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Abelian Instantons and Monopole Scattering

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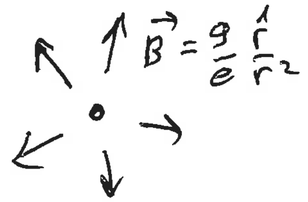
Ofri Telem – The Hebrew University of Jerusalem
w/ C. Csáki, R. Ovadia, J. Terning, and S. Yankielowicz

Definition: $mQED_4$: QED_4 with N_f Dirac fermion, in the background of a static monopole.

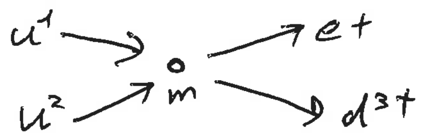


$A_\mu, \bar{\Psi}_f, \Psi_f$

Take Home: $mQED_4$ surprisingly rich

- Topological sectors with abelian instantons
- Abelian instantons ⟶ Callan Rubakov

  e.g.



  Saturating s-wave unitarity bound.

Plan

- Formulation of the problem
- Abelian Instantons
- Fermions in the background of an Abelian Instanton
  [redacted]
  * zero modes
  * 't Hooft vertices
- Exact calculation of Path Integral (PI) for the s-waves of $mQED_4$
- Abelian Instantons ⟶ Callan-Rubakov

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Formulation of the problem

Consider N_f Dirac fermions in the background of a Dirac monopole - A_μ^{mon}

$A_t^{mon} = 0$

$\vec{A}^{mon} = \frac{q}{e} \frac{1-\cos\theta}{r\sin\theta} \hat{\varphi}$

$$\tilde{\Psi}_f = \begin{pmatrix} \chi_f \\ \eta_f^\dagger \end{pmatrix}$$

χ_f η_f LH Weyl fermions of charges $Q = \pm 1$ in units of e.

Let's solve the Dirac equation

$$\gamma^\mu (\partial_\mu - ie A_\mu^{mon}) \Psi_f = 0$$

Catch: angular momentum operator J deformed by A_μ^{mon}

$$\vec{J} = \vec{r} \times (\vec{P} - e\vec{A}^{mon}) - q\hat{r} + \vec{S}$$

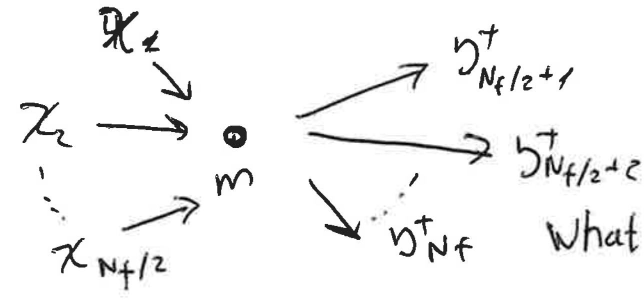
The lowest PW has $j_{min} = |q| - \frac{1}{2}$ and it's wavefunction is

$$\tilde{\Psi}_f = \begin{pmatrix} f_L(t+r)\, \Omega^{(3)}_{q\, j_{min}, m}(\theta, \varphi) \\ f_R(t-r)\, \Omega^{(3)}_{q\, j_{min}, m}(\theta, \varphi) \end{pmatrix}$$

$J^2[\Omega^3] = j_{min}(j_{min}+1)\Omega^3$ monopole spinor harmonic

What does this mean? $\chi_f \sim f_L(t+r)$ can only be incoming. $\eta_f^\dagger \sim f_R(t-r)$ can only be outgoing. Only allowed Process respectin $SU(N_f)$:

WLOG:



Chirality & flavor violation 100% Probability

What is the QFT mechanism?

Reminder: SU(2) Instantons

SU(2) field configuration that wind around Euclidean spacetime infinity:

e.g. BPST instanton $\lim_{|X| \to \infty} A_\mu(x) = g_1^{-1}(x) \partial_\mu g_1(x)$

$$g_1(x) = \frac{1}{r} x_\mu \sigma^\mu \in SU(2)$$

As we go around the s^3 at spacetime infinity, $g_1(x)$ goes 1 full round

Other (multi-instanton) configurations have $g_n(x)$ going n full rounds

$\Rightarrow$ Instantons characterized topologically by $\pi_3[SU(2)] = \mathbb{Z}$

Instanton number: $Ch_2 \equiv n = -\frac{e^2}{8\pi^2} \int \mathrm{tr}\, F \wedge F = \frac{e^2}{16\pi^2} \int \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} d^4x$

a topological invariant

Abelian Instantons?

