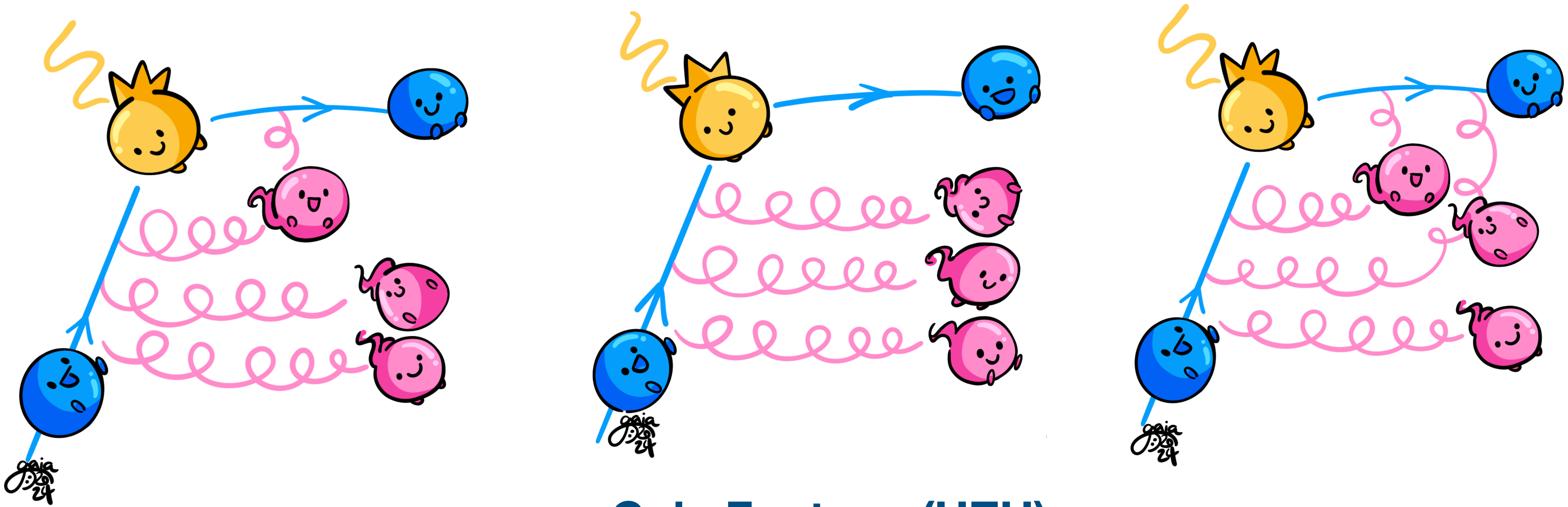
3 loops & 4 cuts:

towards N3LO RRR antenna functions



Gaia Fontana (UZH)

In collaboration with Thomas Gehrmann & Kay Schönwald

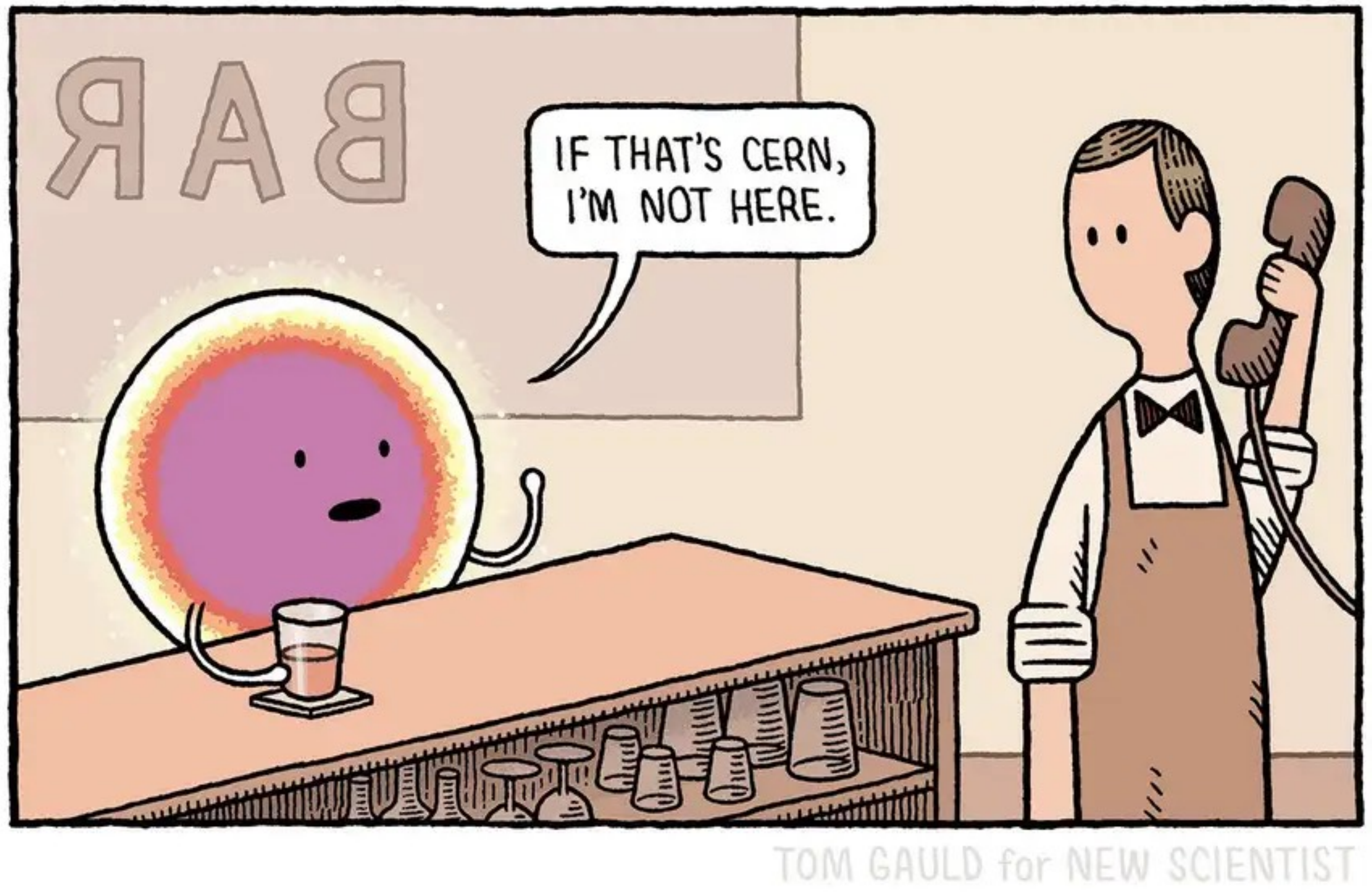
Universität Zürich UZH

Based on JHEP 03 (2024) 159 & upcoming works

35th Rencontres de Blois
Blois, 22/10/2024

Why precision?

