Amplitude methods for gravitational interactions

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Based in work with
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Infrared Photons and Gravitons*

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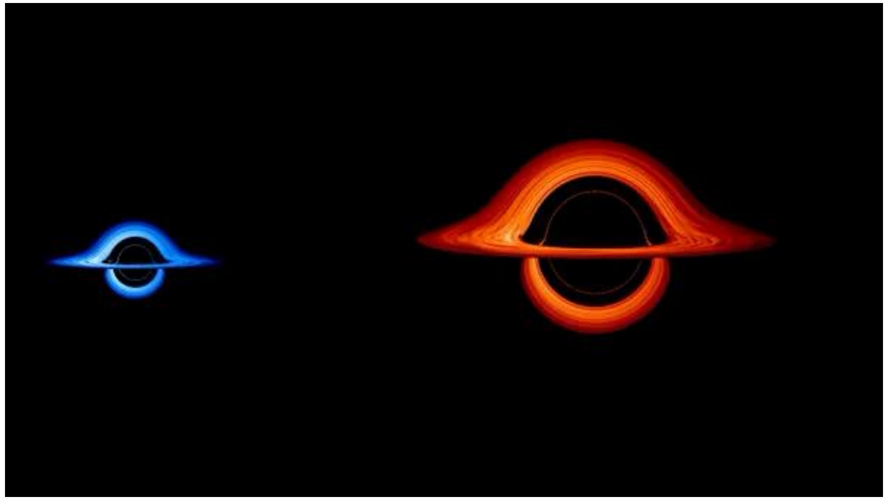
It would be difficult to pretend that the gravitational infrared divergence problem is very urgent. My reasons for now attacking this question are:

(1) Because I can. There still does not exist any satisfactory quantum theory of gravitation, and in lieu of such a theory it would seem well to gain what experience we can by solving any problems that can be solved with the limited formal apparatus already at our disposal. The infrared divergences are an ideal case

(2) Because something might go wrong, and that would be interesting. Unfortunately, nothing does go wrong.

(3) Because in solving the infrared divergence problem we obtain a formula for the emission rate and spectrum of soft gravitons in arbitrary collision processes, which may (if our experience in electrodynamics is a guide) be numerically the most important gravitational radiative correction.

Amplitudes approach to gravity:



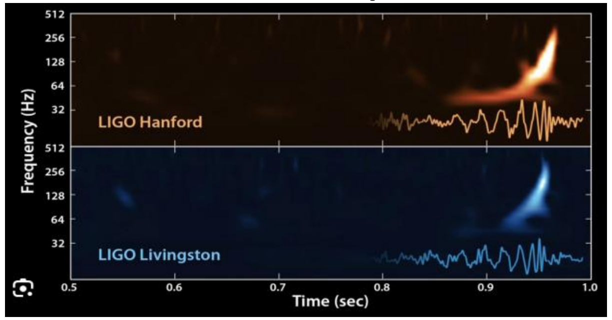
credit:NASA

- Not directly based the space-time geometry and Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

- Main character: the graviton – spin 2 particles; fluctuations around some fixed space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \kappa^2 = 32\pi G$$



- Well suited for: - questions in asymptotically-flat spacetimes
  - perturbation theory; particle physics input on resummation would be useful!
  - gravitational wave physics with (relatively) compact sources

Complementary perspective on gravitational interactions;
Exposes properties not obvious from the Lagrangian *and* has real-world applications

Scattering amplitudes – bread and butter of particle physics calculations

The key of modern amplitudes methods: manifest gauge invariance at all intermediate stages

- On-shell recursion relations — Britto, Cachazo, Feng, Witten
- Generalized unitarity — Bern, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng
- Color-kinematics duality
- Double copy: gravity = YM x YM — Bern, Carrasco, Johansson
- New integration methods
- Massive spinor helicity — Arkani-Hamed, Huang, Huang; in *D=5* Chiodaroli, Gunaydin, Johansson, RR

Some quantum milestones:

4-point amplitudes through 5 loops full color $\mathcal{N}$=4 super-Yang-Mills theory

4-point amplitudes through 5 loops $\mathcal{N}$=8 supergravity

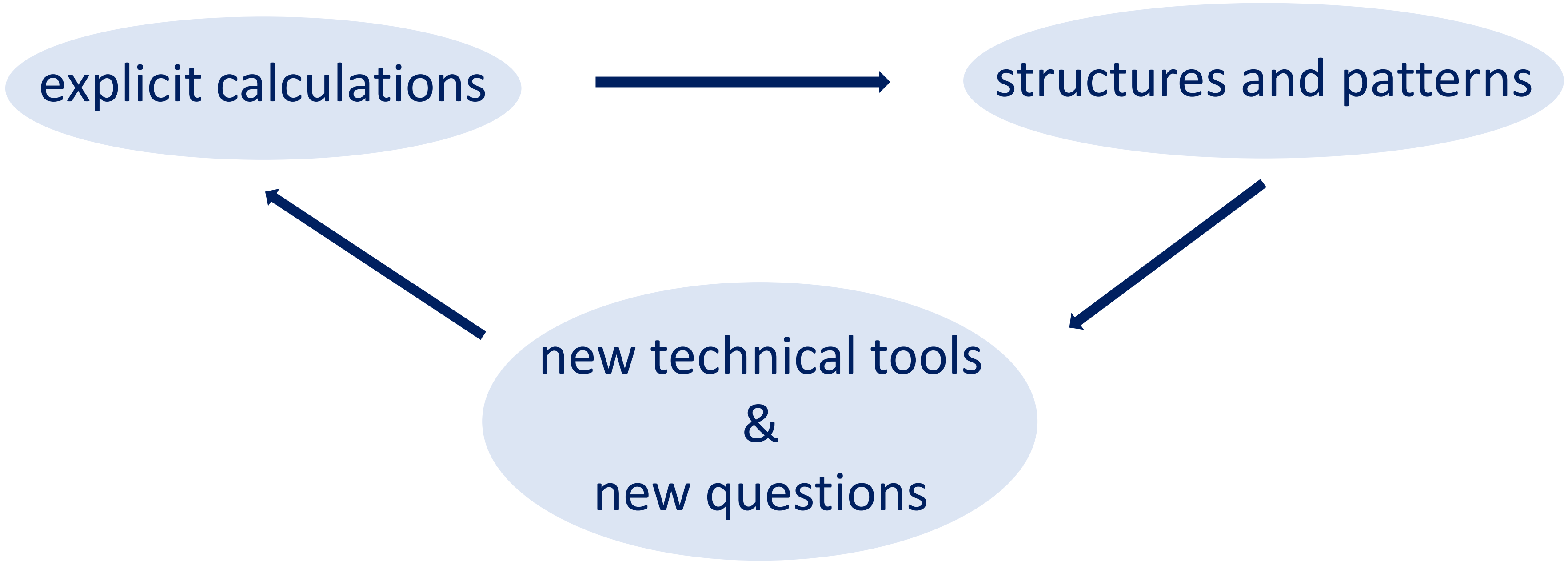
4-point amplitudes through 4 loops $\mathcal{N}$=4 and $\mathcal{N}$=5 supergravity

4-point amplitudes through 2 loops full color $\mathcal{N}$=2 super-QCD $\longrightarrow$ SG+matter

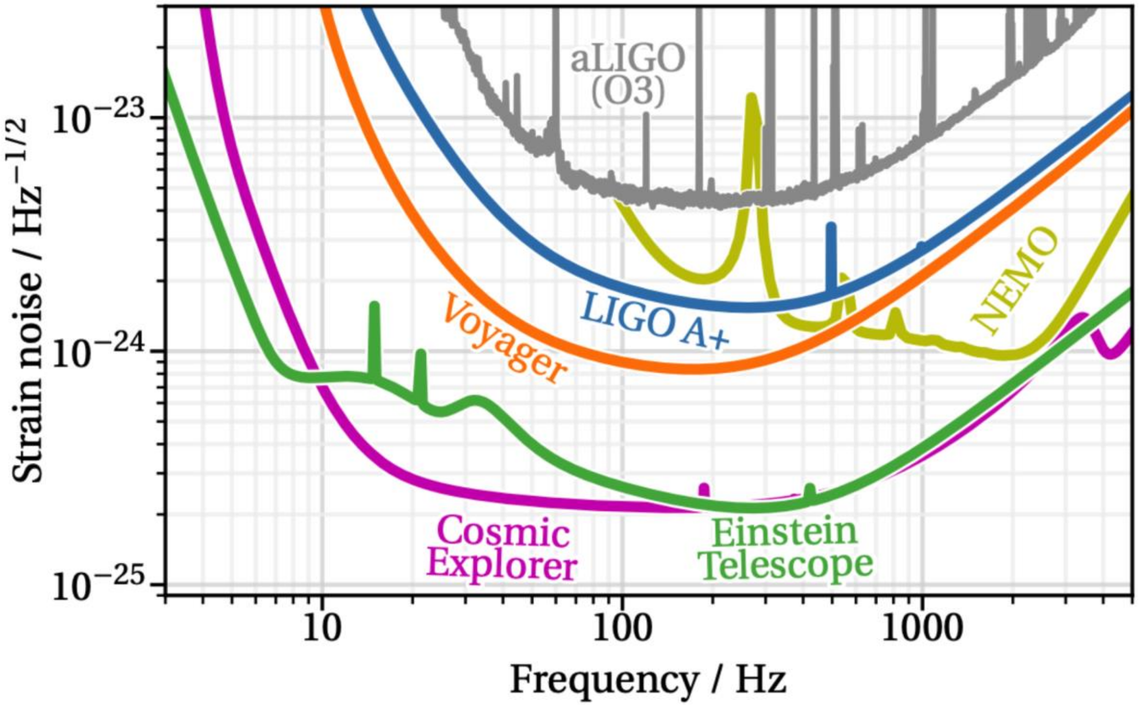
4-point amplitudes through 2 loops full color $\mathcal{N}$=1 super-YM theory $\longrightarrow$ SG+matter

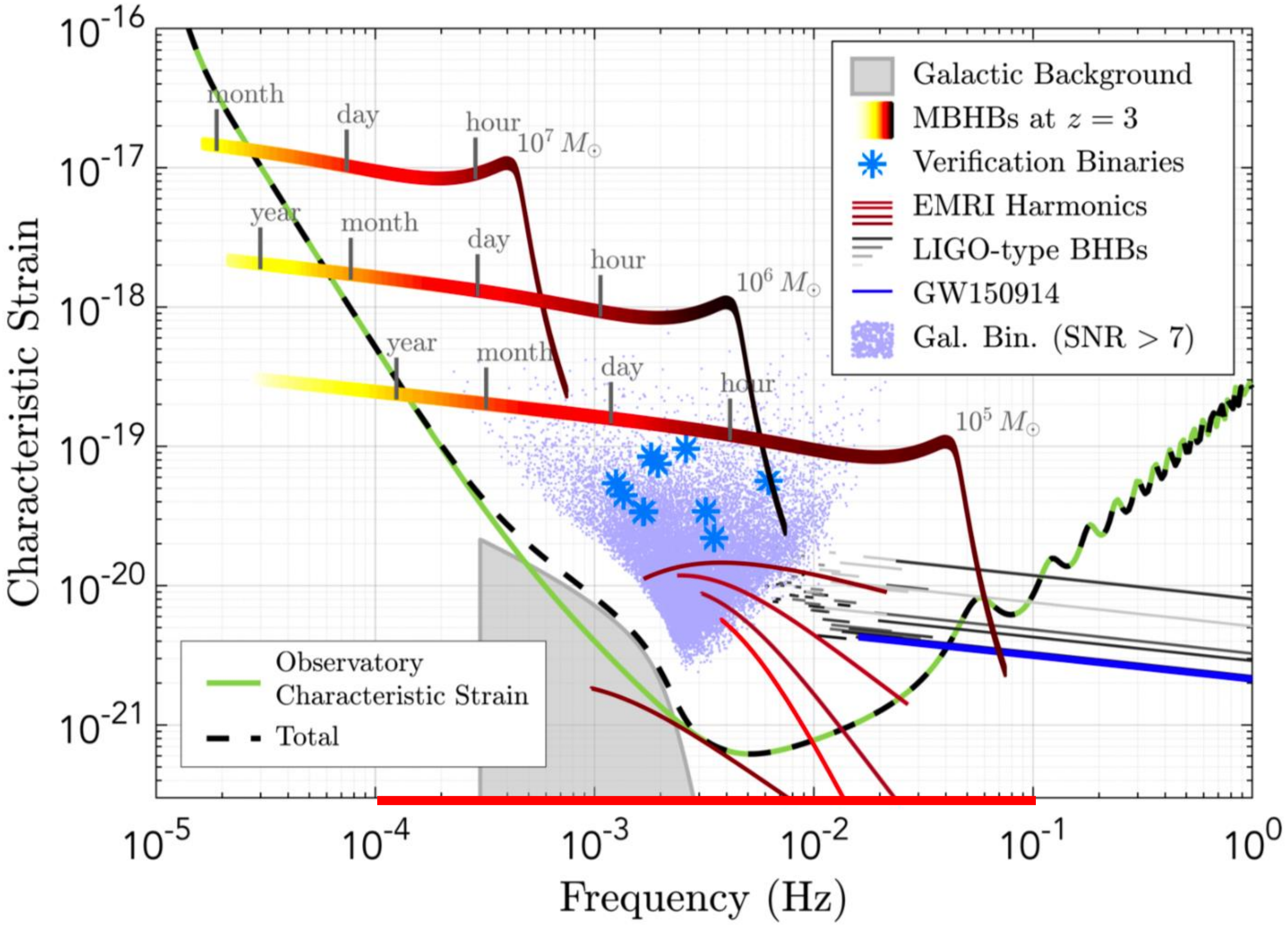
- Supersymmetry is not essential, but it does make things simpler
- Higher-point amplitudes have also been computed through these methods

Exposed enhanced UV cancellations in some of these theories, worthy of further investigation


explicit calculations
structures and patterns
new technical tools
&
new questions

More immediate questions: Can we apply quantum scattering methods to classical gravity, specifically, to gravitational wave physics?