$g(x) \in U(1)$ cannot wind around 4D spacetime infinity! In other words: $\pi_3[U(1)] = \emptyset$.

Secret ingredient: magnetic monopoles!

- Generate a background field

$$A_t^{mon} = 0 \quad \vec{A}^{mon} = \frac{g}{e} \frac{1-\cos\theta}{r\sin\theta} \hat{\varphi} \Longrightarrow \vec{B} = \frac{g}{e} \frac{\hat{r}}{r^2}$$

- Impose boudary conditions at their position

Boundary condition:

$$A_t(r=0)=0 \qquad \tilde{\Psi}_f = e^{-i\varphi(t)}\Psi_f$$

uniquely determined by 't Hooft-Polyakov UV completion

We will take $\varphi(t)\to 0$,
it drops out of all our calculations

Abelian Instantons

Definition: an Abelian Instanton in $mQED_4$ is a field configuration:

$$A_\mu^{tot} = A_\mu^{mon} + A_\mu^{vor}$$

Note: from now on - Euclidean space

So that

$$A_t^{mon}=0 \qquad \vec{A}^{mon} = \frac{g}{e}\frac{1-\cos\theta}{r\sin\theta}\hat{\varphi} \to \vec{B}=\frac{g}{e}\frac{\hat{r}}{r^2}$$

and

$$A_t^{vor} = \frac{1}{\sqrt{4\pi R^2}}a_t(t,r) \qquad \vec{A}^{vor} = \frac{1}{\sqrt{4\pi R^2}}a_r(t,r)\hat{r}.$$

R → arbitrary distance scale

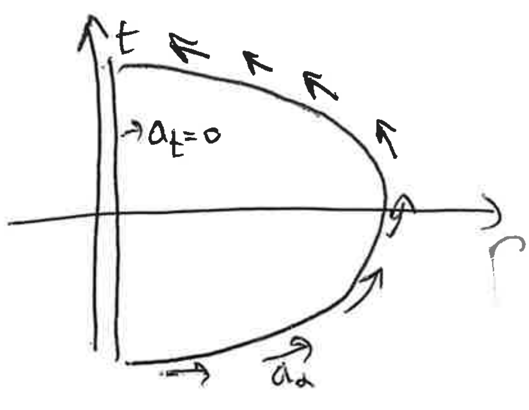
Moreover, $a_t(r=0)=0$, and

$$\lim_{\sqrt{t^2+r^2}\to\infty} a_\alpha = -\frac{n}{e_{2D}}\frac{\epsilon_{\alpha\beta}x^\beta}{x^2}$$

$\alpha\in\{t,r\}$
$e_{2D}=e/\sqrt{4\pi R^2}$
$\epsilon_{\alpha\beta}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

a_α is a ~~fractional~~ half-integer vortex

$$\frac{e_{2D}}{2\pi}\oint a_\alpha dx^\alpha = \frac{n}{2}$$



Intuition: A_μ^{mon} creates "winding" in (θ,φ)

A_μ^{vor} creates "winding" in (t,r)

They combine to form an abelian instanton

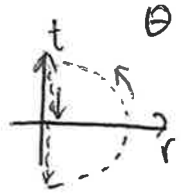
4D (hyper-) spherical coordinates: $\lambda = \sqrt{t^2+r^2}$

$\kappa = \arctan(r/t)$

θ,φ as usual

$A \equiv i A_\mu dx^\mu$, $F = dA = \frac{i}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

$$\lim_{\lambda\to\infty} A^{vor} = i\frac{n}{e} d\kappa$$



$$\lim_{\lambda\to\infty} F^{mon} = i\frac{2q}{e}\sin\theta\, d\theta\wedge d\varphi$$

Instanton number - two ways to calculate

- $$\underset{\text{inst. number}}{Ch_2} = -\frac{e^2}{8\pi^2}\int_{\substack{S_3 \\ \text{at } \lambda\to\infty}} A\wedge F = qn$$