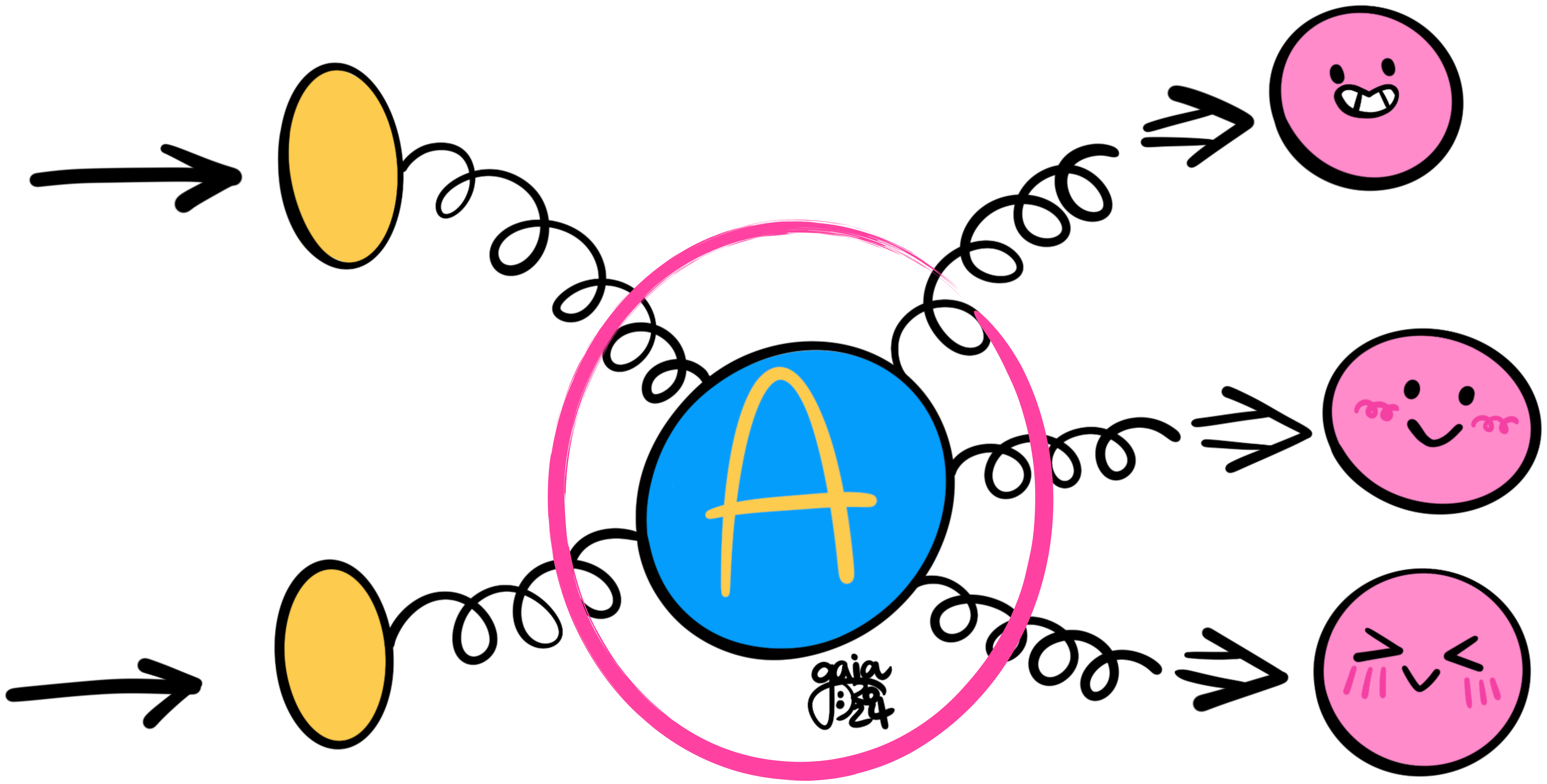
- **Precision physics** as
  - test of the Standard model
  - gate to new physics



- **High-Lumi upgrade of LHC :**
  - theory and experiments must have comparable uncertainties
  - needed: %-level accuracy:

perturbation theory @ **NNLO** and often **N3LO**

Recipe for a theoretical prediction



Many ingredients

- PDFs to describe the proton structure
- Hard scattering
- Radiation and evolution to hadronic states

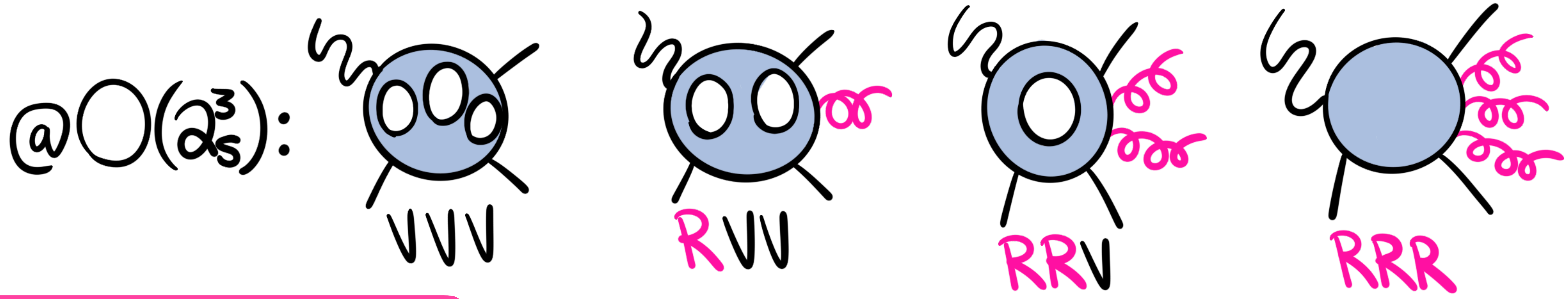
Hard Scattering

Looking @ QCD corrections:

$$\mathrm{d}\sigma = \mathrm{d}\sigma_{LO} + \alpha_S\,\mathrm{d}\sigma_{NLO} + \alpha_S^2\,\mathrm{d}\sigma_{NNLO} + \alpha_S^3\,\mathrm{d}\sigma_{N3LO} + \ldots$$

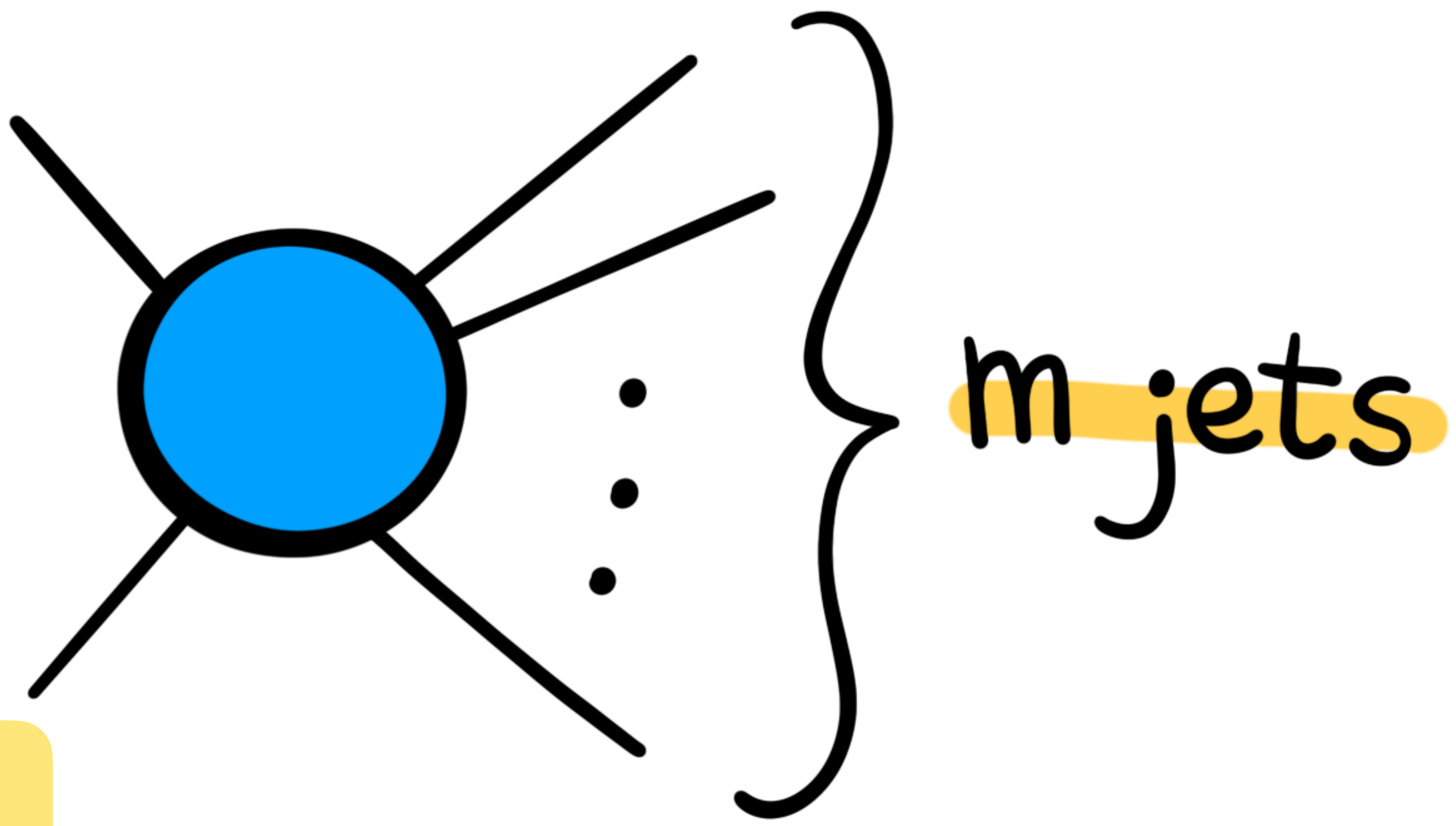
Perturbative series in the strong coupling