- Extended signals → long accurate waveforms are required
  → buildup of theoretical error over long-time evolution must be avoided

More immediate questions: Can we apply quantum scattering methods to classical gravity, specifically, to gravitational wave physics?

Two main issues and blueprints for their resolution

1. quantum amplitudes $\longrightarrow$ classical physics?

e.g. construction of Coulomb/Newton potentials from tree-level amplitudes

$$V(r,p) = -\frac{Gm_1m_2}{r} + \ldots$$

2. Scattering = unbound motion *but* current typical source of gravitational waves are bound systems

"Integrate out the gravitons"

Hamiltonian/observables from scattering $\xrightarrow[\text{continuation*}]{\text{analytic}}$ use to study bound state dynamics

High-energy gravitational scattering and the general relativistic two-body problem

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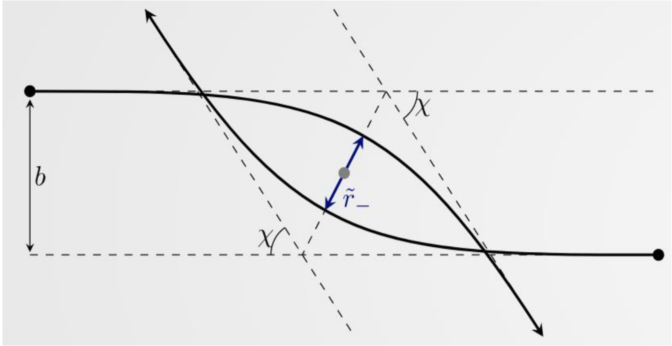
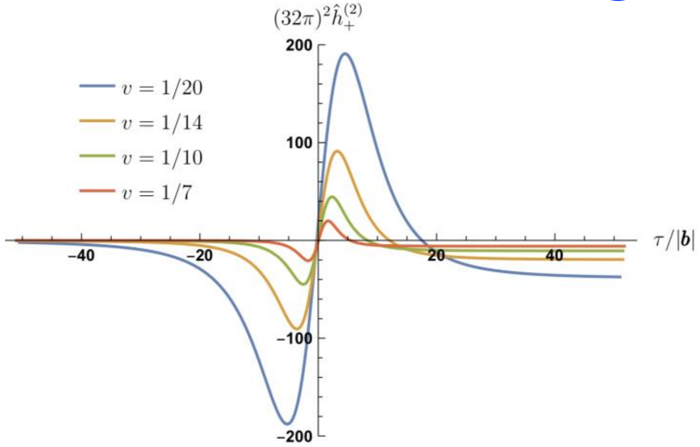
A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

Plan

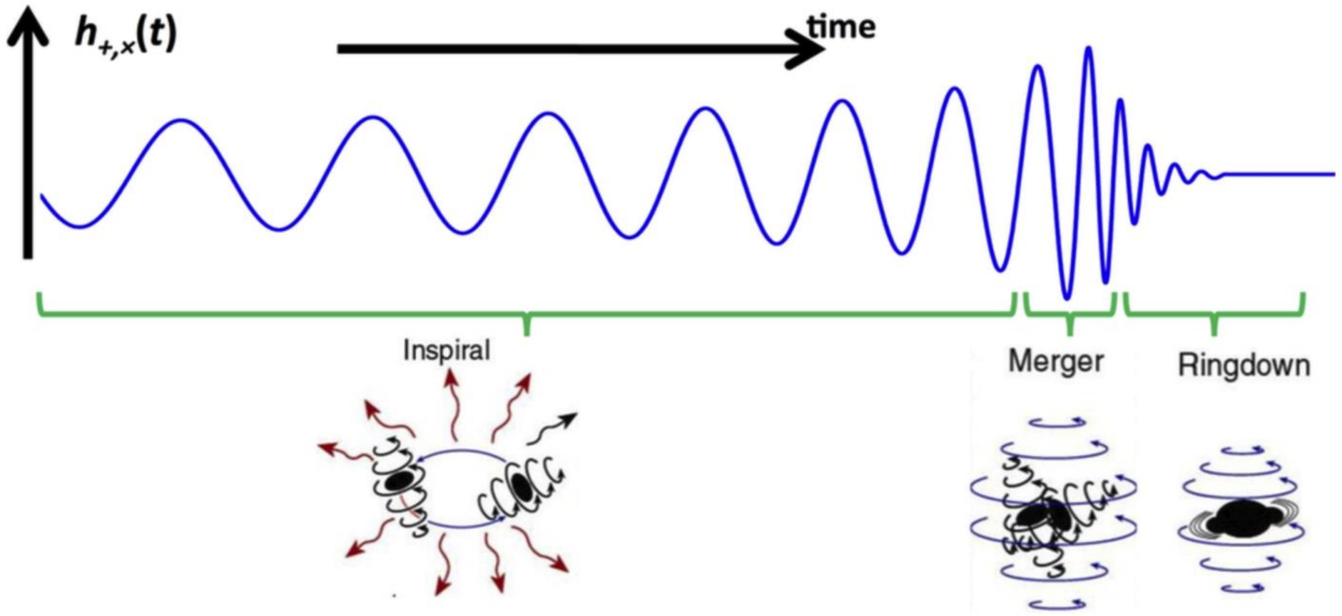
- From scattering amplitudes to effective two-body Hamiltonians with full velocity dependence



- From scattering amplitudes to gravitational scattering waveforms with full velocity dependence



Anatomy of an idealized binary merger



Favata/SXS/K.Thorne

Separation of scales: conservative dynamics vs. radiation emission

- Post-Newtonian expansion (weak field, nonrelativistic):
  - Closed orbits: expansion in two parameters:

Virial thm.

$$\boldsymbol{v}^2 \sim \frac{GM}{|\boldsymbol{r}|} \ll 1$$

- Post-Minkowskian expansion (weak-field, relativistic):
  - Expansion in G

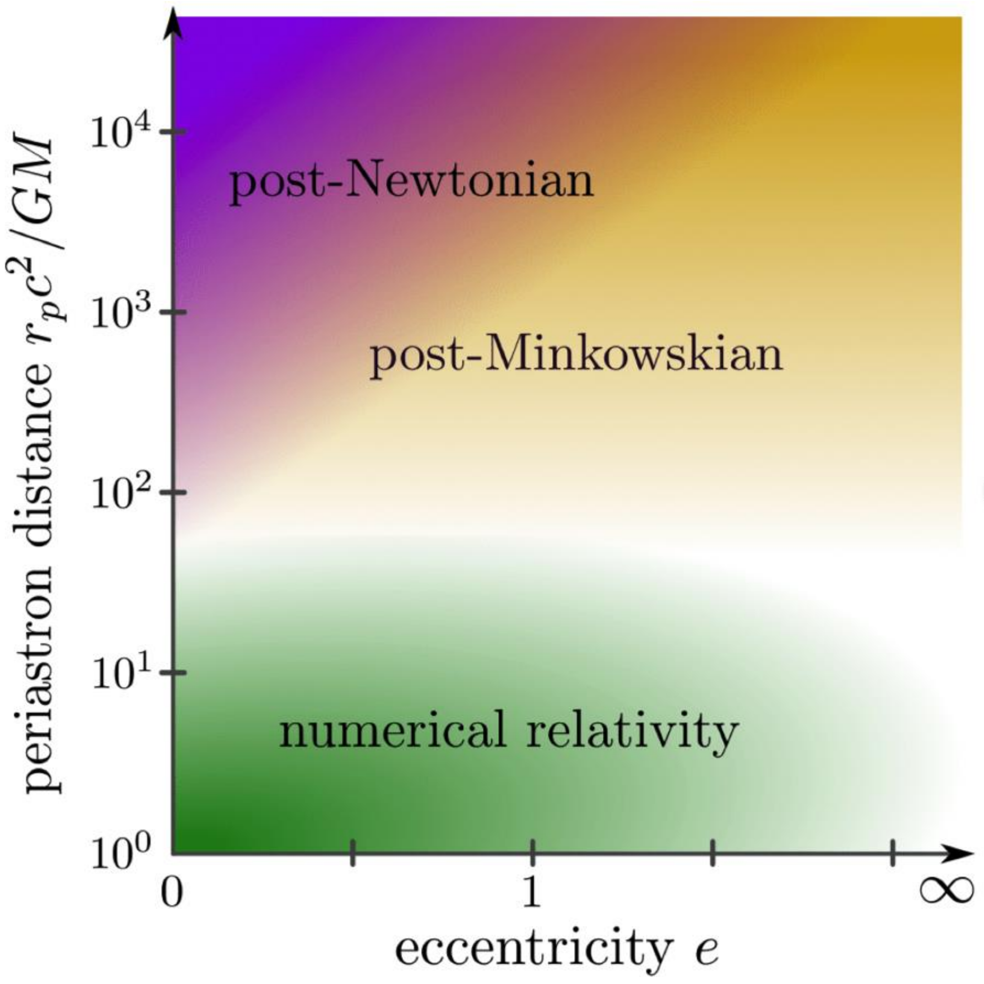
$$\frac{GM}{|\boldsymbol{r}|} \ll \boldsymbol{v}^2 \sim 1$$

- "Self-force" – expansion in a small mass ratio

$$\boldsymbol{v}^2 \sim GM/|\boldsymbol{r}| \sim 1$$

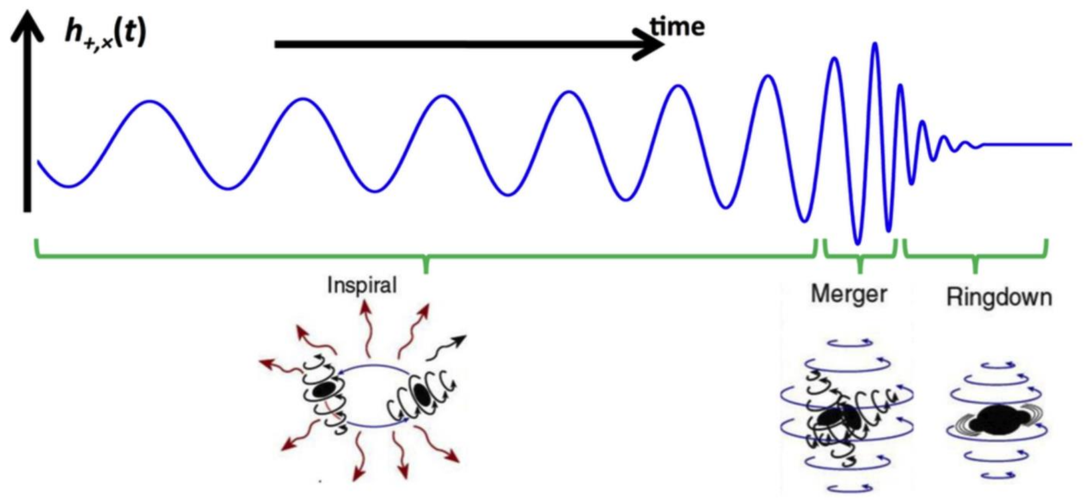
Small theoretical errors accumulate; high precision is needed in long inspiral phase

Why worry about the post-Minkowskian expansion:



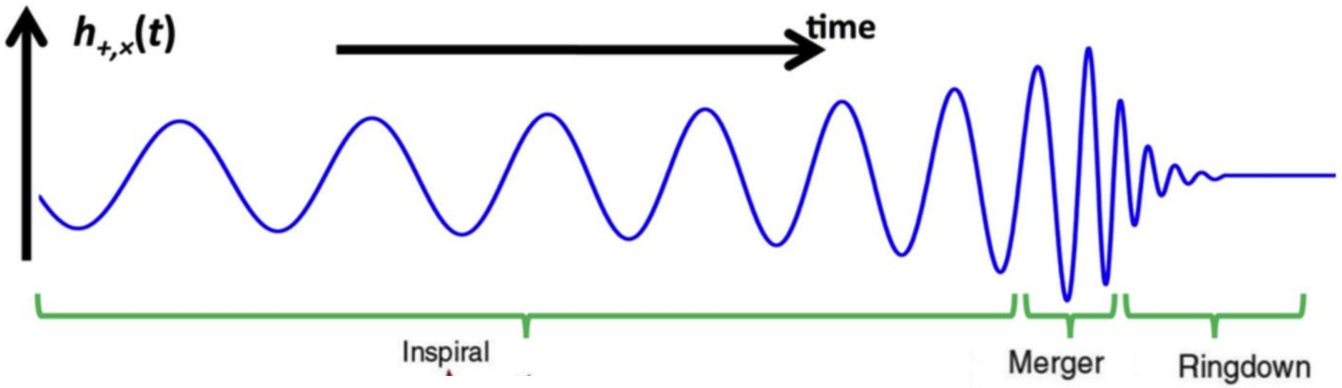
Khalil, Buonanno, Steinhoff, Vines

Favata/SXS/K.Thorne



Post-Minkowskian expansion is the relevant expansion for eccentric bounded motion and for hyperbolic motion

