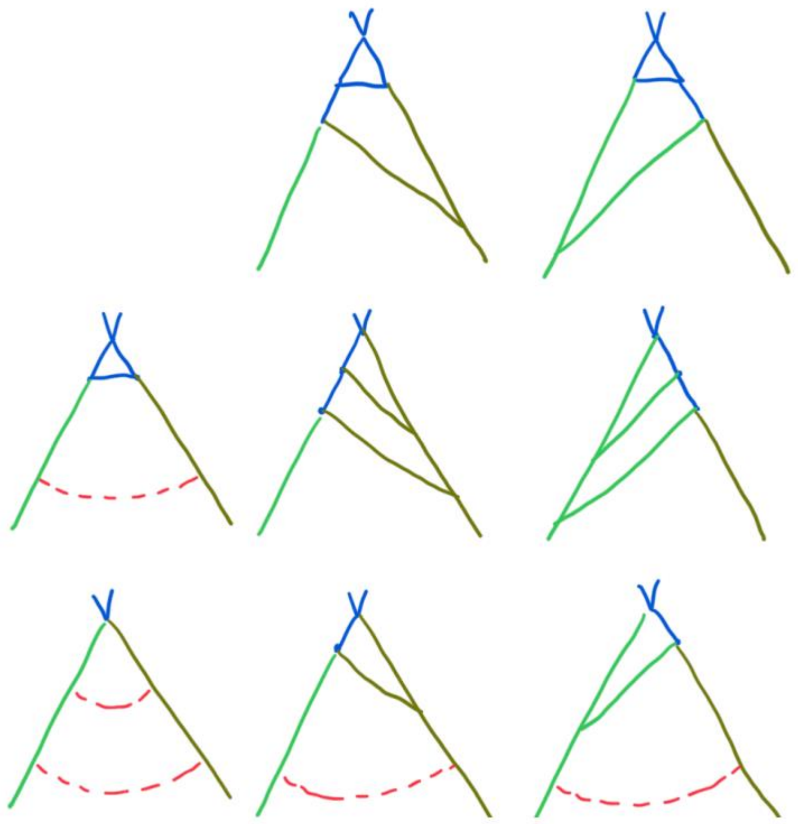
Regions



Relating regions to Landau equations



Solutions of Landau equations →



Regions in the on–shell expansion

E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk, JHEP07(2023)197

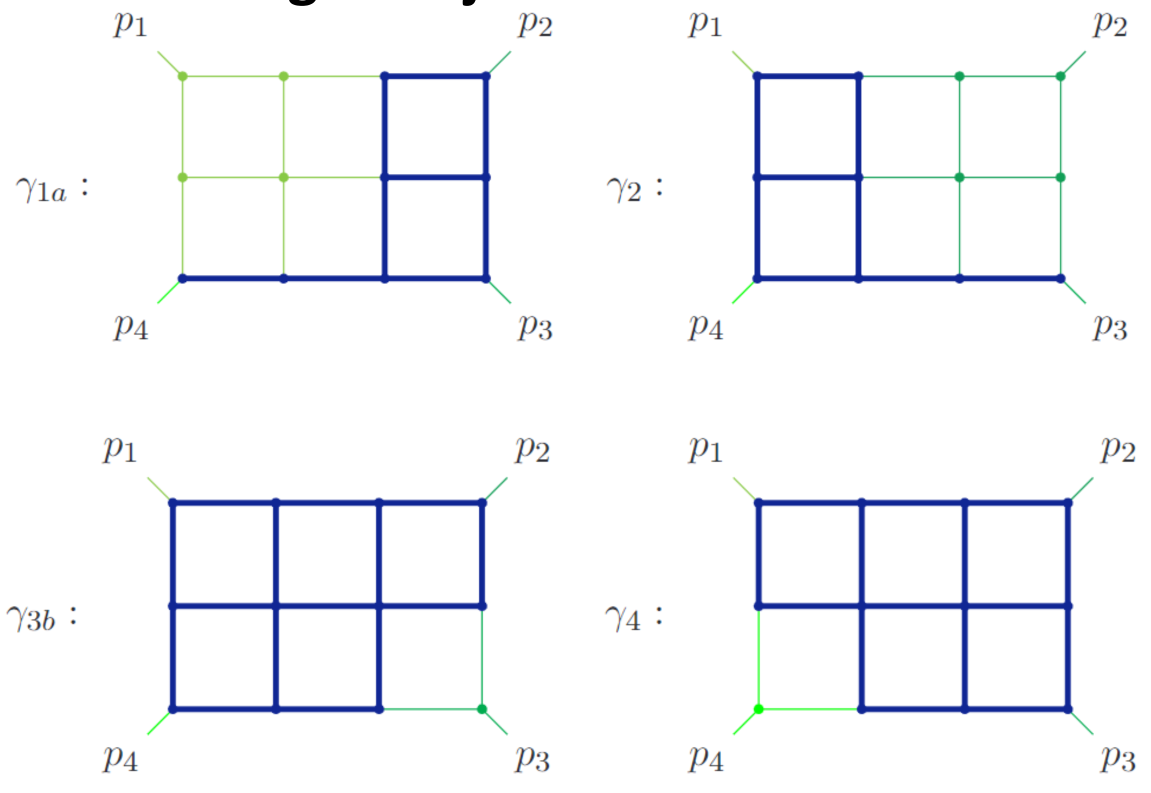
- **Each solution of the Landau equations corresponds to a region, provided that some requirements of H, J, and S are satisfied.**