Beyond LO: contributions from diagrams with increasing loops and legs



Real corrections!

$$d\sigma = d\sigma_{LO} + \alpha_S\, d\sigma_{NLO} + \alpha_S^2\, d\sigma_{NNLO} + \alpha_S^3\, d\sigma_{N3LO} + \ldots$$



@ LO

$$d\sigma_{LO} = \int_{d\Phi_m} d\sigma_{Born}$$

@ NLO

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} d\sigma^{R}_{NLO} + \int_{d\Phi_m} d\sigma^{V}_{NLO}$$

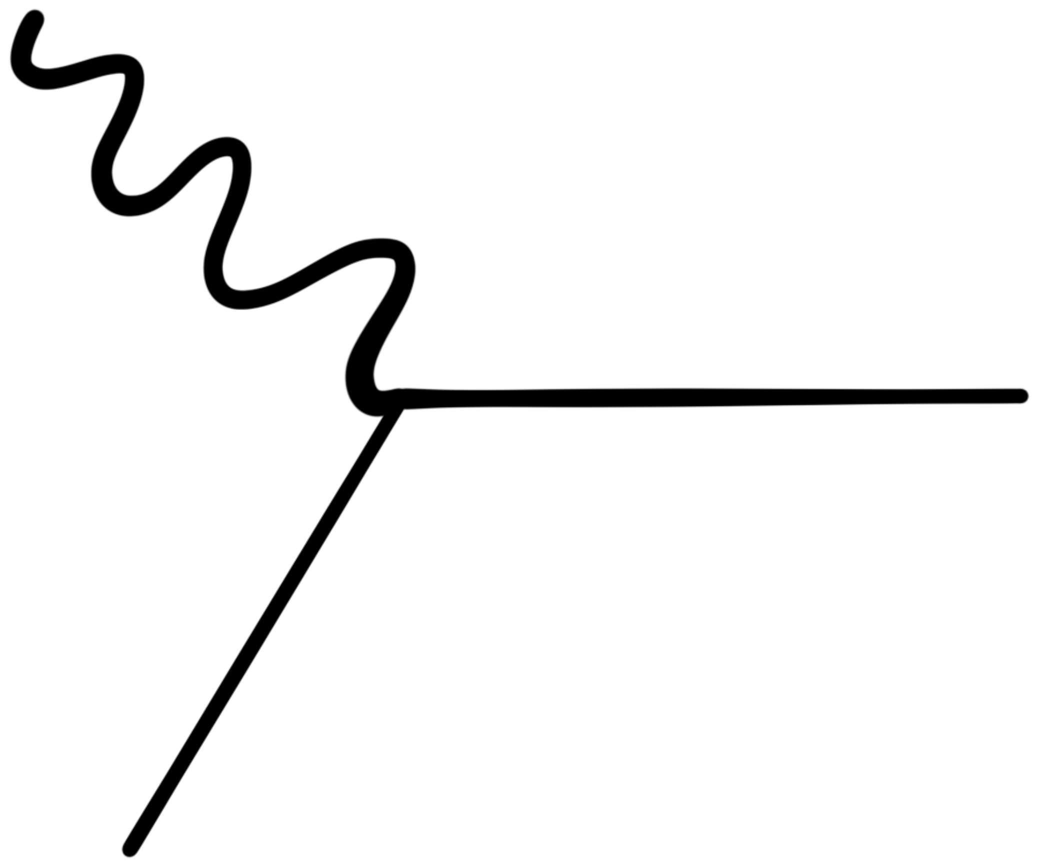
@ NNLO

$$d\sigma_{NNLO} = \int_{d\Phi_{m+2}} d\sigma^{RR}_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^{RV}_{NNLO} + \int_{d\Phi_m} d\sigma^{VV}_{NNLO}$$

@ N3LO

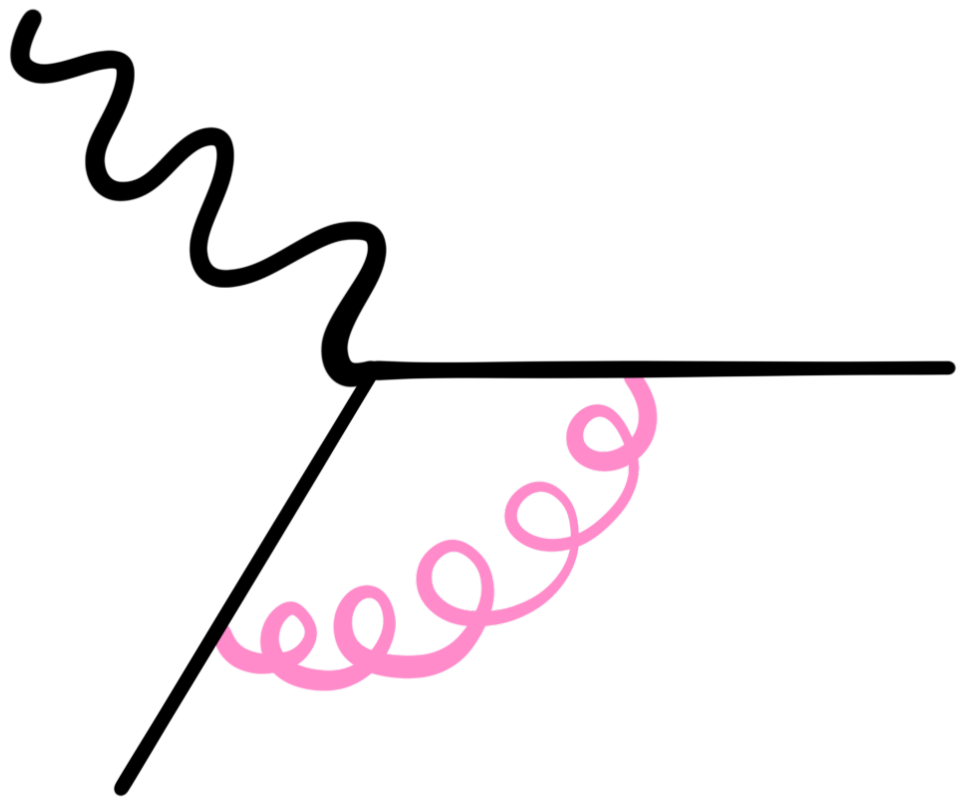
$$d\sigma_{N3LO} = \int_{d\Phi_{m+3}} d\sigma^{RRR}_{N3LO} + \int_{d\Phi_{m+2}} d\sigma^{RRV}_{N3LO} + \int_{d\Phi_{m+1}} d\sigma^{RVV}_{N3LO} + \int_{d\Phi_m} d\sigma^{VVV}_{N3LO}$$

@ LO :