Our goals:



Favata/SXS/K.Thorne

- Improve upon post-Newtonian theory & cover a larger configuration space
- Improve the error accumulation issue by restoring Lorentz invariance
- Organize results so that they are useful for existing GW analysis pipelines

Lots of of problems to choose from (spin, tides, new physics, etc)

choose problems that are clear-cut and difficult with standard methods, but partly within reach of standard methods so that confirmation won’t have to wait

clear winner: conservative and radiative higher orders in perturbation theory

PN vs PM expansion for conservative dynamics of non-spinning compact objects

$$\mathcal{L} = -mc^2 + \frac{1}{2}\mu v^2 + G\frac{m\mu}{r} + \mathcal{O}(c^{-2}) + \mathcal{O}(c^{-4}) + \ldots$$

$$k\mathrm{PN} \sim (v^2)^n \left(\frac{Gm}{r}\right)^{k-n} \longrightarrow k\mathrm{PM} \sim c(p^2, p\cdot e_r)\left(\frac{Gm}{r}\right)^k$$

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ v^{14}	+ $\cdots$)	G
2PM		(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ $\cdots$)	G^2
3PM			(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ $\cdots$)	G^3
4PM				(1	+ v^2	+ v^4	+ v^6	+ v^8	+ $\cdots$)	G^4
5PM					(1	+ v^2	+ v^4	+ v^6	+ $\cdots$)	G^5
6PM						(1	+ v^2	+ v^4	+ $\cdots$)	G^6

1980

2018

2019

2021 - 2023

2024

PM results: Westfahl (79), Westfahl, Goller (80), Portilla (79-80), Bell et al (81), Ledvinka et al (10), Damour (16-17), Guevara (17), Vines (17), Bini, Damour (17-18)

recent PM results: Bern, Cheung, RR, Solon, Shen, Zeng (19), Cheung, Solon (20), Kälin, Porto (20); Parra-Martinez, Ruf, Zeng (20), Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng (21); Herrmann, Parra-Martinez, Ruf, Zeng (21) Dlapa, Kälin, Liu, Neef, Porto (22-23), Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch (24)

From amplitudes to gravitational wave observables

1. Amplitudes → Effective Hamiltonian → Hamilton's equations → Scattering observables

   Amplitudes → Amplitude-radial action relation $\mathcal{M}_4 \sim \int_J e^{iI_r(J)} - 1$ → Scattering observables

   Captures conservative physics (in a specific definition)

2. Observable-based formalism (inspired by QCD event shapes)

   Amplitudes and their cuts → Scattering waveforms and other scattering observables

   Uniformly captures conservative and dissipative physics

3. Related approaches

   - Worldline Quantum Field Theory
   - PM-EFT

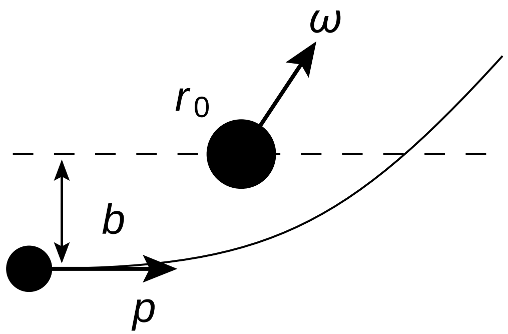
   capture both conservative and radiative physics in scattering regime

Correspondence principle: classical limit emerges when all conserved charges are large

N. Bohr

- In $\hbar = 1$ units:

"classical": de Broglie wave length λ of particles much smaller than their



- size: $\lambda \sim \dfrac{1}{|m r_0 \boldsymbol{\omega}|} \ll r_0 \quad \Longrightarrow \quad |\boldsymbol{S}| = m r_0^2 |\boldsymbol{\omega}| \gg 1$

- separation: $\lambda \sim \dfrac{1}{|\boldsymbol{p}|} \ll |\boldsymbol{b}| \sim \dfrac{1}{|\boldsymbol{q}|} \quad \Longrightarrow \quad |\boldsymbol{L}| = |\boldsymbol{b} \times \boldsymbol{p}| \gg 1 \ ; \ |\boldsymbol{p}| \gg |\boldsymbol{q}|$

Classical physics is in the soft region

Classical limit/expansion: $\mathcal{O}(1/|\boldsymbol{L}|) \sim \mathcal{O}(1/|\boldsymbol{S}|) \sim \mathcal{O}(|\boldsymbol{q}|/m) \sim \mathcal{O}(q/m)$

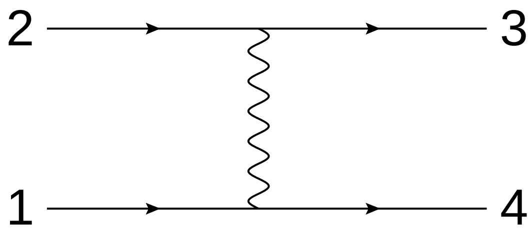
Structure of two-body potential and of classical $2 \to 2 + \mathrm{X}$ amplitude:

$$\mathcal{F}_{\boldsymbol{r}}\Big[V(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{S}), \boldsymbol{q}\Big] \sim \sum_{i \geq 1} \frac{c_i(\boldsymbol{p})}{|\boldsymbol{q}|^3} \left(Gm|\boldsymbol{q}| \right)^i$$

Loop amplitudes contain classical parts!

E.g. the relativistic Newton's potential

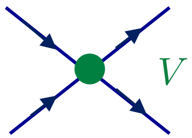
$$\mathcal{M}_{\text{tree}} \propto \frac{G_N}{q^2}\frac{(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{m_1^2 m_2^2} = \frac{G_N}{q^2}(2\sigma^2 - 1)$$



$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$H(\mathbf{p},\mathbf{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})\ , \quad V(\mathbf{p},\mathbf{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G_N}{|\boldsymbol{r}|}\right)^i$$

$$c_1(\boldsymbol{p}^2) = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right)$$



$$V(G_N)\Big|_{G_N^1}$$

$$\xi = \frac{E_1 E_2}{(E_1 + E_2)^2} \qquad \gamma = \frac{E_1 + E_2}{m_1 + m_2}$$

Hamilton's equations: integrals of motion: $E = H \equiv H(r^2, p^2) \qquad \vec{J} = \vec{r} \times \vec{p}$

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = 2\vec{p}F(r^2,p^2)\ , \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -2\vec{r}G(r^2,p^2)\ , \qquad F = \frac{\partial H}{\partial p^2}\ , \qquad G = \frac{\partial H}{\partial r^2}$$

1PM scattering angle:

$$\frac{1}{2}\chi_{\text{class}} = \frac{G_N m_1 m_2}{J}\frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} + \mathcal{O}(G_N^2)$$

Extensive perturbative QFT experience in gauge and gravity theories helps produce relativistic state of the art predictions

- Unitarity methods, recycles trees into loops — Bern, Dixon, Dunbar, Kosower
- Double-copy: gravity from gauge theory — Kawai, Lewellen, Tye; Bern, Carrasco, Johansson
- NRQCD/HEFT and EFT methods — Caswell, Lepage; Luke, Manohar, Rothstein; Golberger, Rothstein; Cheung, Rothstein, Solon
- Method of regions: integrate out potential graviton modes — Beneke, Smirnov
- Reduction to master integrals/Integration-by-parts reduction — Chetyrkin, Tkachov; Laporta
- Method of differential equations for the evaluation of master integrals — Kotikov; Bern, Dixon, Kosower; Gehrmann, Remiddi; Henn, Smirnov
- Methods for evaluation of phase-space integrals — Kosower, Page

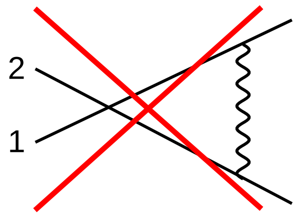
Classical limit helps tremendously

In the classical limit: - particles are always separated $\lambda \sim \frac{1}{|\boldsymbol{p}|} \ll |\boldsymbol{b}| \sim \frac{1}{|\boldsymbol{q}|}$

- specific dependence on $t = q^2$ $\mathcal{M} \propto |\boldsymbol{q}|^{-2+L} \ln |\boldsymbol{q}|^{[(L+1)/2]} \quad L \geq 1$

Some consequences:

1. In contributing diagrams, matter lines should not cross



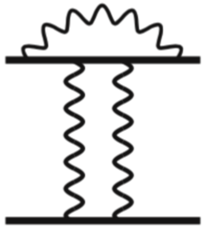
a. $2 \to 2$ before and after reduction to master integrals

b. $2 \to 2 + n$ before reduction

2. Every loop in contributing diagrams should have at least one matter line

Unitarity method provides a means to enforce these and other constraints

e.g. if focus on contribution of potential-region gravitons $\longrightarrow$ no mushrooms

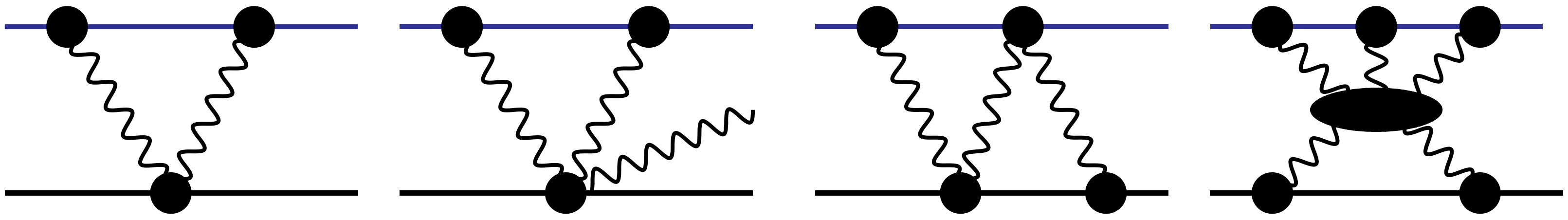


Amplitudes from generalized unitarity and double copy

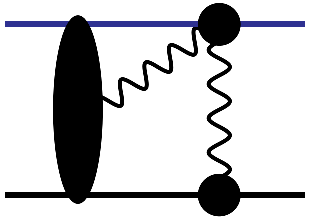
Amplitudes' integrands = rational functions with prescribed poles and residues

Poles = graph structure with given number of external lines and loops

Residues = generalized cuts = products of tree amplitudes



Cuts contributing to 1, 2, and 3-loop classical amplitudes



Cut containing unnecessary pieces

Generalized unitarity: - systematic reconstruction of integrands from this data

- effectively, a reorganization of Feynman graphs making explicit use of gauge invariance

- building blocks are tree amplitudes

Bern, Dixon, Kosower
Britto, Cachazo, Feng
Bern, Carrasco, Johansson, Kosower;…

- Gravity trees from gauge theory trees through KLT relations: Kawai, Lewellen, Tye

$$M_4^{\rm tr}(1,2,3,4) = -is_{12}A_4^{\rm tr}(1,2,3,4)A_4^{\rm tr}(1,2,4,3)$$

$$M_5^{\rm tr}(1,2,3,4,5) = \ \ is_{12}s_{34}A_5^{\rm tr}(1,2,3,4,5)A_5^{\rm tr}(2,1,4,3,5) + (2\leftrightarrow 3)$$

$$M_6^{\rm tr} = 12 \text{ terms of the type } s^3 A_6 A_6$$

- Hold state-by-state for external lines, following addition of helicities

e.g. $\text{scalar} \longleftrightarrow \text{scalar} \times \widetilde{\text{scalar}}$ $\quad h^{++} \longleftrightarrow A^+ \times \tilde{A}^+$ $\quad \varphi^{+-} \longleftrightarrow A^+ \times \tilde{A}^-$

- Hold in any dimension; implements all simplifications required by gauge invariance

- Gravity trees through color/kinematics, double-copy and generalized double-copy Bern, Carrasco, Johansson; Bern, Carrasco, Chen, Johansson, RR

e.g. $$i\mathcal{A}_4^{\rm tree}(1,2,3,4) = g^2\left(\frac{c_s n_s(p,\epsilon)}{s} + \frac{c_t n_t(p,\epsilon)}{t} + \frac{c_u n_u(p,\epsilon)}{u}\right)$$

$$c_s + c_t + c_u = 0 \qquad\qquad n_s + n_t + n_u = 0$$

$$i\mathcal{M}_4^{\rm tree}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^2\left(\frac{n_s(p,\epsilon)\tilde{n}_s(p,\epsilon)}{s} + \frac{n_t(p,\epsilon)\tilde{n}_t(p,\epsilon)}{t} + \frac{n_u(p,\epsilon)\tilde{n}_u(p,\epsilon)}{u}\right)$$