- $$\underset{\text{inst. number}}{Ch_2} = \frac{e^2}{16\pi^2}\int_{R^4} d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} =$$

$$= \frac{1}{2}\int_{R^4} \underbrace{\left(\frac{ie}{2\pi}\frac{iF_{\mu\nu}}{2}dx^\mu dx^\nu\right)}_{ch_1}\wedge\underbrace{\left(\frac{ie}{2\pi}\frac{iF_{\mu\nu}}{2}dx^\mu dx^\nu\right)}_{ch_1}$$

$$= \underbrace{\left(\frac{e}{2\pi}\int d\Omega\, r^2 \vec{B}^{mon}\right)}_{Ch_1(\theta,\varphi)}\cdot\underbrace{\left(\frac{e}{2\pi}\int dt\,dr\, \vec{E}^{vor}\right)}_{Ch_1(t,r)}$$

$$= (2q)\cdot(n/2)$$

2D winding number in (θ,φ) | 2D winding number in (t,r)

Half-integer!

(we are on a 2D half-plane $(t, r\geq 0)$ with $a_t(r=0)=0$)

What EOM do they solve?

Later: Maxwell + Schwinger mass, in the presence of sources.

Fermions in the background of an Abelian Instanton

Dirac eq. $\quad \not{D}\psi = 0 \qquad \bar{\psi}\overleftarrow{\not{D}} = 0$

B.C. $\qquad \bar{\psi} = \psi\big|_{r=0}$

$$\not{D} \equiv \gamma^\mu\left(\partial_\mu - ieA_\mu^{mon} - ieA_\mu^{vor}\right)$$

Warm-up:

$$\not{D}_{mon}\tilde{\psi} = 0 \qquad \bar{\tilde{\psi}}\,\not{D}_{mon} = 0$$

$$\bar{\tilde{\psi}} = \tilde{\psi}\big|_{r=0}$$

$$\not{D}_{mon} = \gamma^\mu\left(\partial_\mu - ieA_\mu^{mon}\right)$$

Focus on lowest partial wave with $j = j_{min} \equiv |q| - \frac{1}{2}$:

$$-j_{min} \leq m \leq j_{min} \qquad \begin{pmatrix} \tilde{\psi}_m \\ \bar{\tilde{\psi}}_m \end{pmatrix} = \sqrt{\frac{M}{2\pi}}\,\frac{1}{r}\begin{pmatrix} \begin{pmatrix} f_L(t+ir)\,\Omega^3_{q,m}(\theta,\varphi) \\ f_R(t-ir)\,\Omega^3_{q,m}(\theta,\varphi)\end{pmatrix} \\ \begin{pmatrix} f_L(t-ir)\,\Omega^3_{-q,m}(\theta,\varphi) \\ f_R(t+ir)\,\Omega^3_{-q,m}(\theta,\varphi)\end{pmatrix}\end{pmatrix}$$

Full solution:

$$-j_{min} \leq m \leq j_{min} \qquad \begin{pmatrix} \psi_m \\ \bar{\psi}_m \end{pmatrix} = \exp\left(-e\gamma_5\partial^{-2}E_r(t,r)\right)\begin{pmatrix} \tilde{\psi}_m \\ \bar{\tilde{\psi}}_m \end{pmatrix}$$

$\partial^{-2} \to$ Inverse Laplacian in (t,r), $\quad E_r = \partial_t A_r^{vor} - \partial_r A_t^{vor}$

$$\partial^{-2}E_r(t,r) = \int dt'dr'\,D(t,r;t',r')E_r(t',r')$$

$$\partial^2 D(t,r;t',r') = \delta(t-t')\delta(r-r')$$

$$D(t,r;t',r') = -\frac{1}{4\pi}\log\left(\frac{1}{M^2(\Delta t^2 + \Delta r^2)}\right)$$

Requirement: log-normalizability

$$\int d^4x\, |\Psi| \quad \text{at most log-divergent}$$

$\Rightarrow$ Can regulate by putting in (t,r) "box" in a regulator independent way.

Noting $\sqrt{t^2+r^2} \to \infty$ behavior of E_r for the vortex, $2|qn|$ zero modes:

$n>0$

$-j_{min} \leq m \leq j_{min}$

$0 \leq \ell < n$

$$\begin{pmatrix} \psi^{(0)}_{m\ell} \\ \bar{\psi}^{(0)}_{m\ell} \end{pmatrix} = \sqrt{\frac{\mu}{2\pi}}\, \frac{1}{r} \exp(-e\partial^{-2}E_r) \begin{pmatrix} [\mu(t+ir)]^{\ell} \begin{pmatrix} \Omega^3_{q,m}(\theta,\varphi) \\ 0 \end{pmatrix} \\ [\mu(t-ir)]^{\ell} \begin{pmatrix} \Omega^3_{-q,m}(\theta,\varphi) \\ 0 \end{pmatrix} \end{pmatrix}$$

and similar for $n<0$.