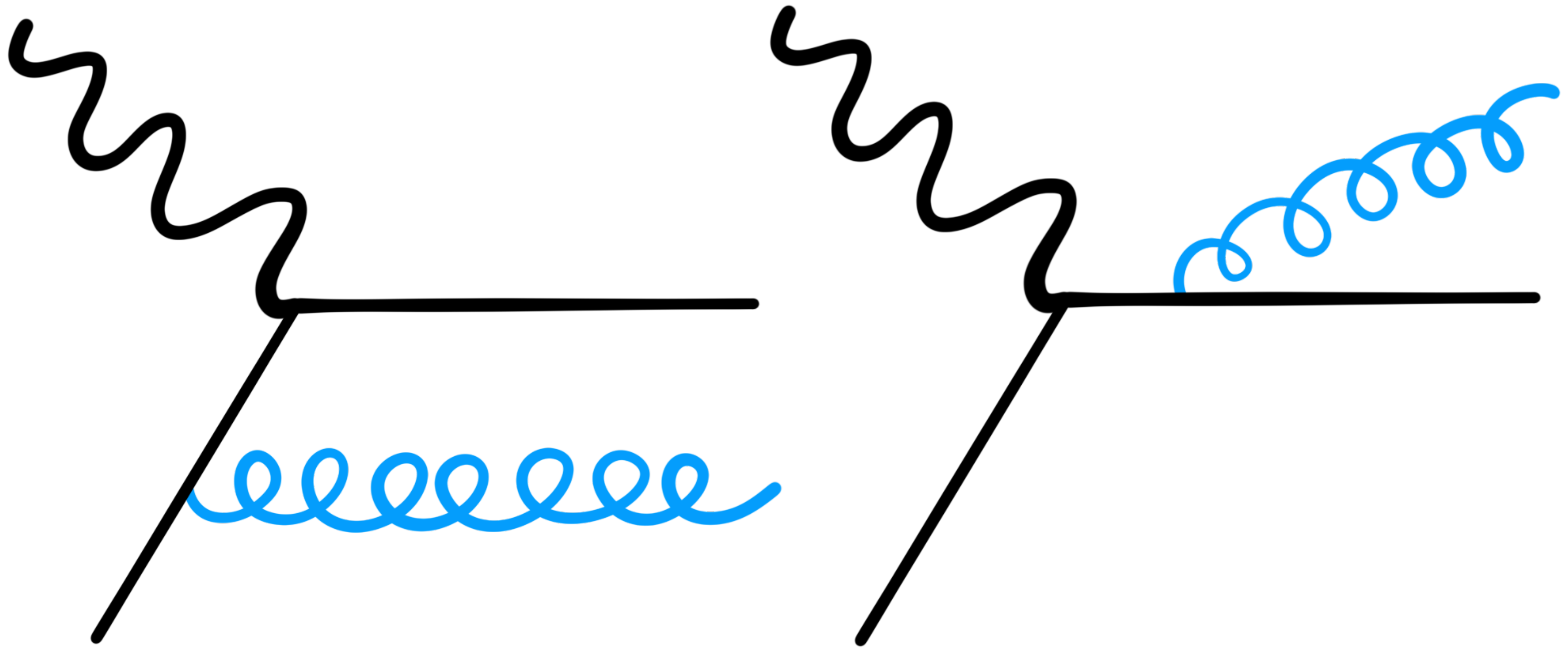


$\gamma q \to q$

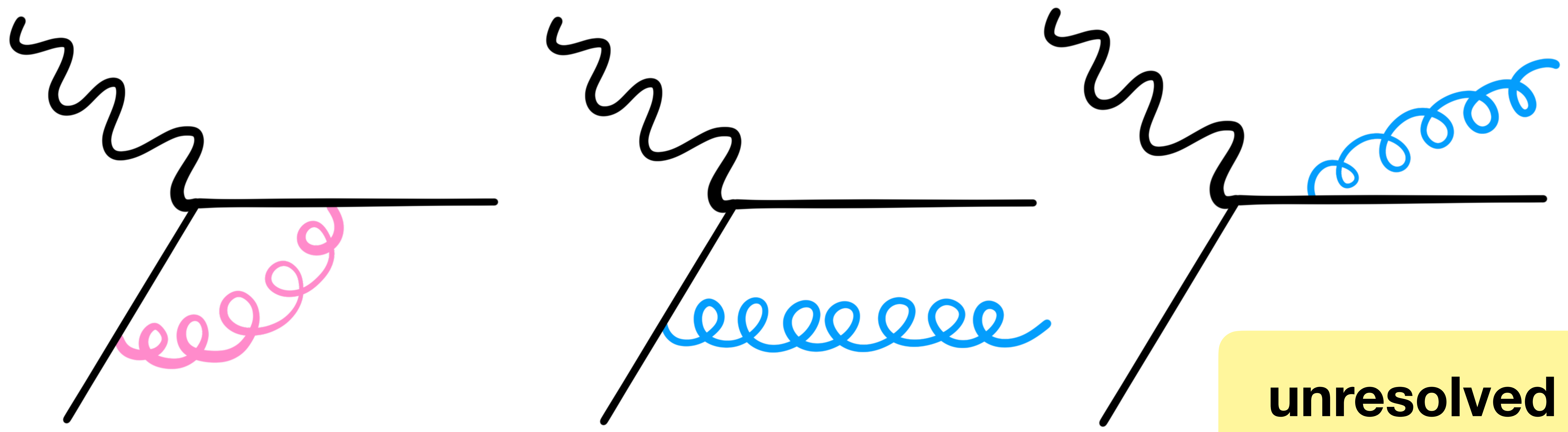
@ NLO corrections: all contributions must be taken into account



Integrate over the loop momentum k



Radiation of an extra gluon



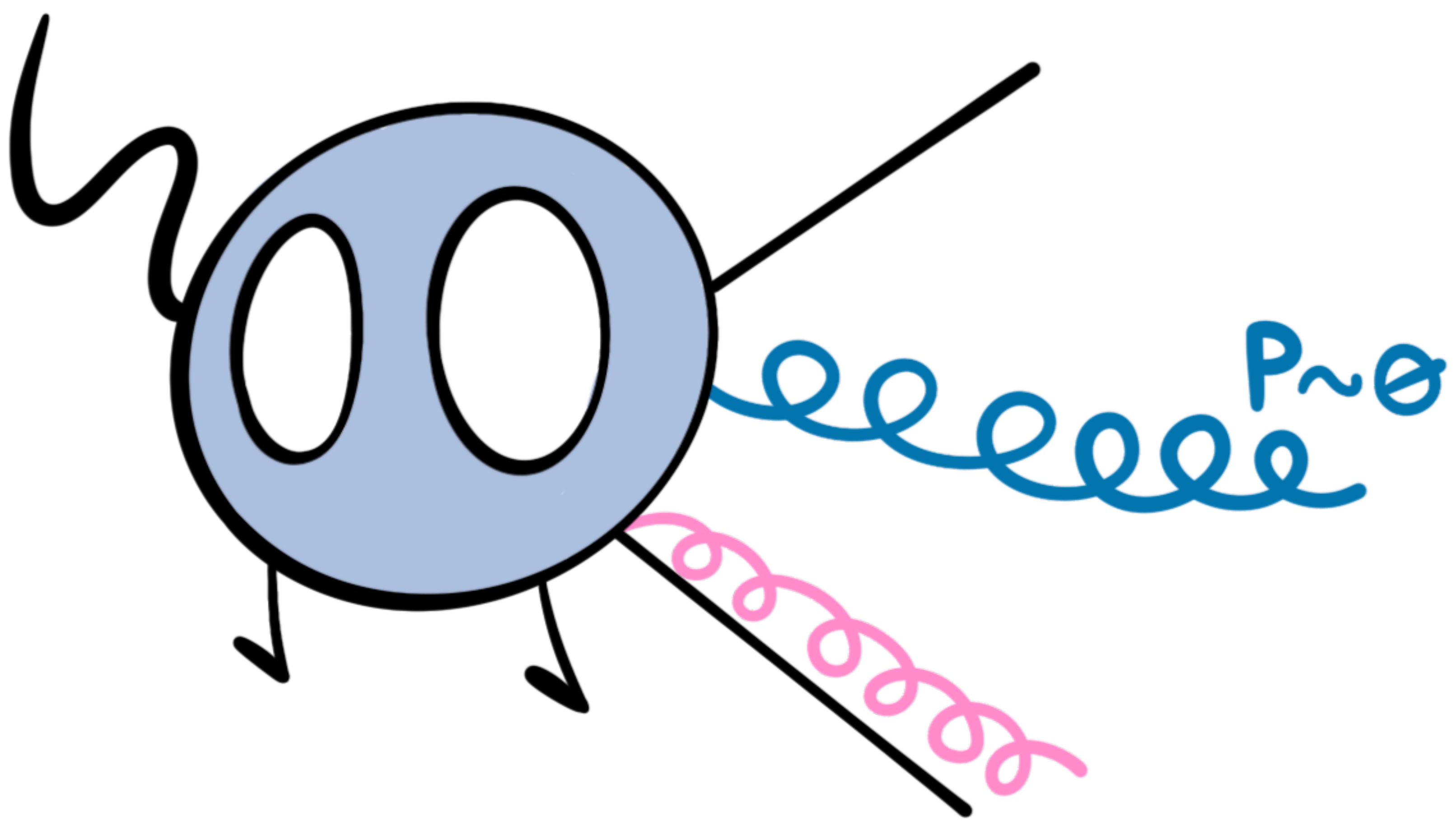
unresolved : soft or collinear

the extra gluon can become unresolved :
must add real radiation!