- Full power in relating loop level amplitudes; implications beyond amplitudes and gravitational theories

- Here used cut-by-cut; important for obtaining a graph-based organization of amplitudes

Integral evaluation – 4 points

Sudakov
Beneke, Smirnov;
HQET

1. Expand truncated integrand in the soft region; use special choice of momenta

$$\bar{p}_1 = p_1 + q/2 \qquad \bar{p}_2 = p_2 - q/2 \qquad \bar{p}_i^2 = \bar{m}_i^2 \qquad \bar{p}_i \cdot q = 0 \qquad y = \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2} = \sigma + \mathcal{O}(q^2)$$

$$\ell_i^2 \to \ell_i^2 \qquad (\ell_i + p_j)^2 - m_j^2 \to 2\bar{p}_j \cdot \ell_i + \mathcal{O}(q^2) \qquad I(p_1, p_2, q) \to \sum_k (q^2)^{a_k} \bar{m}_1^{b_{1k}} \bar{m}_2^{b_{2k}} I_k(y)$$

a. Single parameter b. Specific mass dependence

2. Differential equations for master integrals $\frac{dI_i}{dy} = A_{ij}(y,\epsilon) I_i$

Henn;
Parra-Martinez, Ruf, Zeng

3. Solve them; boundary conditions (i.e. values for some velocity) determine the region

Integral evaluation at 5 points is more involved

$$\bar{p}_1 = p_1 - q_1/2 \qquad \bar{p}_2 = p_2 - q_2/2 \qquad q_1 + q_2 = k \qquad \bar{p}_i^2 = \bar{m}_i^2 \qquad \bar{p}_i \cdot q_i = 0 \qquad \bar{p}_j \cdot q_i = \bar{p}_j \cdot k$$

- Mass dependence can be factorized
- Soft-region scaling is manifest $q_i \to \lambda q_i \quad k \to \lambda k \quad \bar{p}_i \to \bar{p}_i$; multi-parameter
- Linear propagators can be linearly dependent already at 1 loop

State of the art for spinless conservative PM scattering dynamics through G^4

Hamiltonian for hyperbolic motion: $H^{\text{hyp}} = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{r^n} c_n(\boldsymbol{p}^2)$ $\quad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} \quad m = m_1 + m_2$

$$\left(\gamma = \frac{E}{m},\ \xi = \frac{E_1 E_2}{E^2}\right)$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right]$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &-\frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+\frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg] \end{aligned}$$

Bern, Cheung, RR, Solon, Shen, Zeng

$$\begin{aligned} c_4^{\text{hyp}} = &\frac{m^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\text{p}} + \nu\left(4\mathcal{M}_4^{\text{t}}\log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2 c_1^2 c_2\right] \\ &+\left(\mathcal{D} + \frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4 + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2c_2\right] \end{aligned}$$

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

Parts of 4PM Hamiltonian from amplitudes methods:

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)}$$

$$\mathcal{M}_4^{\mathrm{t}} = r_1 + r_2 \log\left(\tfrac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4\pi^2 + r_5\,\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\,\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7\,\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)$$

$$\mathcal{M}_4^{\mathrm{rem}} = r_8 + r_9\log\left(\tfrac{\sigma+1}{2}\right) + r_{10}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}\log(\sigma) + r_{12}\log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\tfrac{\sigma+1}{2}\right) + r_{14}\tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}$$
$$+ r_{15}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17}\tfrac{1}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right]$$

- Derived with Feynman prescription $i\epsilon$
- Real part same as Damour's principal-value prescription for graviton prop's for conservative dyn.
- Remarkably compact given that it is the result of higher-loop GR calculation
- Amplitudes and scattering angle have simple mass dependence

$$\mathcal{M} \sim 1 \oplus \nu \oplus \cdots \oplus \nu^{[L/2]} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

  - Initially observed by Antonelli, Buonanno, Steinhoff, van de Meent, Vines in the 3PM angle
  - Thorough understanding by Damour: *good mass polynomiality rule*
  - From amplitudes perspective it is a consequence of Lorentz invariance

Parts of 4PM Hamiltonian from amplitudes methods:

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)}$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\mathcal{M}_4^{\pi^2} = r_4\pi^2 + r_5\,\mathrm{K}\left(\frac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\frac{\sigma-1}{\sigma+1}\right) + r_6\,\mathrm{K}^2\left(\frac{\sigma-1}{\sigma+1}\right) + r_7\,\mathrm{E}^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

$$\begin{aligned}\mathcal{M}_4^{\rm rem} = {} & r_8 + r_9\log\left(\frac{\sigma+1}{2}\right) + r_{10}\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}\log(\sigma) + r_{12}\log^2\left(\frac{\sigma+1}{2}\right) + r_{13}\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\frac{\sigma+1}{2}\right) + r_{14}\frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} \\ & + r_{15}\,\mathrm{Li}_2\left(\frac{1-\sigma}{2}\right) + r_{16}\,\mathrm{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) + r_{17}\frac{1}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right)\right]\end{aligned}$$

- Reproduces available 6PN results in the overlap and through $\mathcal{O}(\nu)$

Bini, Damour, Geralico

Blümlein, Maier, Marquard, Schäfer

- All-order-in-velocity verification

3PM: Cheung, Solon; Kälin, Liu, Porto
4PM: Dlapa, Kälin, Liu, Porto

- $\mathcal{O}(\nu^2)$ difference with Blümlein et al. and Foffa, Sturani remains to be fully resolved, though consensus is that good mass polynomiality should hold

- Analytic continuation to bound motion is nontrivial because of the tail effect; complete understanding is an important open problem
  for recent attempts see Dlapa, Kalin, Liu, Neef, Porto

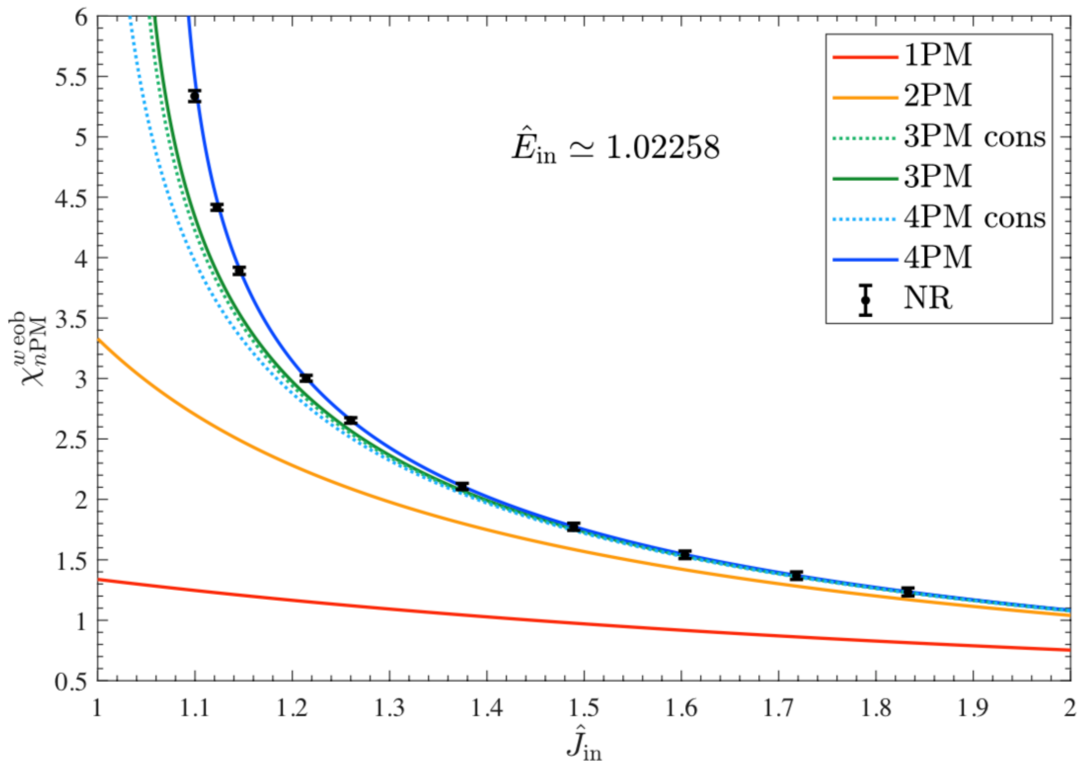
Bini, Damour

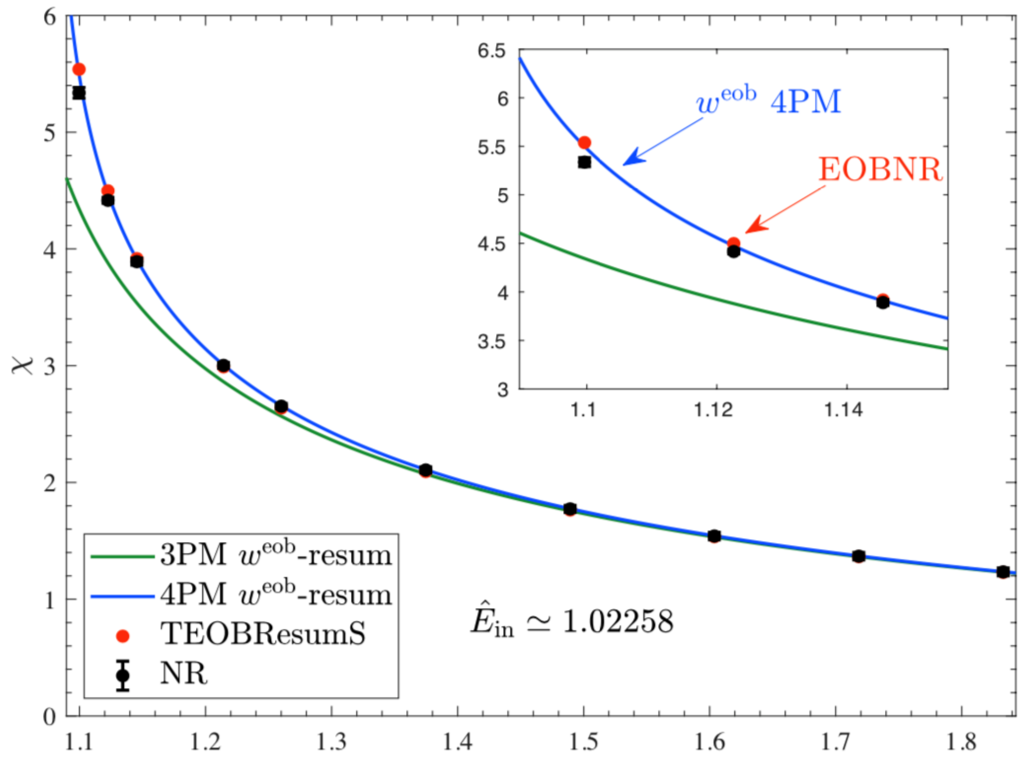
Kälin, Porto; +Cho

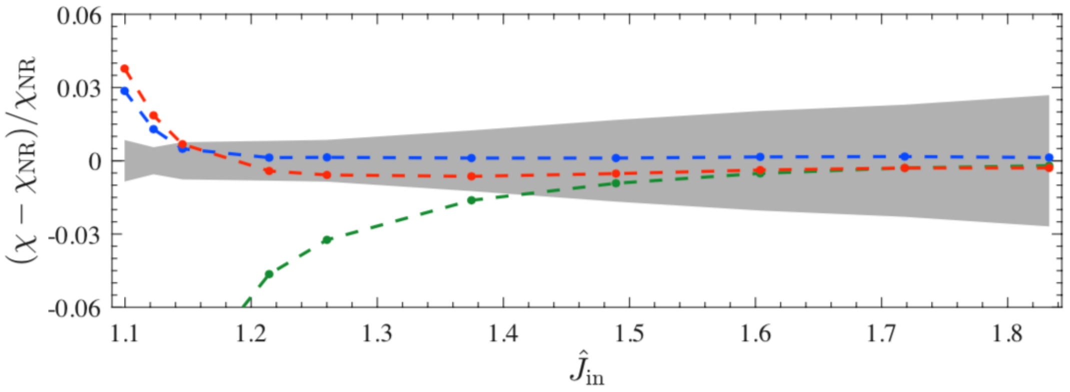
Comparison with Numerical GR via EOB Damour, Retegno

Dissipative radiation effects from Dlapa, Kalin, Liu, Neef, Porto

$$\chi^{w\ eob}_{\mathrm{nPM}}(\sigma, j) \equiv 2j \int_0^{\bar{u}_{\max}(\sigma,j)} \frac{d\bar{u}}{\sqrt{p_\infty^2 + w_{\mathrm{nPM}}(\bar{u},\sigma) - j^2\bar{u}^2}} - \pi$$