  - *Requirement of H: all the internal propagators of H_{red}, which is the reduced form of H, are off-shell.*
  - *Requirement of J: all the internal propagators of $\widetilde{J}_{i,\text{red}}$, which is the reduced form of the contracted graph $\widetilde{J}_i$, carry exactly the momentum p_i^μ.*
  - *Requirement of S: every connected component of S must connect at least two different jet subgraphs J_i and J_j.*

Application 1: graph–finding algorithm

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

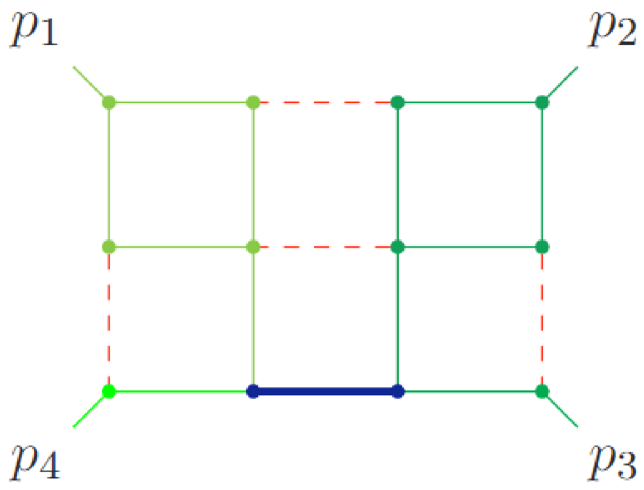
Step 1: constructing the "jets":



Application 1: graph–finding algorithm

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

Step 2: overlaying the "jets":



This algorithm does not involve constructing Newton polytopes, and can be much faster.

Step 3: removing pathological configurations.

Application 2: analytic structures of $\mathcal{I}$

- In addition, one can use this knowledge to study the analytic structure of wide-angle scattering, which further leads to properties regarding the commutativity of multiple on-shell expansions.

Theorem 4. *If R is a jet-pairing soft region that appears in the on-shell expansion of a wide-angle scattering graph G, then the all-order expansion of $\mathcal{I}(G)$ in this region can be written as follows:*

$$\mathcal{T}_{\boldsymbol{t}}^{(R)}\mathcal{I}(\boldsymbol{s}) = \left(\prod_{p_i^2\in\boldsymbol{t}} (p_i^2)^{\rho_{R,i}(\epsilon)}\right) \cdot \sum_{k_1,\ldots,k_{|\boldsymbol{t}|}\geqslant 0} \left(\prod_{p_i^2\in\boldsymbol{t}} (-p_i^2)^{k_i}\right) \cdot \overline{\mathcal{I}}_{\{k\}}^{(R)}(\boldsymbol{s}\setminus\boldsymbol{t})\,, \tag{5.8}$$

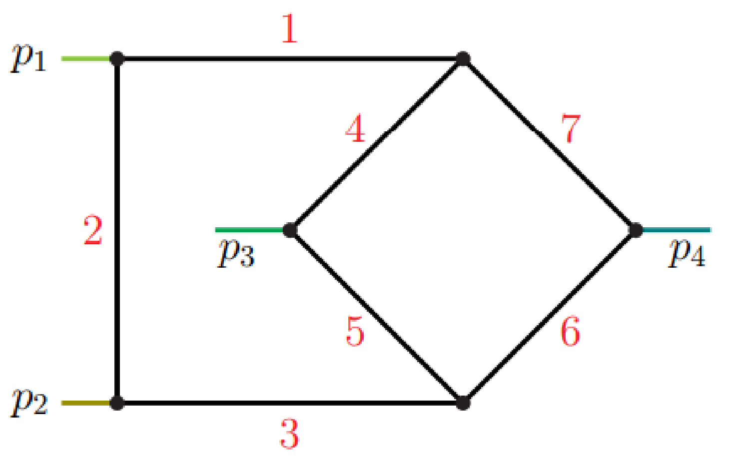
where $\rho_{R,i}(\epsilon)$ is a linear function of ϵ, k_i are non-negative integer powers and $\overline{\mathcal{I}}_{\{k\}}^{(R)}(\boldsymbol{s}\setminus\boldsymbol{t})$ is a function of the off-shell kinematics, independent of any $p_i^2\in\boldsymbol{t}$.