KLN thm [Kinoshita 1962 ; Lee, Nauenberg 1964]

finiteness when summing over all unresolved configurations

- Separate pieces are IR-divergent:
  - Explicit poles in ϵ after loop integration
  - Implicit divergencies from real radiation



How do we deal with these divergencies?

NLO example

⛔ Hard to solve analytically ⛔

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO}$$

- finite

$$\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} \quad + \quad \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO}$$

∞ $\qquad$ ∞

NLO example

⛔ Hard to solve analytically ⛔

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO}$$

- finite

$$\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} - \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{S}_{NLO} \quad + \quad \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{S}_{NLO}$$

- finite
- finite

Subtraction Schemes

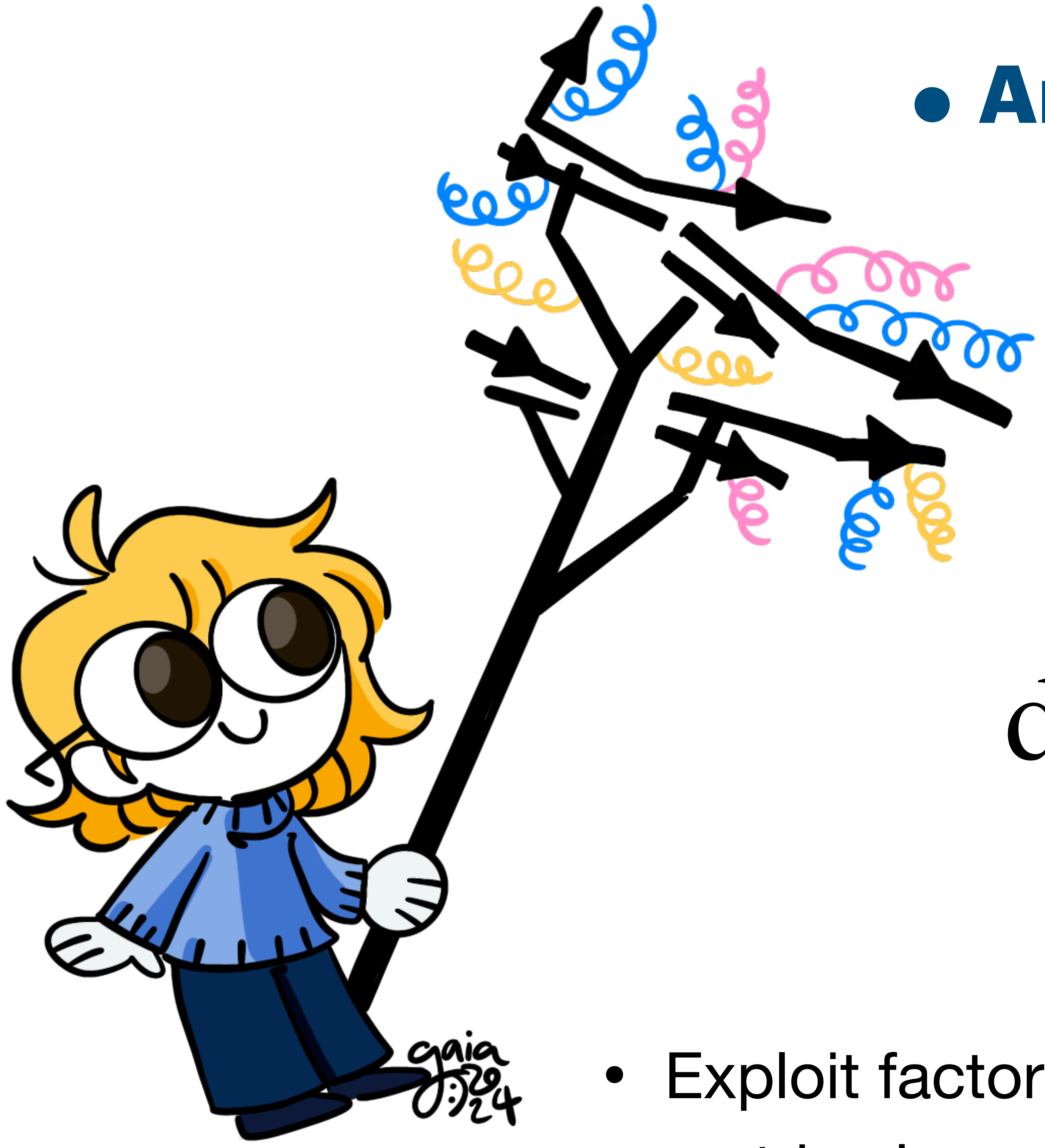
$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} \left(d\sigma^R_{NLO} - d\sigma^S_{NLO}\right) + \int_{d\Phi_m} \left[d\sigma^V_{NLO} + \int_1 d\sigma^S_{NLO} \right]$$

- finite
- finite

- Add and subtract the same quantity $d\sigma^S$
  - Mimics singular behaviour in IR-limits of $d\sigma^V, d\sigma^R$
  - Makes the integrals individually finite
  - Simple enough to be analytically integrated over $d\Phi_1$

Antenna Subtraction scheme

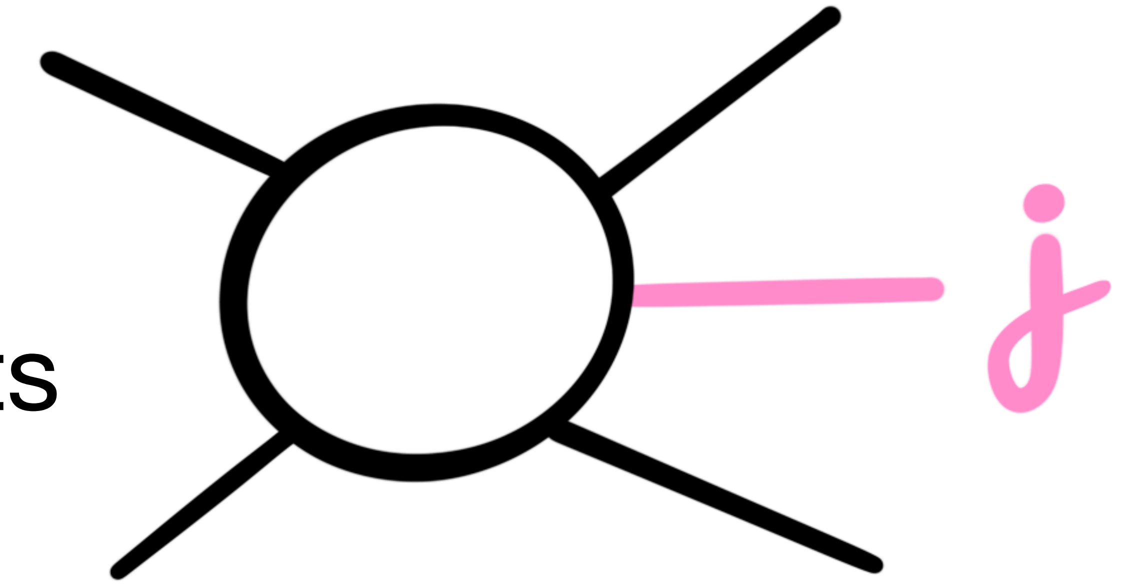
[Gehrmann-De Ridder, Gehrmann, Glover (2007)]