- PM EOB with radiation reaction improves comparison w/ NR to percent level precision for spinless scattering
- Precision depends somewhat on parameters
- New motivation for EOB studies of PM data for a precise analytic description of binary systems

see Buonanno, Jakobsen, Mogull for recent progress

Gravitational radiation from hyperbolic encounters

2. The observable-based formalism (KMOC formalism)

Kosower, Maybee, O'Connell

An observable = change in the expectation value of a suitable operator between initial and final state

$$\langle \mathcal{O} \rangle = \langle \psi_{\rm out} | \mathcal{O} | \psi_{\rm out} \rangle - \langle \psi_{\rm in} | \mathcal{O} | \psi_{\rm in} \rangle \qquad |\psi_{\rm out}\rangle = \mathbb{S} |\psi_{\rm in}\rangle \quad \text{where} \quad \mathbb{S} = \mathbb{I} + i\mathbb{T}$$

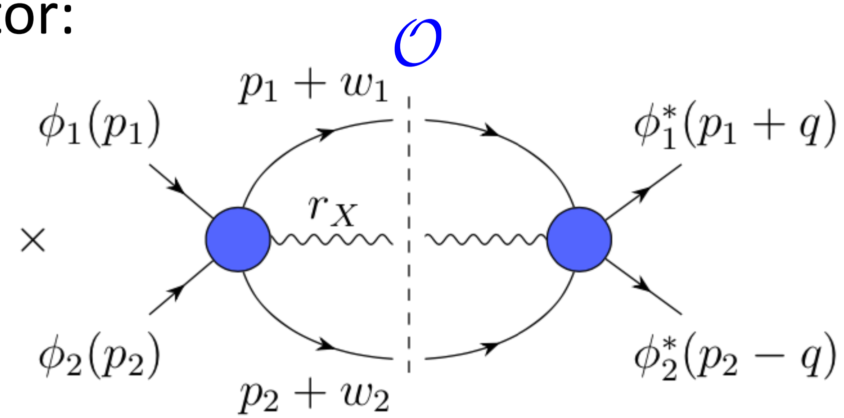
In terms of the transition matrix:

$$\langle \mathcal{O} \rangle = i \langle \psi_{\rm in} | \hat{\mathcal{O}} \mathbb{T} - \mathbb{T}^\dagger \hat{\mathcal{O}} | \psi_{\rm in} \rangle + \langle \psi_{\rm in} | \mathbb{T}^\dagger \mathcal{O} \mathbb{T} | \psi_{\rm in} \rangle \qquad |\psi_{\rm in}\rangle = \int d\Phi[p_1] d\Phi[p_2] \phi(p_1) \phi(p_2) e^{ip_1 \cdot b_1 + ip_2 \cdot b_2} |p_1 p_2\rangle$$

Evaluation: insert final-state identity operator:

$$\mathbb{I} = \sum\!\!\!\!\!\!\!\int |p_1 p_2 X\rangle \langle p_1 p_2 X|$$

$$\langle \text{out} | \mathbb{T} | \text{in} \rangle = \mathcal{M}(\text{in} \to \text{out})$$



Quantum mechanical construction; classical limit can be taken at any stage

classical observables $\longleftrightarrow$ expand to classical order w/ care Re: phase space

2. The observable-based formalism (KMOC formalism)

Kosower, Maybee, O'Connell

Inclusive: $\mathcal{O}$ vs. Observable

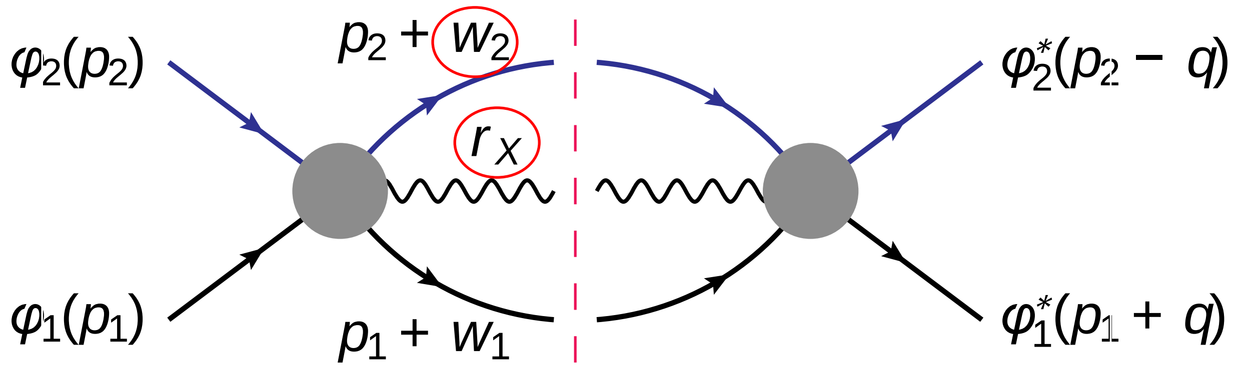
Δp of a matter particle ⟶ (momentum) impulse

Total momentum of messengers ⟶ momentum loss

Total angular momentum of messengers ⟶ angular momentum loss

through 3PM: Herrmann, Parra-Martinez, Ruf, Zeng

through 3PM: Manohar, Ridgway, Shen



The observable-based formalism (KMOC formalism)

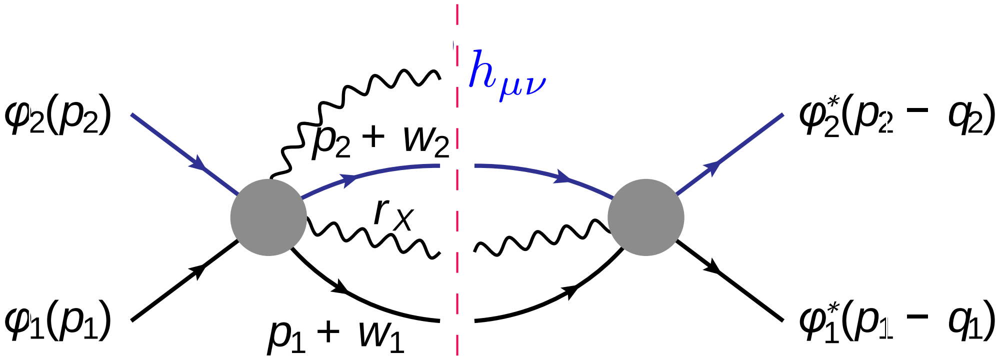
Kosower, Maybee, O'Connell

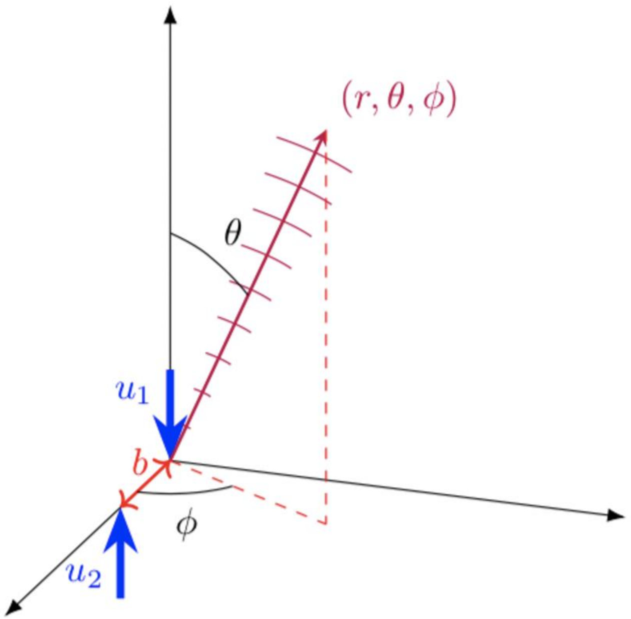
Local/differential:

Cristofoli, Gonzo, Kosower, O'Connell

asymptotic field operator $\longrightarrow$ classical field/waveform

e.g. linearized metric $\quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad W(T_R,\theta,\phi) = \frac{1}{4G}\lim_{r\to\infty} r\epsilon_+^\mu \epsilon_+^\nu h_{\mu\nu} = \frac{1}{4G}(h_+^\infty - ih_\times^\infty)$





$$\mathcal{O} = h_{\mu\nu}(x) = \sum_{h=\pm}\int \hat{\mathrm{d}}\Phi(k)\left[\varepsilon_\mu^{(h)*}(k)\varepsilon_\nu^{(h)*}(k)\,e^{-ik\cdot x}\,\hat{a}_{hh}(k) + \varepsilon_\mu^{(h)}(k)\,\varepsilon_\nu^{(h)}(k)\,e^{+ik\cdot x}\,\hat{a}_{hh}^\dagger(k)\right]$$

If no gravitational radiation in initial state: $\langle h_{\mu\nu}\rangle = -i\int d\Phi[k]\left[\tilde{J}(k)e^{-ik\cdot x} - \mathrm{cc}\right]$ $\qquad \tilde{J}(k) = i\langle\psi_{\rm in}|\mathbb{S}^\dagger a_{--}(k)\mathbb{S}|\psi_{\rm in}\rangle$

Assuming source of gravitational radiation is a localized current

$$\langle h_{\mu\nu}\rangle = \frac{-i}{4\pi^2}\int d^4y\frac{J(y)}{(x-y)^2 - i\epsilon} = \frac{-i}{4\pi^2}\int d\omega d\boldsymbol{k} dy^0 d^3\boldsymbol{y}\frac{\tilde{J}(\omega,\boldsymbol{k})e^{-i\omega y^0 + i\boldsymbol{k}\cdot\boldsymbol{y}}}{(x^0-y^0)^2 - |\boldsymbol{x}-\boldsymbol{y}|^2 - i\epsilon}$$

LO departure from Minkowski metric at infinity: $\xrightarrow{|x|\to\infty}$ $\displaystyle\frac{1}{4\pi|\boldsymbol{x}|}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}W(\omega,\boldsymbol{n},b_1,b_2,p_1,p_2,\epsilon)e^{-i\omega\tau}$: the waveform

$\tau = x^0 - |\boldsymbol{x}|$

In frequency space: $W^{\rm KMOC}(\omega,\boldsymbol{n},b_1,b_2,p_1,p_2,\epsilon) = (-i)\langle\psi_{\rm in}|\mathbb{S}^\dagger a_{--}(k)\mathbb{S}|\psi_{\rm in}\rangle\Big|_{k=(\omega,\omega\boldsymbol{n})}$

Not an IR-safe observable! However, IR divergence of the matrix element is absorbed in the definition of the retarded time.

$$W^{\rm KMOC}(\omega,\boldsymbol{n},b_1,b_2,p_1,p_2,\epsilon) \stackrel{?}{=} W^{\rm GR\ methods}(\omega,\boldsymbol{n},b_1,b_2,p_1,p_2,\epsilon)$$

$$W^{\mathrm{KMOC}}(\omega, \boldsymbol{n}, b_1, b_2, p_1, p_2, \epsilon) = (-i)\langle\psi_{\mathrm{in}}|\mathbb{S}^\dagger a_{--}(k)\mathbb{S}|\psi_{\mathrm{in}}\rangle\Big|_{k=(\omega,\omega\boldsymbol{n})}$$

$$= \int \prod_{i=1}^{2} \frac{d^d q_i}{(2\pi)^d}\hat{\delta}(2p_i \cdot q_i)e^{iq_i\cdot b_i}\Big[\langle p_1' p_2' k_{--}|\mathbb{T}|p_1 p_2\rangle - i\langle p_1' p_2'|\mathbb{T}^\dagger a_{--}(k)\mathbb{T}|p_1 p_2\rangle\Big]$$

certain IR divergences and finite parts cancel between the two terms

Surviving IR divs: drifts of worldlines

Caron-Huot, Giroux, Hannesdottir, Mizera

Amplitude

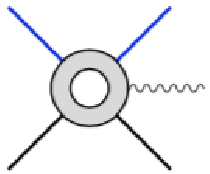
Cut

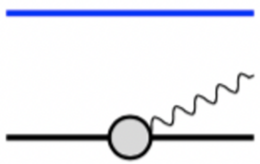
Connected

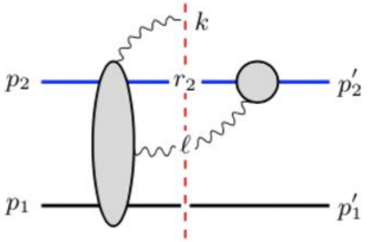
Disconnected

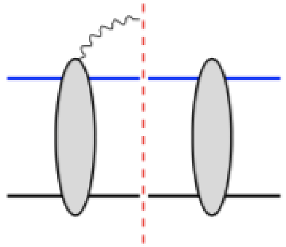
Disconnected

Connected









$$W^{\mathrm{KMOC}}(\omega, \boldsymbol{n}) = W_{\mathrm{constant}} + W^{\mathrm{tree}} + (M_{\mathrm{fin}}^{1\ \mathrm{loop}} - \mathcal{S}_{\mathrm{fin}}^{1\mathrm{loop}}) + \mathcal{S}_{\mathrm{disconnected}}^{1\mathrm{loop}}$$

