# A new special property of Schwarzschild quasinormal modes

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## 1. Background

### 1.1 Quasi normal modes (QNMs)

The merger of two black holes results in a gravitational wave signal that experiences a **ringdown** characteristic of many dissipative systems. This portion of the signal is typically represented using a superposition of damped sinusoids with complex-valued **quasi-normal modes** that correspond to the characteristic frequencies of the system. The measurement of multiple modes or insight into nonlinear effects in the signal could provide meaningful **tests of general relativity**, for example of the no-hair theorem [1].

### 1.2 Natural polynomials and the radial Teukolsky equation

The Teukolsky formalism describes the **linearized Einstein equations** and provides a set of separable solutions, $R(r)$ and $S(\theta)$ [2]. Previous work established a spectral solution to the unsourced radial equation using a newly constructed sequence of **natural polynomials**. These polynomials are orthogonal with respect to a scalar product on a compactified domain between the event horizon and spatial infinity. Crucially, the associated weight function does not diverge in either limit, allowing for the QNM boundary conditions to be imposed *a priori*. All solutions are thus inherently physical and satisfy the relevant criteria to be QNMs from black hole ringdown [3].

**Motivation: We want to learn about fundamental physics using this special overtone property of Schwarzschild quasi-normal modes.**

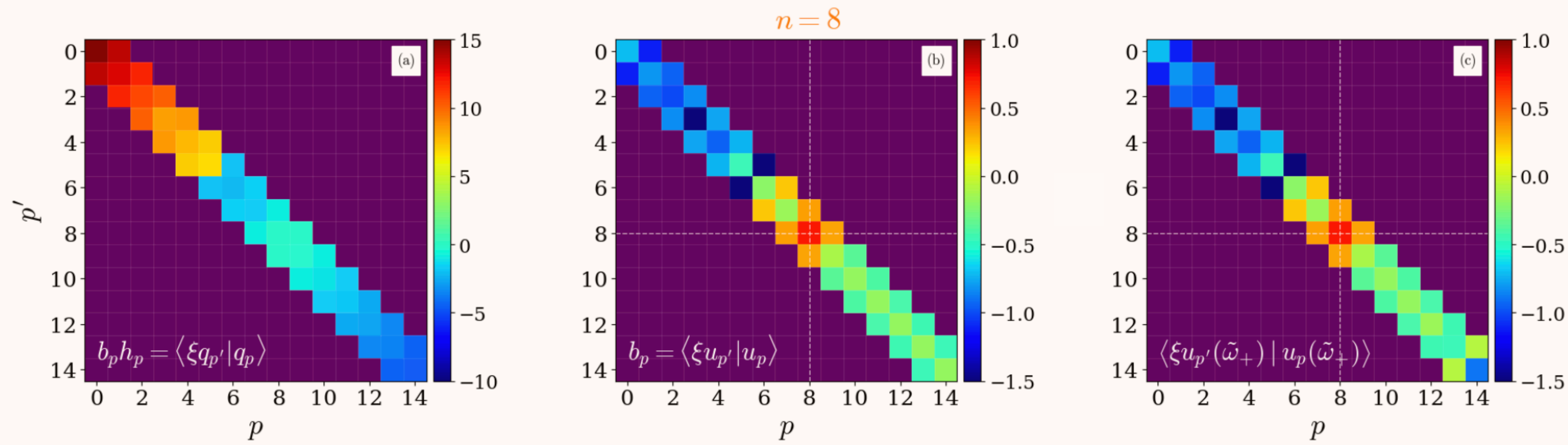


**FIG. 1.** These matrices of the three-term recurrence relation for different polynomial scenarios show no peak and a distinct peak at the overtone label $n$ for $(s,\ell,m,n)=(-2,2,2,8)$, $a/M=0$, $\tilde{\omega}_{\ell mn}=-2i$, and $\omega_n^+=-\frac{1}{4}i(p+1)$. First, panel (a) shows the recurrence relation of the monic orthogonal polynomials has no such peak. This may be contrasted with panels (b) and (c) which both peak at the overtone index $n$ for the polynomials of the quasi-normal mode frequency $\tilde{\omega}$ and constrained frequency, $\omega_n^+$. All color bars correspond to the $\log_{10}$ of absolute values of the relevant quantities and all purple tiles are exactly zero.

## 2. Results

### 2.1 Algebraically special modes

QNMS are labelled with three indices, where $l$ is closely associated to the angular problem and $n$ orders the **overtones**. We find a more physical interpretation for $n$ linked to the index of eigenvalues of the radial solution, analogous to the way $l$ tracks the eigenvalues of solutions in the angular case.

We find also **constrained frequencies** that arise from a condition on the confluent Heun polynomials

$$\tilde{\omega}_+ = -\frac{1}{4}i(p+1) \qquad \tilde{\omega}_- = -\frac{1}{4}i(p+1+s)$$

which are closely associated with algebraically special modes (**total transmission modes**) and expand on previous work in this regime [4].