- **Antenna functions**
  - Built from simple matrix elements
  - Mimic the divergent behaviour in singular limits
  - Can be easily integrated over phase space

$$\mathrm{d}\sigma^{S}_{NLO} \sim X^{\ell}_{2+\,extra\ radiation}\,\tilde{M}^{\ell}_{hard\,partons}\,J_m$$

- Exploit factorisation of matrix elements in IR limits



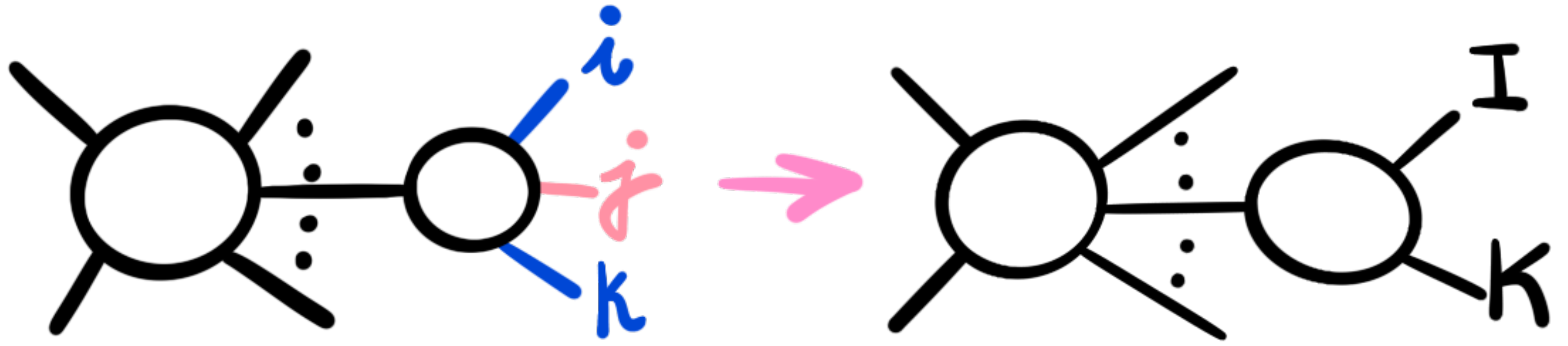
What can happen?

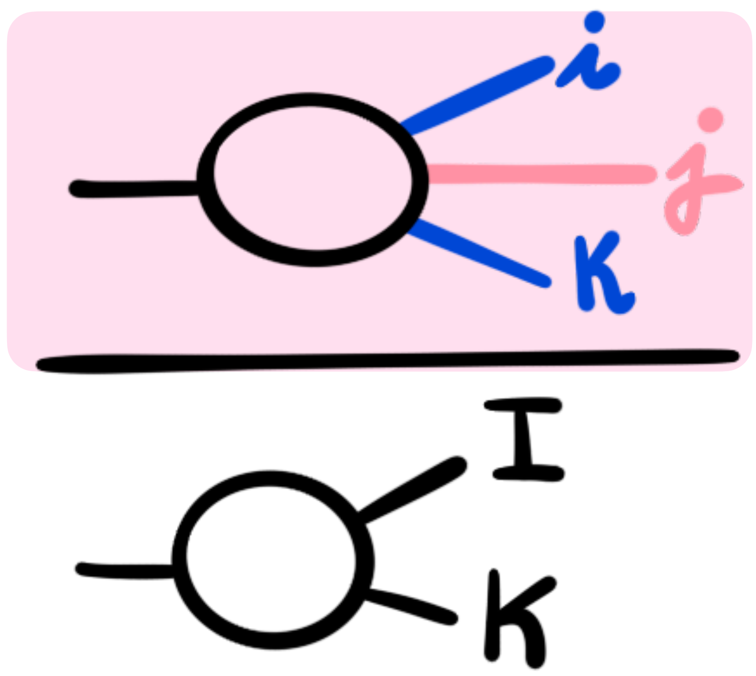
Matrix element with an extra radiation j

IR limit factorization

$j \parallel i, \quad j \parallel k, \quad j$ soft

Final state hard radiators

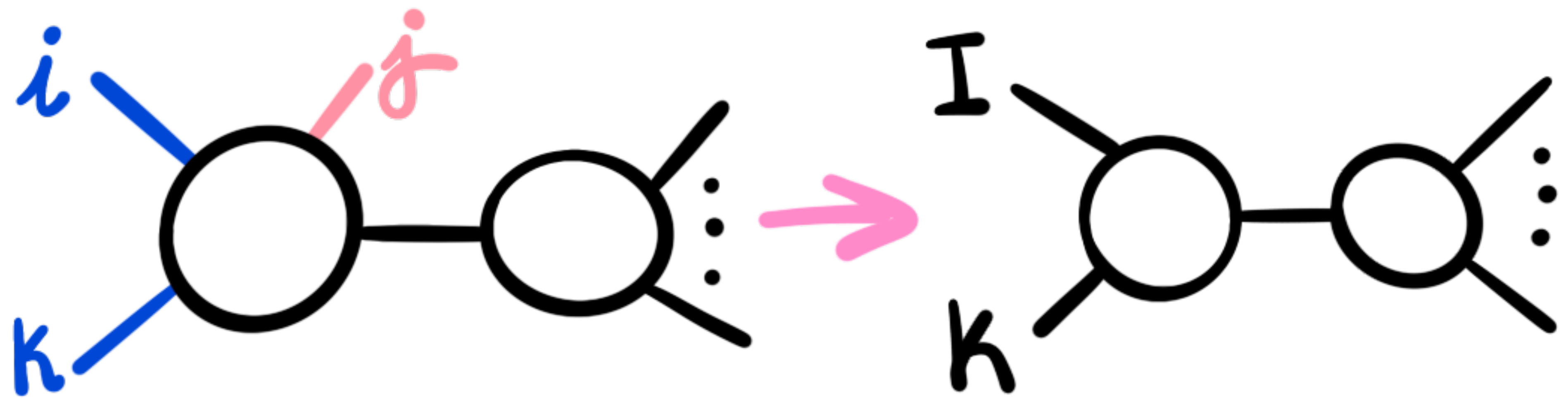


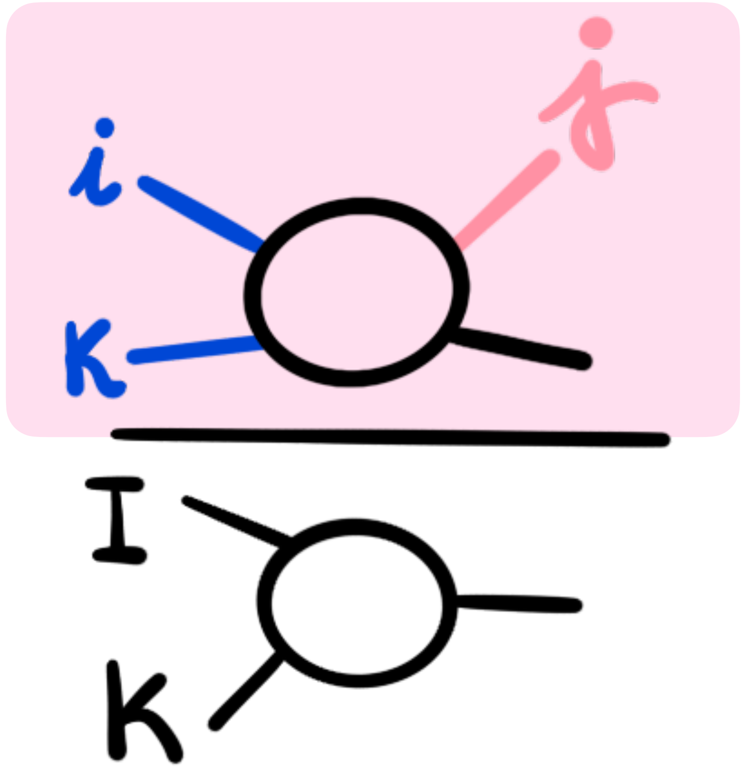


Annihilation into hadrons

Solved! :)
[Chen,Jakubčík,Marcoli, Stagnitto '23]