- Finite terms in connected cut restore causality of connected amplitude

Caron-Huot, Giroux, Hannesdottir, Mizera

The surviving (2PMI) part of 1-loop 5-point amplitude

Quantum amplitude: Carrasco, Vazquez-Holm
Herderschee, RR, Teng; Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini;
Elkhidir, O'Connell, Sergola, Vazquez-Holm; Georgoudis, Heissenberg, Vazquez-Holm
+ linear in in spin: Bohnenblust, Ita, Kraus, Schlenk

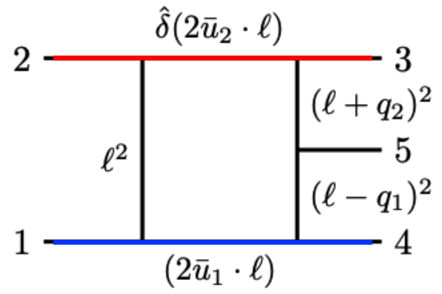
In terms of master integrals

$$M^{\text{2MPI,cl.}}_{5,m_1^2m_2^3} = c_1 I_{0,0,1,0} + c_2 I_{1,1,0,0} + c_3 I_{1,0,1,0} + c_4(I^+_{0,1,0,1} + I^-_{0,1,0,1}) + c_5(I^+_{0,0,1,1} + I^-_{0,0,1,1}) + c_6 I_{1,1,1,0}$$
$$+ c_7(I^+_{1,1,0,1} + I^-_{1,1,0,1}) + c_8(I^+_{1,0,1,1} + I^-_{1,0,1,1}) + c_9(I^+_{0,1,1,1} + I^-_{0,1,1,1}) + c_{10}(I^+_{1,1,1,1} + I^-_{1,1,1,1})$$

$$M^{\text{2MPI,cl.}}_{5,1\text{ loop}} = M^{\text{2MPI,cl.}}_{5,m_1^2m_2^3} + (\bar{u}_1 \leftrightarrow \bar{u}_2, \bar{m}_1 \leftrightarrow \bar{m}_2, q_1 \leftrightarrow q_2)$$

Integral definition:

$$I^{\pm}_{a_1,a_2,a_3,a_4} = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\hat{\delta}(2\bar{u}_2\cdot\ell)}{[\ell^2]^{a_1}[(\ell+q_2)^2]^{a_2}[(\ell-q_1)^2]^{a_3}(2\bar{u}_1\cdot\ell \pm i0)^{a_4}}$$



Some features:
- One matter line is cut
- When present, the second matter line has PV/retarded propagator
- Bubbles contribute; their coefficient provides the requisite *q* dependence
- Unlike the subtracted 4-point amplitude, it is infrared divergent

Importance and consequence emphasized by Caron-Huot, Giroux, Hannesdottir, Mizera

The connected cut part

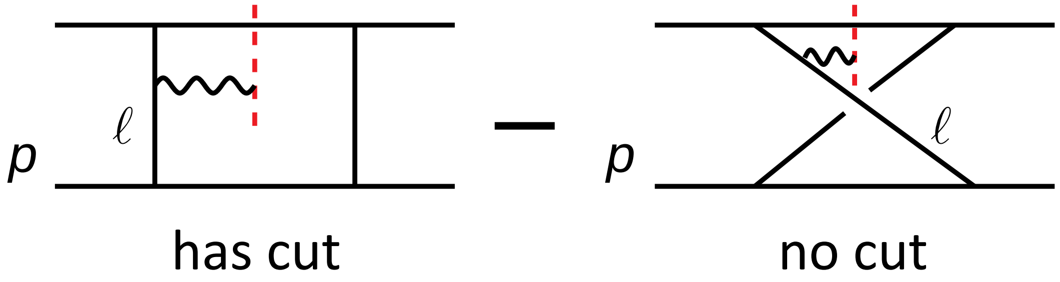
Herderschee, RR, Teng; Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini; Georgoudis, Heissenberg, Vazquez-Holm

+ linear in in spin: Bohnenblust, Ita, Kraus, Schlenk

- In the quantum theory simply take the 2-particle cut of the amplitude.
- In the classical theory extract from the PV after IBP reduction

$$\mathrm{PV}\frac{1}{2p\cdot\ell} \propto \frac{1}{2p\cdot\ell + i\epsilon} - \frac{1}{-2p\cdot\ell + i\epsilon}$$

e.g.



In terms of master integrals:

$$\begin{aligned} M^{\text{2MPI+Conn. Cut,cl.}}_{5,m_1^2 m_2^3} &= c_1 I_{0,0,1,0} + c_2 I_{1,1,0,0} + c_3 I_{1,0,1,0} + c_4\left(\cancel{I^+_{0,1,0,1}} + I^-_{0,1,0,1}\right) + c_5\left(\cancel{I^+_{0,0,1,1}} + I^-_{0,0,1,1}\right) + c_6 I_{1,1,1,0} \\ &+ c_7\left(\cancel{I^+_{1,1,0,1}} + I^-_{1,1,0,1}\right) + c_8\left(\cancel{I^+_{1,0,1,1}} + I^-_{1,0,1,1}\right) + c_9\left(\cancel{I^+_{0,1,1,1}} + I^-_{0,1,1,1}\right) + c_{10}\left(\cancel{I^+_{1,1,1,1}} + I^-_{1,1,1,1}\right) \end{aligned}$$

$$M^{\text{2MPI+Conn. Cut,cl.}}_{5,1\ \text{loop}} = M^{\text{2MPI+Conn. Cut,cl.}}_{5,m_1^2 m_2^3} + (\bar{u}_1 \leftrightarrow \bar{u}_2, \bar{m}_1 \leftrightarrow \bar{m}_2, q_1 \leftrightarrow q_2)$$

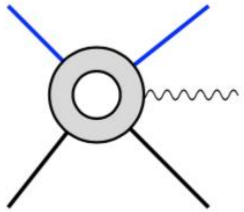
The conclusion is that all matter propagators become causal

Caron-Huot, Giroux, Hannesdottir, Mizera

Putting together the connected amplitude and cut

Bini, Damour, Di Angelis, Geralico, Herderschee, RR, Teng; Georgoudis, Heissenberg, Russo

+ linear in in spin: Bohnenblust, Ita, Kraus, Schlenk



$$\mathcal{M}^{1\text{ loop}} = -\frac{i\,\kappa^2}{32\pi}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{\mu^2}\right]\mathcal{M}^{\text{tree}}_{d=4}$$

$$+\,\kappa^4\left[A^R_{\text{rat}} + \frac{A^R_1}{\sqrt{w_2^2 - q_1^2}} + \frac{A^R_2}{\sqrt{w_1^2 - q_2^2}} + \frac{A^R_3}{\sqrt{-q_1^2}} + \frac{A^R_4}{\sqrt{-q_2^2}}\right]$$

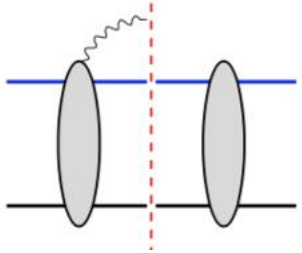
$$+\,i\,\kappa^4\left[A^I_{\text{rat}} + A^I_1\frac{\operatorname{arcsinh}\frac{w_2}{\sqrt{-q_1^2}}}{\sqrt{w_2^2-q_1^2}} + A^I_2\frac{\operatorname{arcsinh}\frac{w_1}{\sqrt{-q_2^2}}}{\sqrt{w_1^2-q_2^2}} + A^I_3\log\frac{q_2^2}{q_1^2} + A^I_4\log\frac{w_1}{w_2} + A^I_5\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}}\right]$$

$$= -\frac{i\,\kappa^2}{32\pi\epsilon}(m_1 w_1 + m_2 w_2)\mathcal{M}^{\text{tree}}_{d=4} + \widetilde{\mathcal{M}}^{\text{fin}}_{1\text{ loop}}$$

$$\mathcal{S}^{1\text{ loop}} = \frac{i\,\kappa^2\Gamma}{64\pi}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{(\sigma^2-1)\mu^2}\right]\mathcal{M}^{\text{tree}}_{d=4} \qquad \Gamma = \frac{3\sigma - 2\sigma^3}{(\sigma^2-1)^{3/2}}$$

$$+\,\frac{i\,\kappa^4}{256\pi}\frac{(2\sigma^2-1)^2}{(\sigma^2-1)^{3/2}}(m_1 w_1 + m_2 w_2)\left[\frac{1}{\epsilon} - \log\frac{w_1 w_2}{(\sigma^2-1)\mu^2}\right]\frac{m_1^2 m_2^2 (u_1\cdot f\cdot u_2)^2(w_1^2 + \sigma w_1 w_2 + w_2^2)}{w_1^3 w_2^3}$$

$$+\,i\,\kappa^4\left[A^{\text{cut}}_{\text{rat}} + A^{\text{cut}}_1\log\frac{w_1}{w_2} + A^{\text{cut}}_2\log\frac{w_1 w_2}{-q_1^2} + A^{\text{cut}}_3\log\frac{w_1 w_2}{-q_2^2} + A^{\text{cut}}_4\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}}\right]$$



$$= \frac{i\,\kappa^2\Gamma}{64\pi\epsilon}(m_1 w_1 + m_2 w_2)\mathcal{M}^{\text{tree}}_{d=4} + \widetilde{\mathcal{S}}^{\text{fin}}_{1\text{ loop}} + \mathcal{S}^{\text{local}}_{1\text{ loop}}$$

Putting together the connected amplitude and cut and a surprise

Bini, Damour, Di Angelis, Geralico, Herderschee, RR, Teng; Georgoudis, Heissenberg, Russo

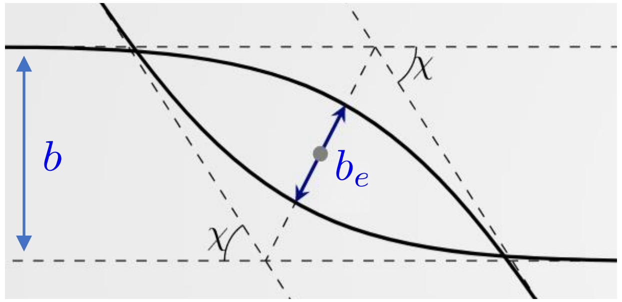
+linear in in spin: Bohnenblust, Ita, Kraus, Schlenk

$$M^{\text{2MPI+Conn. Cut,cl.}}_{5,m_1^2m_2^3} = c_1 I_{0,0,1,0} + c_2 I_{1,1,0,0} + c_3 I_{1,0,1,0} + c_4(\cancel{I^+_{0,1,0,1}} + I^-_{0,1,0,1}) + c_5(\cancel{I^+_{0,0,1,1}} + I^-_{0,0,1,1}) + c_6 I_{1,1,1,0}$$
$$+ c_7(\cancel{I^+_{1,1,0,1}} + I^-_{1,1,0,1}) + c_8(\cancel{I^+_{1,0,1,1}} + I^-_{1,0,1,1}) + c_9(\cancel{I^+_{0,1,1,1}} + I^-_{0,1,1,1}) + c_{10}(\cancel{I^+_{1,1,1,1}} + I^-_{1,1,1,1})$$

$$M^{\text{2MPI+Conn. Cut,cl.}}_{5,1\text{ loop}} = M^{\text{2MPI+Conn. Cut,cl.}}_{5,m_1^2m_2^3} + (\bar{u}_1 \leftrightarrow \bar{u}_2, \bar{m}_1 \leftrightarrow \bar{m}_2, q_1 \leftrightarrow q_2)$$

At 1 loop, connected cut $=$ rotation of connected amplitude; does it hold beyond 1 loop?