Logic so far

Consider $A^{tot}_\mu = A^{mon}_\mu + A^{vor}_\mu$

- A^{tot}_μ has instanton number $n/2$
- $\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$ have $2|qn|$ zero modes in A^{tot}_μ background

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Fermionic PI in A^{tot}_μ background

$$Z^{(n/2)}_{\psi}[A^{tot}_\mu, \underbrace{\bar{\eta}, \eta}_{\text{sources}}] \equiv \int \prod_{f=1}^{N_f} D\bar{\psi}_f D\psi_f \exp\left\{ \sum_f \int d^4x \left[i\bar{\psi}_f \not{D} \psi_f + \bar{\eta}_f \psi_f + \bar{\psi}_f \eta_f \right] \right\}$$

To compute, expand $\begin{pmatrix} \psi_f \\ \bar{\psi}_f \end{pmatrix}$ as:

$$\begin{pmatrix} \psi_f \\ \bar{\psi}_f \end{pmatrix} = \sum_{m\ell} \underbrace{c_{m\ell f}}_{\text{Grassmann}} \overbrace{\begin{pmatrix} \psi^{(0)}_{m\ell} \\ \bar{\psi}^{(0)}_{m\ell} \end{pmatrix}}^{\text{functions}} + \text{non-zero modes}$$

Result:

$$Z_{\psi}^{(n/2)}[A_\mu^{tot}, \bar{\eta}, \eta] = \underbrace{F^{(n/2)}}_{\text{'t Hooft vertex}} \cdot \underbrace{e^{-\Gamma'}}_{\substack{\text{Exponential of non-zero}\\ \text{mode Green's function - unimportant}}} \cdot \underbrace{(\det(i\not{D}))^{N_f}}_{\text{Schwinger mass term}}$$

$$F^{(n/2)} = \int \prod_{mlf} c_{mlf} \exp\left\{\int d^4x \sum_{mlf} c_{mlf} (\bar{\eta}_f \psi_{ml}^{(0)} + \eta_f \psi_{ml}^{(0)})\right\}$$

$\downarrow$ Grassmannian integration

$$F^{(n/2)} = \epsilon_{N_f}(X_1, \dots, X_{N_f}) \equiv \epsilon^{f_1, \dots, f_{N_f}} X_{f_1} \cdots X_{f_{N_f}}$$

$$X_f \equiv \prod_{ml} \left[(\bar{\eta}_f \psi_{ml}^{(0)}) + (\bar{\Psi}_{ml}^{(0)} \eta_f) \right] \qquad (fg) \equiv \int d^4x\, f(x) g(x)$$

't Hooft vertex:

- Antisymmetric product of all flavors
- Generates

all flavors $\times\ 2|q_n|$

with $2|q_n|N_f$ legs.

- This is the vertex responsible for Callan-Rubakov

Finally:

$$\left[\det(i\not{D})\right]^{N_f} = \exp\left\{-\int d^2x \; \tfrac{1}{2} m_a^2 \, a_\alpha \underbrace{\left(\eta^{\alpha\beta} - \frac{\partial^\alpha \partial^\beta}{\partial^2}\right) a_\beta}_{\text{Gauge invariant Schwinger term}}\right\}$$

$$m_a^2 = \frac{2|q| N_f e_{2D}^2}{\pi}$$

Can recast into AdS_2 metric $g^{2D}_{\alpha\beta} = \left(\frac{R}{r}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

as

$$\left[\det(i\not{D})\right]^{N_f} = \exp\left\{-\int d^2x \sqrt{g_{2D}} \; \tfrac{1}{2} m_a^2 a_\alpha \left(g^{\alpha\beta} - \frac{\nabla^\alpha \nabla^\beta}{\Box}\right) a_\beta\right\}$$

∇^α - AdS_2 covariant der

$\Box = \nabla_\alpha \nabla^\alpha$

So far: in A_μ^{tot} background

magic happens: zero modes, 't Hooft vertex

Now: full gauge PI to show A_μ^{tot} generated and unsuppressed. To calculate, we keep only the photon s-wave (ask me later).