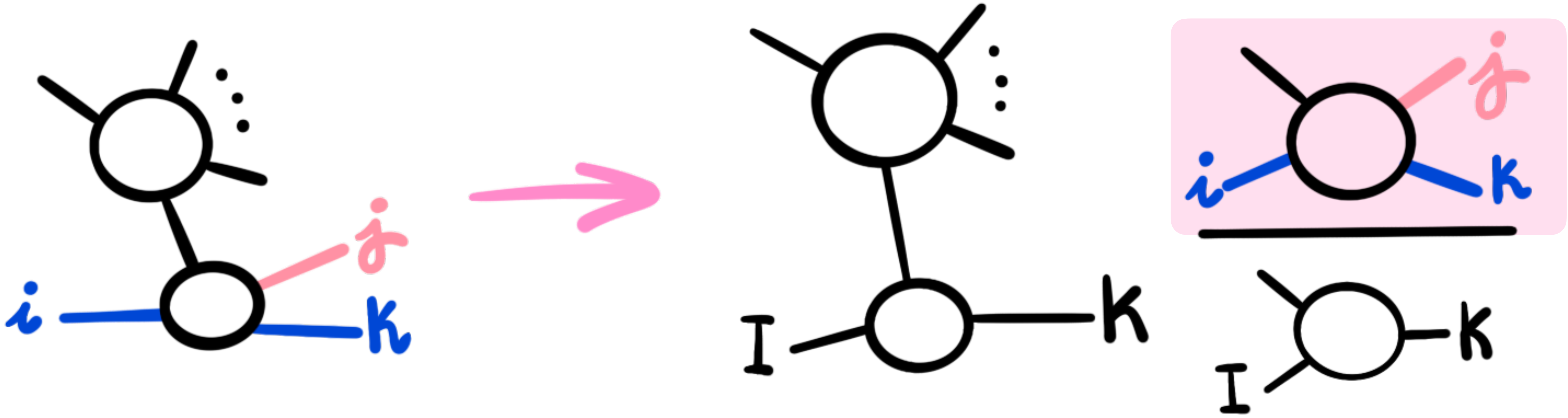
Initial state hard radiators





Drell-Yan

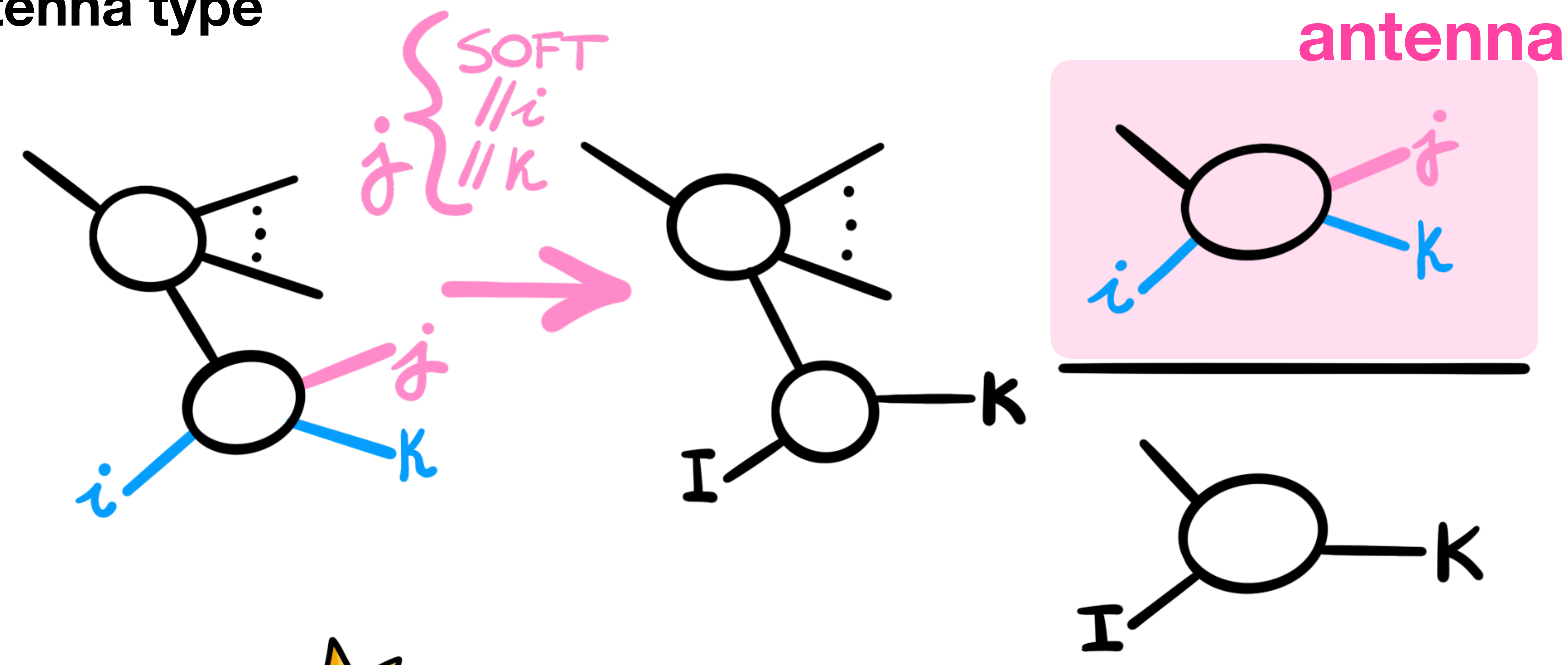
Initial-final state hard radiators

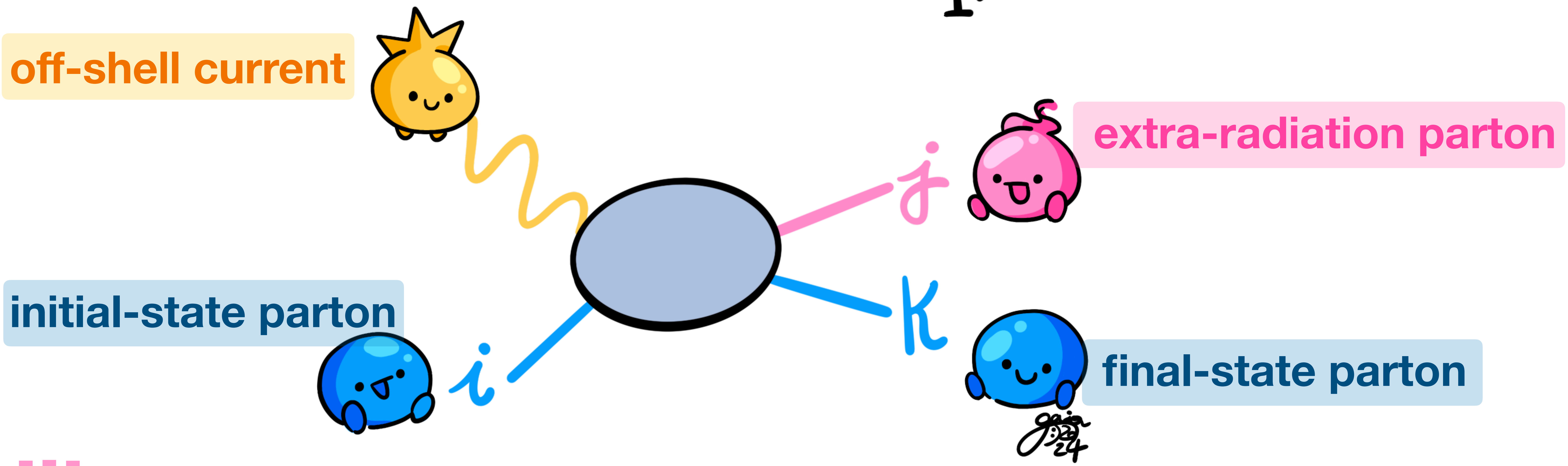


Deep Inelastic Scattering

- **Antenna functions**

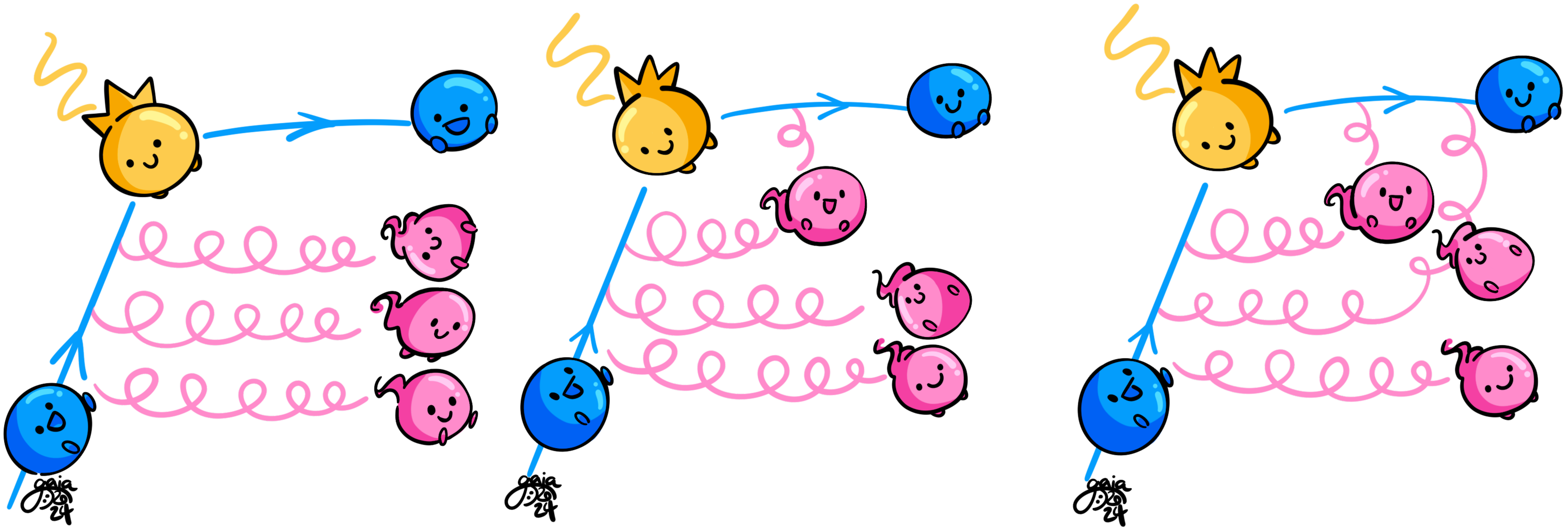
Focus: initial-final antenna type





⇒DIS-like process

Towards N3LO RRR antennae

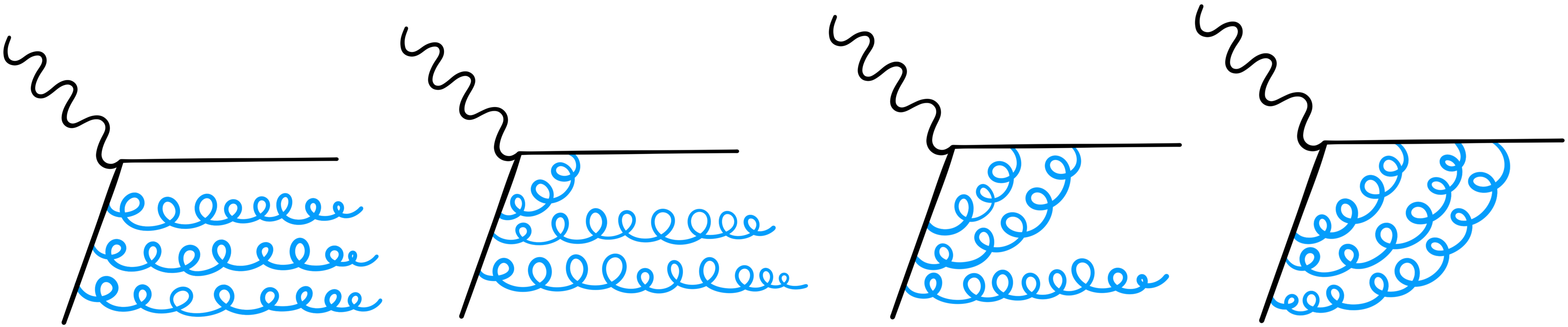


Deep Inelastic Scattering N3LO

Representative contributions at order α_s^3

$$q_1 + q_2 \to p_1 + p_2 + (p_3) + (p_4)$$

$$q_2^2 = -Q^2 < 0,\ q_1^2 = 0,\ p_i^2 = 0, \quad i = 1,2,3,4$$

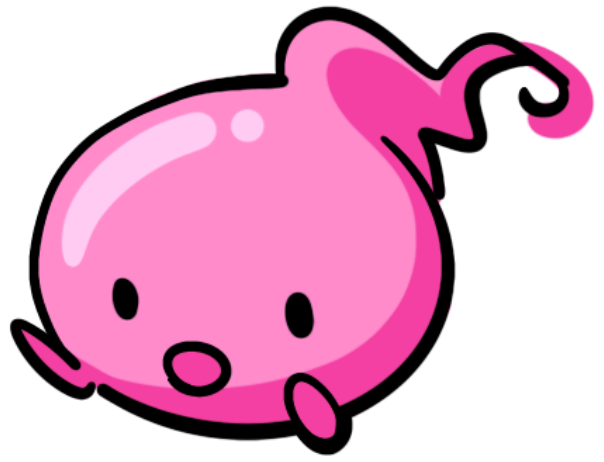