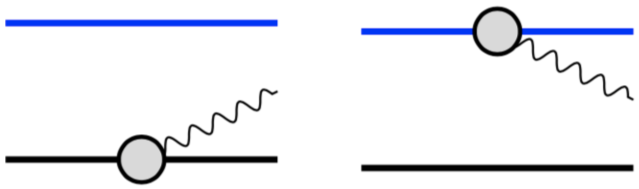
$$(\text{Conn. Amplitude} + \text{Conn. Cut})(\phi) = (\text{Conn. Amplitude})(\phi + \chi/2) + \delta\tau W^{\text{tree}}$$



Disconnected contributions:

Bini, Damour, Di Angelis, Geralico, Herderschee, RR, Teng

Amplitude:



$$\mathcal{M}_3(p_1,k) = 2\delta(2p_1\cdot k)(\varepsilon(k)\cdot p_1)^2 = \delta(\omega) m_1 \frac{(\varepsilon(k)\cdot u_1)^2}{n\cdot u_1}$$

$$W_{\rm const}(\omega,\boldsymbol{n}) = \int \Phi[p_1]\mathcal{M}_3(p_1,k)e^{ik\cdot b_1} + \int \Phi[p_2]\mathcal{M}_3(p_2,k)e^{ik\cdot b_2} = \delta(\omega)\left[m_1\frac{(\varepsilon(k)\cdot u_1)^2}{n\cdot u_1} + m_2\frac{(\varepsilon(k)\cdot u_2)^2}{n\cdot u_2}\right]$$

- Localized on gravitons with zero frequency $\longrightarrow$ time independent $\quad k=\omega n \quad n=(1,\boldsymbol{n})$
- Also the asymptotic metric for two Schwarzschild black holes moving with velocities u_1 and u_2
- Can be modified by large gauge transformations (BMS transformations) Veneziano, Vilkovisky

$$\delta_T\left[\frac{h^\infty_{ab}}{r}\right] = \frac{1}{r}(2D_aD_b - \gamma_{ab}\Delta)T(n) - T(n)\partial_\tau\left[\frac{h^\infty_{ab}}{r}\right]$$

- Set $W_{\rm const}=0$ by choosing $\quad T(n) = 2G\,[m_1 n\cdot u_1 \ln n\cdot u_1 + m_2 n\cdot u_2 \ln n\cdot u_2]$

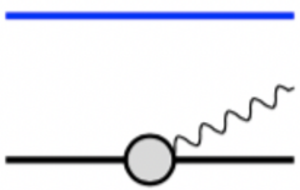
This is referred to as the canonical BMS frame

$W_{\rm const}\neq 0$ is referred to as the intrinsic BMS frame

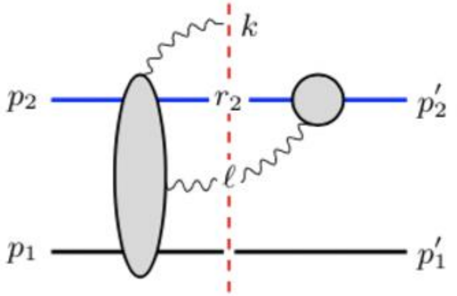
What is the amplitudes way of changing the BMS frame?

Unitarity: disconnected amplitude(s) contribute to disconnected cut!

$$\mathcal{M}_{\text{disc}} = i\langle p_1' p_2' | \mathbb{T}^\dagger a_{--}(k) \mathbb{T} | p_1 p_2 \rangle \Big|_{\text{disc}}$$



$$\longrightarrow \int d\Phi[\ell] d\Phi[r_2]$$

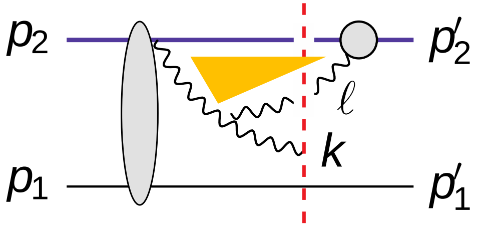


Only zero-energy support $\longrightarrow$ soft limit of the 6-point amp.

$$\mathcal{M}_6 \simeq \kappa \mathcal{M}_5 \left[\sum_a \frac{\eta_a \varepsilon_{\mu\nu}(\ell) p_a^\mu p_a^\nu}{2\ell \cdot p_a + i\eta_a \epsilon} + \frac{\varepsilon_{\mu\nu}(\ell) k^\mu k^\nu}{2\ell \cdot k + i\epsilon} \right]$$

$$\mathcal{M}_{\text{disc}} = -i\kappa^2 \mathcal{M}_5^{\text{tree}} \left[m_1^2 (u_1 \cdot k)^2 I(u_1, k) + m_2^2 (u_2 \cdot k)^2 I(u_2, k) \right]$$

Integrals require additional regularization:



$$2u_2 \cdot \ell = (u_2 + \ell)^2 - 1 \to ((u_2 - \beta_n)^2 - 1 \simeq 2u_2 \cdot \ell - 2\beta u_2 \cdot n$$

$$I(u_2, k) = \frac{1}{m_2 \omega} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_2 \cdot \ell - 2\beta u_2 \cdot n) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n} = \frac{1}{32\pi m_2 (u_2 \cdot n)} \left[\frac{1}{\epsilon} - \ln u_2 \cdot n - \ln \frac{\beta^2}{\pi} \right]$$

IR divergence and regulator β are absorbed into shift of the retarded time.

$$\mathcal{M}_{\text{disc}} = -2i\omega G \mathcal{M}_5^{\text{tree}} \left[m_1 (u_1 \cdot n) \ln(u_1 \cdot n) + m_2 (u_2 \cdot n) \ln(u_2 \cdot n) \right] = -i\omega T(n) \mathcal{M}_5^{\text{tree}}$$

Summary and comparison with GR results

Bini, Damour, De Angelis, Geralico, Herderschee, RR, Teng

KMOC/QFT-based waveform to 1 loop: $W^{\mathrm{KMOC}} = W_{\mathrm{const}} + W^{\mathrm{tree}} + \mathrm{FT}\left[\mathcal{M}^{\mathrm{fin}}_{1\ \mathrm{loop}} - \mathcal{S}^{\mathrm{fin}}_{1\ \mathrm{loop}} + \mathcal{M}_{\mathrm{disc}}\right] + \text{higher loops}$

- Analytic results agree with available GR calculations of Bini, Damour and Geralico (through 2.5PN)

$$W^{\mathrm{KMOC}} \sim 1 + \frac{GM^2\nu}{p_\infty}\left(1 + p_\infty + \cdots + p_\infty^5 + \ldots\right) + \frac{GM^2\nu}{p_\infty}\frac{GM}{bp_\infty^2}\left(1 + p_\infty + \cdots + p_\infty^5 + \ldots\right)$$

$$M = m_1 + m_2 \qquad \nu = m_1 m_2/M^2 \qquad p_\infty = \sqrt{\sigma^2 - 1}$$

- Analytic results in the soft ω expansion agree with results of Sahoo and Sen

$$W^{\mathrm{KMOC}} \sim \frac{\mathcal{A}}{\omega} + \mathcal{B}\ln\omega + \mathcal{C}\omega(\ln\omega)^2 + \mathcal{D}\omega\ln\omega + \ldots$$

- BMS frame is important: agreement w/ GR calculations of Bini, Damour, Geralico requires the intrinsic frame. Consistently change BMS frame in QFT?

- Connected part of the cut is equivalent to $\phi \to \phi + \chi/2$ in amplitude; how far does it go?
  1-loop analytic argument by Georgoudis, Heissenberg, Russo

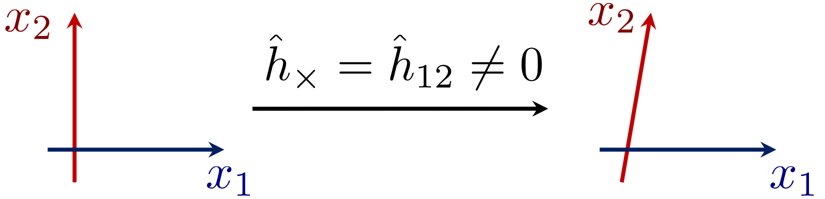
- Numerical transform to time domain; recent analytic progress by Brunello, De Angelis
  + reinterpret transform as additional loop(s) and use standard IBP & DE methods

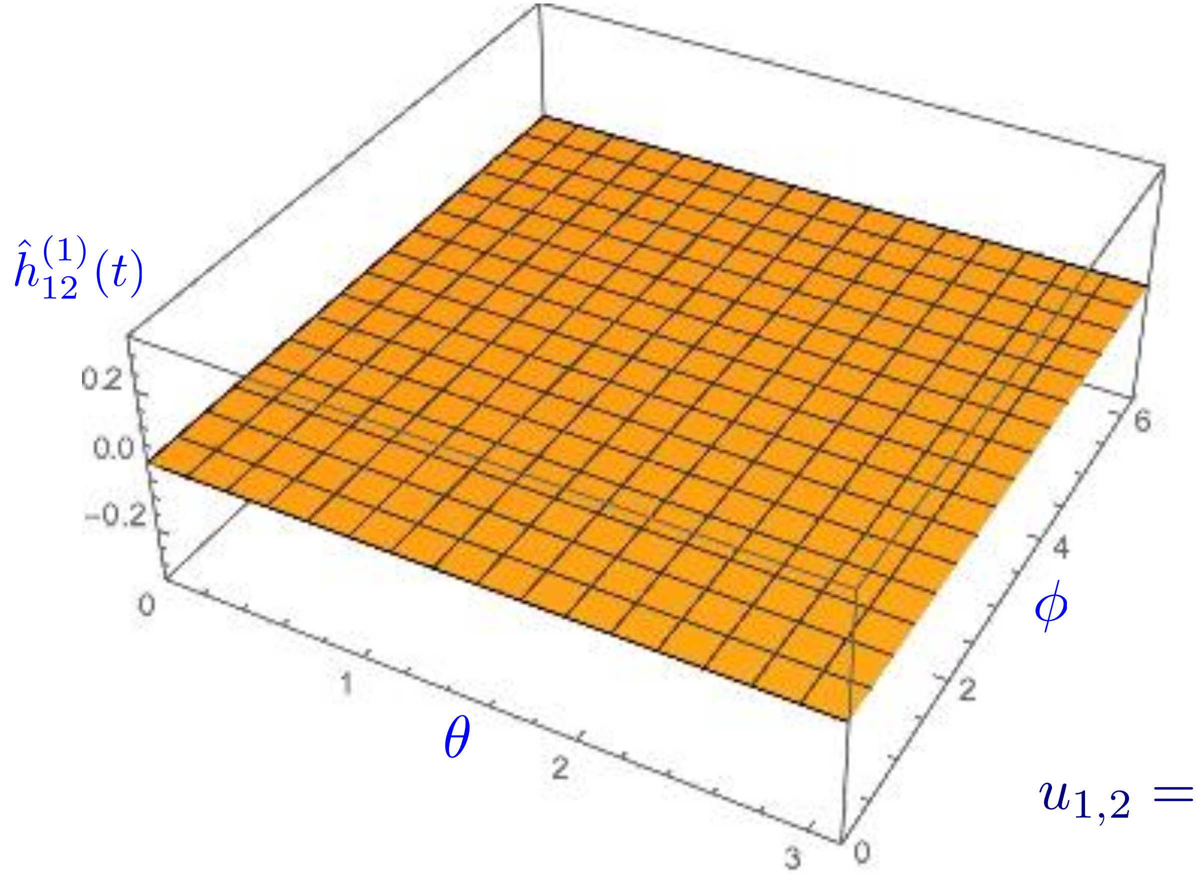
What would we "see" at a fancy gravitational wave detector

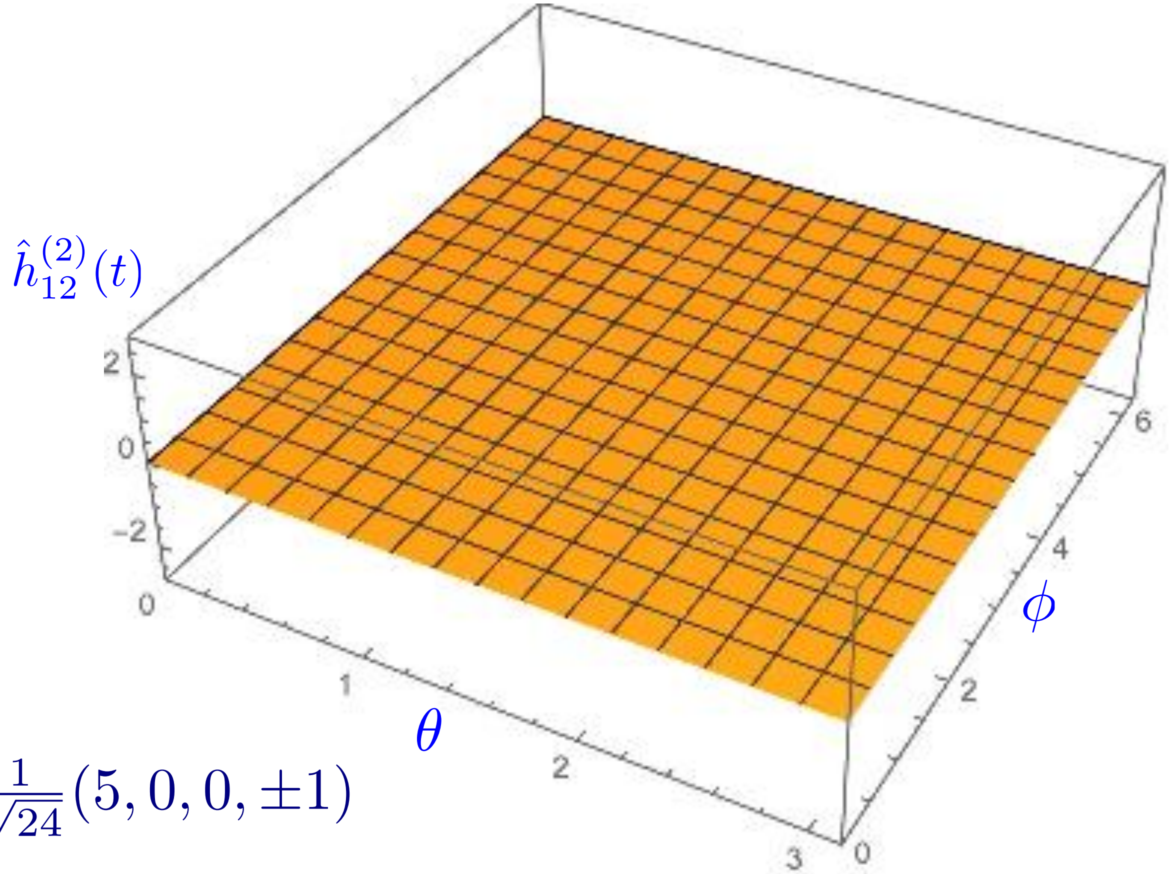
Herderschee, RR, Teng

LO also by Jakobsen, Mogull, Plefka by different means

$$g_{\mu\nu}\Big|_{|\boldsymbol{x}|\to\infty} = \eta_{\mu\nu} + \frac{\kappa^2 \mathrm{M}}{8\pi|\boldsymbol{x}|}\left[\frac{\kappa^2\mathrm{M}}{\sqrt{-b^2}}\hat{h}^{(1)}_{\mu\nu} + \left(\frac{\kappa^2\mathrm{M}}{\sqrt{-b^2}}\right)^2 \hat{h}^{(2)}_{\mu\nu} + \dots\right]$$







$$u_{1,2} = \frac{1}{\sqrt{24}}(5,0,0,\pm 1)$$

$$\hat{h}_+ = \frac{1}{2}(\hat{h}_{11} - \hat{h}_{22})$$
$$\hat{h}_\times = \hat{h}_{12}$$