Landau analysis of cancellations

- Each region (except the hard region) must correspond to an infrared singularity, satisfying the Landau equations:

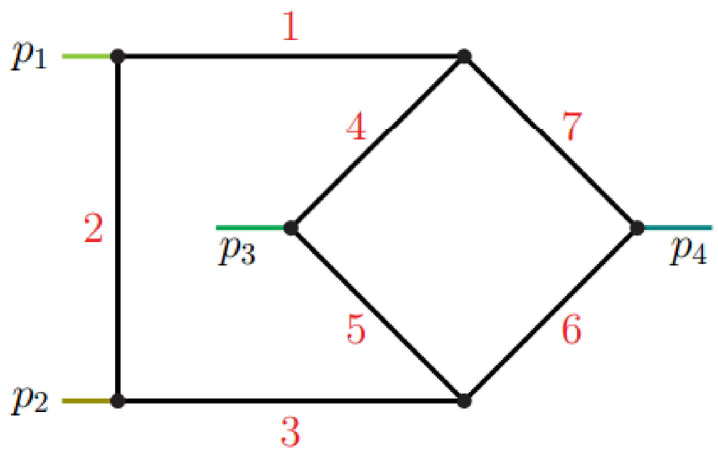
$$\mathcal{F}(\boldsymbol{\alpha};\boldsymbol{s}) = 0,$$
$$\forall i, \quad \alpha_i = 0 \ \text{ or } \ \partial\mathcal{F}/\partial\alpha_i = 0.$$

- Therefore, $\mathcal{F}$ having both positive and negative terms does not necessarily imply a region, because the Landau equation above may not be satisfied.
- For example,



Landau analysis of cancellations

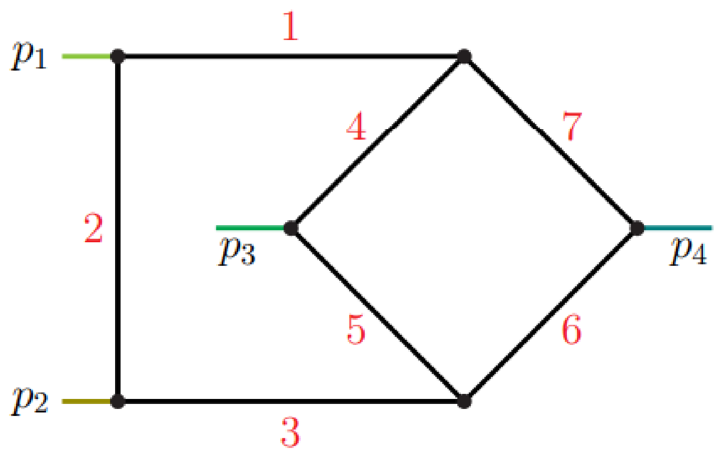
- For example,



$$
\begin{aligned}
\mathcal{F}(\boldsymbol{\alpha};\boldsymbol{s}) = \quad & (-p_1^2)\left[\alpha_1\alpha_2(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_2\alpha_4\alpha_7\right] \\
& + (-p_2^2)\left[\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_2\alpha_5\alpha_6\right] \\
& + (-p_3^2)\left[\alpha_4\alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_6+\alpha_7)+\alpha_1\alpha_5\alpha_7+\alpha_3\alpha_4\alpha_6\right] \\
& + (-p_4^2)\left[\alpha_6\alpha_7(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+\alpha_1\alpha_4\alpha_6+\alpha_3\alpha_5\alpha_7\right] \\
& + (-q_{12}^2)\left[\alpha_1\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_3\alpha_4\alpha_7+\alpha_1\alpha_5\alpha_6\right] \\
& + (-q_{13}^2)\alpha_2\alpha_5\alpha_7 + (-q_{14}^2)\alpha_2\alpha_4\alpha_6.
\end{aligned}
$$

Landau analysis of cancellations

- For example,



One can check that any possible cancellation within $\mathcal{F}$ is not compatible with the Landau equations.

- Therefore, all the regions are from the lower facets of the Newton polytope.

- Actually, as one can check in this way, most cases where $\mathcal{F}$ is indefinite does not have regions due to cancellations.