$$Z^{s\text{-wave}}_{mQED_4}[J_\alpha, \bar{\eta}, \eta] = \sum_{n=-\infty}^{\infty} Z^{(n/2)}_{mQED_4}[J_\alpha, \bar{\eta}, \eta]$$

$$Z^{(n/2)}_{mQED_4} = \int_{c_1 = n/2} Da_\alpha \, e^{-S_a} \underbrace{Z_4^{(n/2)}[A_\mu^{mon} + A_\mu, \bar{\eta}, \eta]}_{\text{already computed}}$$

$$S_a = \int d^2x \sqrt{g_{2D}} \left(\tfrac{1}{4} f_{\alpha\beta} f^{\alpha\beta} - J^\alpha a_\alpha\right) \qquad f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$$

massless Schwinger model* in AdS_2!

* axial

Substituting $Z_4^{(n/2)}$,

$$Z^{(n/2)}_{mQED_4} = \int_{Ch_1=n/2} Da_\alpha \, F^{(n/2)} e^{-\Gamma'} e^{-\bar{S}_a}$$

$$\bar{S}_a = \int d^2x \sqrt{g_{2D}} \left\{ \frac{1}{4} f^{\alpha\beta} f_{\alpha\beta} + \frac{m_a^2}{2} a_\alpha \left(g^{\alpha\beta}_{2D} - \frac{\nabla^\alpha \nabla^\beta}{\Box} \right) a_\beta - J_\alpha a^\alpha \right\}$$

a_α integral – Gaussian!

Exactly solvable.

Finally: change variables $a_\alpha \to \sigma$

$\epsilon_{\alpha\beta} = \left(\frac{R}{r}\right)^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ via $a_\alpha = \frac{1}{e_{2D}} \epsilon_{\alpha\beta} \nabla^\beta \sigma$ $\quad \partial_r \sigma|_{r=0} = 0$

$$\Rightarrow Z^{(n/2)}_{mQED_4} = \int_{Ch_1=n/2} D\sigma \, F^{(n/2)} e^{-\Gamma'} e^{-S_\sigma}$$

$$S_\sigma = \int d^2x \sqrt{g_{2D}} \left\{ \frac{1}{2e_{2D}^2} \sigma (\Box - m_a^2) \Box \sigma - \bar{\lambda} \sigma \right\} \qquad \bar{\lambda}\sigma \equiv J_\alpha a^\alpha$$

Now we have everything to compute correlators.

Let's first show that sources for σ generate vortices.

σ Green's function:

$$G_\sigma(x, x') = \frac{e_{2D}^2}{m_a^2} \left[P(x, x'; m_a) - D^N(x, x') \right]$$

P: massive 2D scalar GF in AdS_2

D^N: Neumann massless 2D scalar GF

Vortices

Consider a source $\rho(x) = 2|q|nN_f\,\delta(x-x_1)$ for σ. What is the generated field?

$$a_\alpha = \frac{n\pi}{e_{2D}}\,\epsilon_{\alpha\beta}\nabla^\beta\left(P - D^N\right)$$

$$\Rightarrow \lim_{|x|\to\infty} a_\alpha = -\frac{n}{e_{2D}}\,\frac{\epsilon_{\alpha\beta}x^\beta}{x^2}$$

a vortex!

Conclusion: sources for σ generate vortices
→ abelian instantons.

but $\psi^{(0)}_{m\ell}(x_1) \sim e^{\sigma(x_1)}$ a source for σ

in the PI. Similarly:

$$\psi^{(0)}_{m_1\ell_1}(x_1)\ldots\psi^{(0)}_{m\ell}(x_{2|q|nN_f}) \sim e^{\sigma(x_1)+\ldots+\sigma(x_{2|q_n|N_f})}$$

Each zero mode insertion generates a fractional vortex around it. They all combine to an $n/2$ vortex.
Schematically,

$$\langle\psi_1(x_1)\ldots\psi_{N_f}(x_{2|q_n|N_f})\rangle \sim \int D\sigma\, e^{-S_\sigma}\, e^{\sigma(x_1)+\ldots+\sigma(x_{2|q_n|N_f})}$$

~~fermionic cor~~

charge conserving correlators generate the instantons/vortices that fit them.
These are the leading and only gauge saddle point in a Gaussian PI.