Direct probe of gravitational scattering with possible observations

Tools that enabled state of the art calculations

Developing (computational) tools

Effective field theories: - positive energy

- relativistic, spin-dependent

- heavy mass

Cheung, Rothstein, Solon;
Bern, Cheung, RR, Shen, Solon, Zeng
Bern, RR, Shen, Zeng; + Kosmopoulos, Teng
Brandhuber, Chen, Travaglini, Wen; Damgaard, Haddad, Helset

Observable-based formalism: - inclusive observables

- differential observables

Kosower, Maybee, O'Connell
Cristofoli, Gonzo, Kosower, O'Connell

Generating functions for observables: - radial action

- eikonal

- exponential rep. of *S* matrix

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng
Amati, Ciafaloni, Veneziano;
di Vecchia, Heissenberg, Russo, Veneziano
Damgaard, Planté, Vanhove

Amplitude building blocks & techniques in the classical limit

- Generalized unitarity + tree-level double copy + generalized gauge symmetry
- Loop-level double copy + dilaton projection

Carrasco, Vazquez-Holm

- New amplitudes for higher-spin particles/minimal amplitudes/…

Arkani-Hamed, Huang, O'Connell
Chiodaroli, Johansson, Pichini

- New amplitudes from classical scattering

Bautista, Guevara, Kavanagh, Vines

Developing (computational) tools

2d QFTs (worldline): NRGR, +spin, PM EFT, WQT

Goldberger, Rothstein; Levi Steinhoff; Kälin, Porto; Mogull, Plefka, Steinjoff; +Jakbosen;

new tools: in-in formalism in the PM EFT and WQFT

Dlapa, Kälin, Liu, Porto; Jakobsen, Mogull, Plefka, Sauer

Multiloop integration technology: - automated programs for IBP

Anastasiou, Lazaropoulos; Smirnov, Smirnov; Maierhöffer, Usovitsch, Uwer; Studerus; Lee; Mandal, Mastrolia, Patil; etc

- differential equations

Bern, Dixon, Kosower; Henn, Smirnov

Synergy with traditional GR approached to the two-body problem

- Exploration of simpler theories
  - QED

    De la Cruz, Maybee, O'Connell, Ross; Bern, Gatica, Herrmann, Luna, Zeng Saketh, Vines, Steinhoff, Buonanno; Bern, Herrmann, RR, Ruf, 2xSmirnov, Zeng

  - a charged scalar model -- SF + PM extension

    Barack, Bern, Herrmann, Long, Parra-Martinez, RR, Ruf, Shen, Solon, Teng, Zeng

- Boundary-to-bound map

  Kälin, Liu, Porto

- Good mass polynomiality, developed into the Tutti-Frutti method

  Vines, Steinhoff, Buonanno; Damour

- EOB – PM resummation

  Khalil, Buonanno, Steinhoff, Vines; Damgaard, Vanhove; Damour, Retegno; Buonanno, Jakobsen, Mogull

Outlook

- Scattering offers a different perspective on gravity, is a clean regime and we have powerful tools: we should use them to assist our GR friends, push the state of the art and to explore the subtleties awaiting in the bound regime
- Relate bound-state and scattering observables beyond the reach of direct analytic continuation?
  - including bypass use of Hamiltonian and radiation reaction forces
  - can we learn from bound state analysis in QCD?
- Structure of high order gravitational perturbation theory and resummation
  - QCD-style resummation?
  - Interface with gravitational self-force/QFT in nontrivial asymptotically-flat backgrounds
  - Assist with EOB-style resummation
- Are the other particle-physics-inspired observables that are useful for GW physics?
- Countless interesting questions regarding physics of spinning and finite-size bodies

There is great momentum in the community; we should expect renewed progress in the future

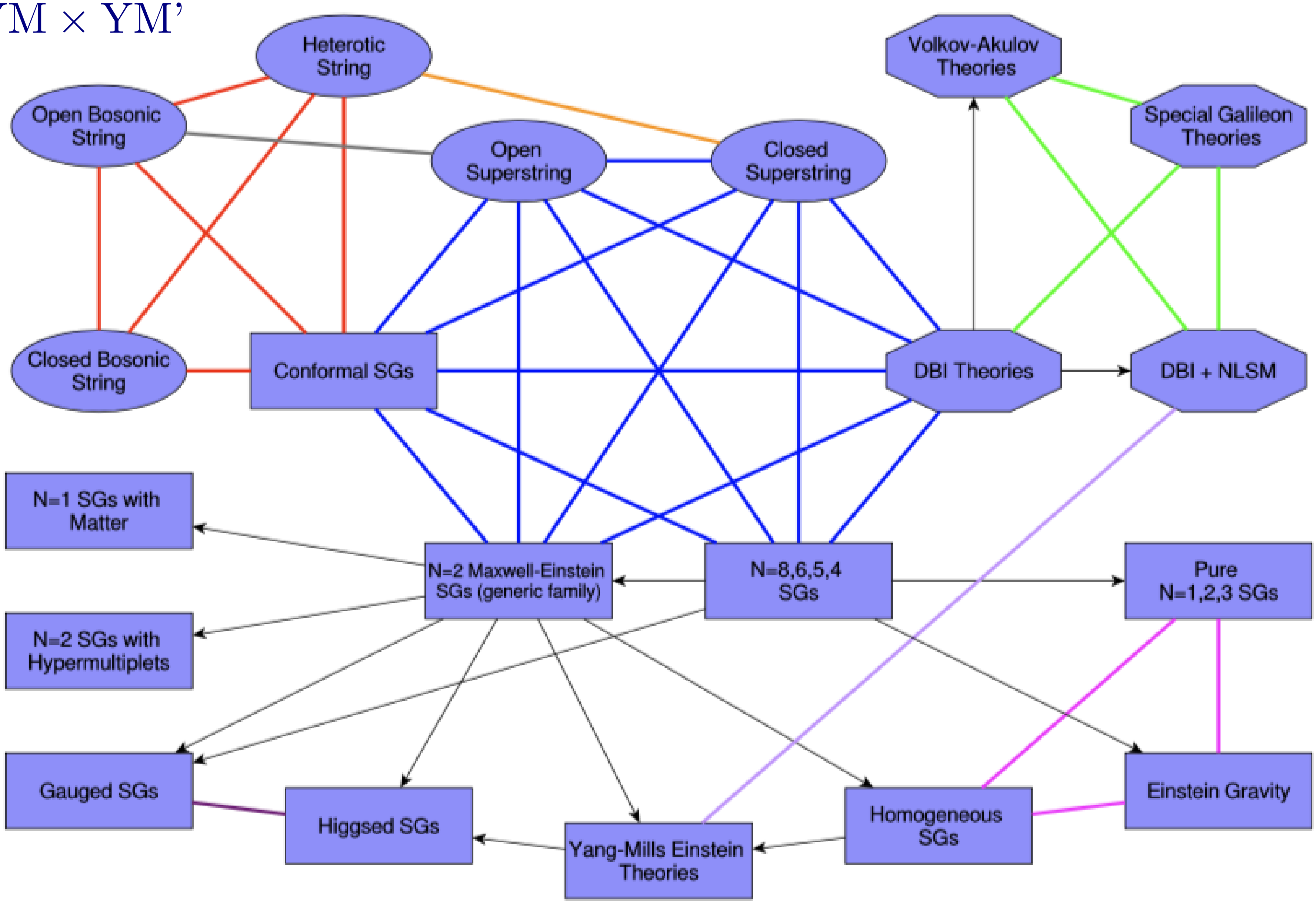
Computational methods developed for GW applications feed back to quantum gravity questions!

EXTRA

Web of double-copy-constructible theories

Bern, Carrasco, Chiodaroli, Johansson, RR

$$\text{Gravity} = \text{YM} \times \text{YM'}$$



Double copy links various gravitational (and nongravitational) theories through their building blocks

Theoretical structures relevant in the classical limit

Theoretical structures relevant in the classical limit

- High energy limit – exposes interplay between inclusive and exclusive observables, soft graviton theorems and universality of gravitational interactions

  e.g. absence of collinear/mass singularities in inclusive observables

Block, Nordsiek;
Kinoshita, Lee, Nauenberg

$$\mathcal{M} \to -8\pi G^3 s^2 \log(-t) \log\left(\frac{m_1 m_2}{s}\right)$$

soft graviton theorem: Di Vecchia, Heissenberg, Russo, Veneziano

linear response: Damour

More information from 4 and 5-point amplitudes in simplifying limits?

- Nonperturbative structures: (1) exponentiation of amplitudes

  - Eikonal exponentiation with and without spin; relation to observables: $i\mathcal{M} = e^{i\delta} - 1$

$$\Delta\mathcal{O} = e^{-i\delta\mathcal{D}}[\mathcal{O}, e^{i\delta\mathcal{D}}] \qquad \delta\mathcal{D}g \equiv \delta g + \mathcal{D}_{SL}(\delta, g) \qquad \mathcal{D}_{SL}(\delta, g) \equiv -\sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial \delta}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

Di Vecchia, Heissenberg, Russo, Veneziano; Damgaard, Plante, Vanhove;
Bern, Ita, Parra-Martinez, Ruf; Bern, Luna, RR, Shen, Zeng

  - Amplitude-(radial) action relation: $i\mathcal{M} = e^{iI_r} - 1 \qquad dI_r = \frac{\chi}{2\pi} dJ + \tau dE$

Bern, Parra-Martinez, RR, Ruf, Shen, Solon, Zeng

curved space: explicit demonstration in probe approx. to all orders in *G* — Kol, O'Connell, Telem

- Nonperturbative structures: (2) "impetus formula" (direct connection trajectory $\longleftrightarrow$ amplitude)

$$\mathcal{M}(r) = \frac{p(r)^2 - p(\infty)^2}{2E}$$

observed in Bern, Cheung, RR, Shen, Solon, Zeng

formalized by Kälin, Porto

e.g. derive (finite parts of) amplitudes from classical motion

(3) Newman-Janis shift $\phi_{\mathrm{Kerr}}(z) = \phi_{\mathrm{Schw}}(z + ia)$

Relation between Schwarschild and Kerr solutions through a complex shift

Used for leading order calculations impulse calculation — Arkani-Hamed, Huang, O'Connell

Other uses? (e.g. intriguing w.s. for Kerr) — Guevara, Maybee, Ochirov, O'Connell, Vines

- Analytic continuation and time non-locality

- Newtonian mechanics: one Hamiltonian determines both bound and unbound motion

- GR: suitable analytic continuation yields bound observables from unbound ones

"Boundary to Bound" or "B2B", e.g. $\Delta\Phi(\mathcal{E}, J) = \Delta\chi(\mathcal{E}, J) + \Delta\chi(\mathcal{E}, -J) \quad \mathcal{E} < 0$ — Kälin, Porto

Applicable for instantaneous + universal (log) nonlocal in time parts of *H* — Cho, Kälin, Porto

and not for the rest of nonlocal *H* — Damour, Jaranowski, Schäfer

- Relations between amplitude fragments

Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White

- Relation to gravitational self-force: mass dep. of classical amplitude and angles (up to factor)

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \qquad \mathcal{M} \sim 1 \oplus \nu \oplus \cdots \oplus \nu^{[L/2]}$$

"good mass polynomiality rule"

observed at 3PM by Vines, Steinhoff, Buonanno

thorough understanding provided by Damour

- Search for structure in simpler theories:

  QED

  De la Cruz, Maybee, O'Connell, Ross; Bern, Gatica, Herrmann, Luna, Zeng
  Saketh, Vines, Steinhoff, Buonanno

  a charged scalar model: SF + PM aim to extend both beyond respective validity regime

  Barack, Bern, Herrmann, Long, Parra-Martinez, RR, Ruf, Shen, Solon, Teng, Zeng