### 2.2 Mathematical properties

It is generally true that orthogonal polynomials obey a **three-term recurrence relation** which, combined with raising and lowering operators, allows us to relate consecutive polynomial indices [5]. In the case of the natural polynomials, the relation is symmetric and the coefficient $b_p$ of the central term peaks with overtone label $n$ for polynomial order $p$,

$$\xi u_p(\xi) = a_{p+1}u_{p+1}(\xi) + b_p u_p(\xi) + a_p u_{p-1}(\xi)$$

**BONUS: Can you deduce the property these Minecraft swords share with our matrices? Hint: see panels (b-c) of Fig. 1.**

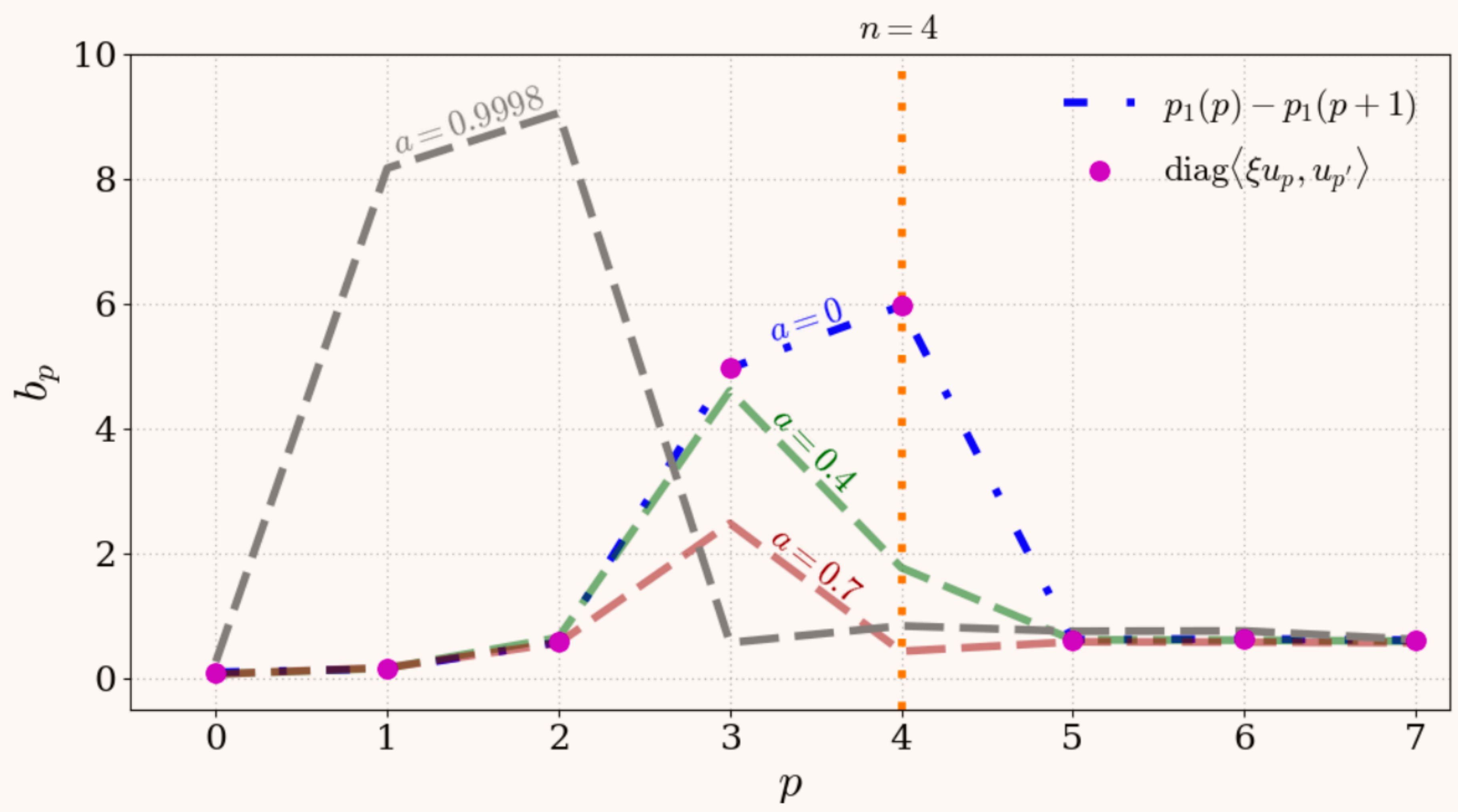


FIG. 2. The two formulations of $b_p$ are plotted: first as the difference in leading coefficients of polynomials of order $p$ and $p+1$, and then as the diagonal values of the three-term recurrence relation matrix, $\langle \xi u_p, u_{p'} \rangle$. This corresponds to the physical scenario $(s,\ell,m,n)=(-2,2,2,4)$, $a/M=0$, and $\tilde{\omega}_{\ell mn}=0.2075-0.9468i$. There is a visible peak at the polynomial order corresponding to overtone label, $n=4$. In grey, red, and green are plotted alternate spins $a/M=0.9998, 0.7, 0.4$ respectively.

## References

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