Results:

- $N_f = 2 \quad q = 1/2$ Chirality & flavor changing

$$\Psi_{1L} + \text{mon} \longrightarrow \Psi_{2R} + \text{mon}$$

100% of s-wave

$$C_{1/2,2} = \frac{1}{2} \langle \epsilon^{fg} \bar{\Psi}_{fL}(x_1) \underset{\text{wilson line}}{\underset{\downarrow}{W(x_1,x_2)}} \Psi_{gL}(x_2) \rangle$$

$$= \frac{1}{2} \epsilon^{fg} P_L \frac{\partial}{\partial y_f} \exp\left\{ i \int_{x_1}^{x_2} dy^\alpha \epsilon_{\alpha\beta} \nabla^\beta \partial_{\bar{x}} \right\} P_L \frac{\partial}{\partial \bar{y}_g} \log Z \Big|_{\bar{x}=\bar{y}=y=0}$$

$$= \frac{F_{12}^{\theta\varphi}}{2\pi r_1 r_2} \frac{1}{\Delta t_{12} - i\Sigma r_{12}} \qquad F_{12}^{\theta\varphi} = \Omega^3(\theta_1, \varphi_1) \otimes \Omega^3(\theta_2, \varphi_2)$$

Saturates s-wave unitarity bound generated by abelian instanton / 't Hooft Vertex.

- Callan – Rubakov

Next page

Callan–Rubakov

$$u^1 + u^2 + \text{mon} \longrightarrow e^+ + (d^3)^\dagger + \text{mon}$$

$N_f = 4$
$q = 1/2$

$$\psi_1 = \begin{pmatrix} -\bar{u}^2 \\ (u^1)^\dagger \end{pmatrix} \quad \psi_2 = \begin{pmatrix} \bar{u}^1 \\ (u^2)^\dagger \end{pmatrix} \quad \psi_3 = \begin{pmatrix} e \\ -(\bar{d}_3)^\dagger \end{pmatrix} \quad \psi_4 = \begin{pmatrix} d^3 \\ (\bar{e})^\dagger \end{pmatrix}$$

$$C^{CR}_{1/2,4} = \left\langle \frac{1}{4!} W(x_1,x_3) W(x_2,x_4)\, \epsilon_4(\bar{\psi}_{L1}(x_1), \bar{\psi}_{L2}(x_2), \psi_{L3}(x_3), \psi_{L4}(x_4) \right.$$

only $n/2 = 1/2$ sector of PI contributes

$$\Rightarrow C^{CR}_{1/2,4} = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{(\Delta t_{13} - i\Sigma r_{13})(\Delta t_{14} - i\Sigma r_{14})(\Delta t_{23} - i\Sigma r_{23})(\Delta t_{24} - i\Sigma r_{24})}}$$

Saturating the s-wave unitarity bound and consistent with Rubakov, Callan, Polchinski, ...

General Lesson for Monopole-catalysis

- Every charge-conserving correlator with $2|qn|$ fermionic insertions per flavor leads to a 't Hooft vertex with $2|qn|N_f$ legs. "The fermions generate the instanton that fits them". The instanton number is qn. Half of the insertions are $\bar{\psi}$ and half are ψ for charge conservation.
- The $n/2$-vortex generated in each correlator is the combination of $2|qn|N_f$ fractional vortices, each around an insertion of $\bar{\psi}$ or ψ. Each fractional vortex has $1/(4|q|N_f)$ vortex number

- The solutions of the Dirac eq. in the monopole + Vortex background are not strictly left or right moving, but vortex bound. ~~...~~ Moreover each fermionic wavefunction is equally dispersed among all fractional vortices. This might have implications for the unitarity puzzle.

Conclusions

- The Callan-Rubakov effect is generated by Abelian Instantons in $mQED_4$: QED_4 in the background of a 't Hooft line.
- The 2D EFT is the massless Axial Schwinger model in AdS_2:
  * Exactly solvable (gaussian PI)
  * PI splits into topological sectors
  * In $n \neq 0$ sectors: fermionic zero modes, 't Hoof vertex
- In monopole-catalysis correlators, each fermionic insertion generates a fractional vortex around it. The fractional vortices combine into an $n/2$-vortex generating the 't Hooft vertex relevant for the correlator
- Monopole catalysis is an Abelian IR process outside the core.