RRR $2 \to 4$ | **RRV** $2 \to 3$ | **RVV** $2 \to 2$ | **VVV** $2 \to 1$

Workflow

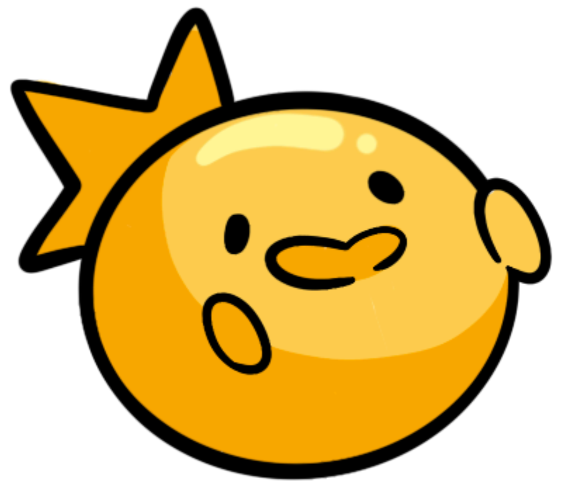
Phase space integral

→ Reverse unitarity



→ Reduction to master integrals

→ DE & canonical form



→ Boundaries

Reverse Unitarity

[Anastasiou, Melnikov 2002]

phase space → (cut) loops

$$-2\pi i\delta^+(p_i^+) = \frac{1}{p_i^2 + i0^+} - \frac{1}{p_i^2 - i0^+} = \frac{1}{[p_i^2]_{cut}}$$

$$I_{RRR} = \int d\Phi_4\,(2\pi)^D\delta^D\left(q_1 + q_2 - \sum_{i=1}^{4} p_i\right)\prod_j \frac{1}{D_j^{\alpha_j}}$$

Notice!

$$d\Phi_n = \prod_{i=1}^{n}\frac{d^d p_i}{(2\pi)^d}\delta^+\left(p_i^2\right)$$

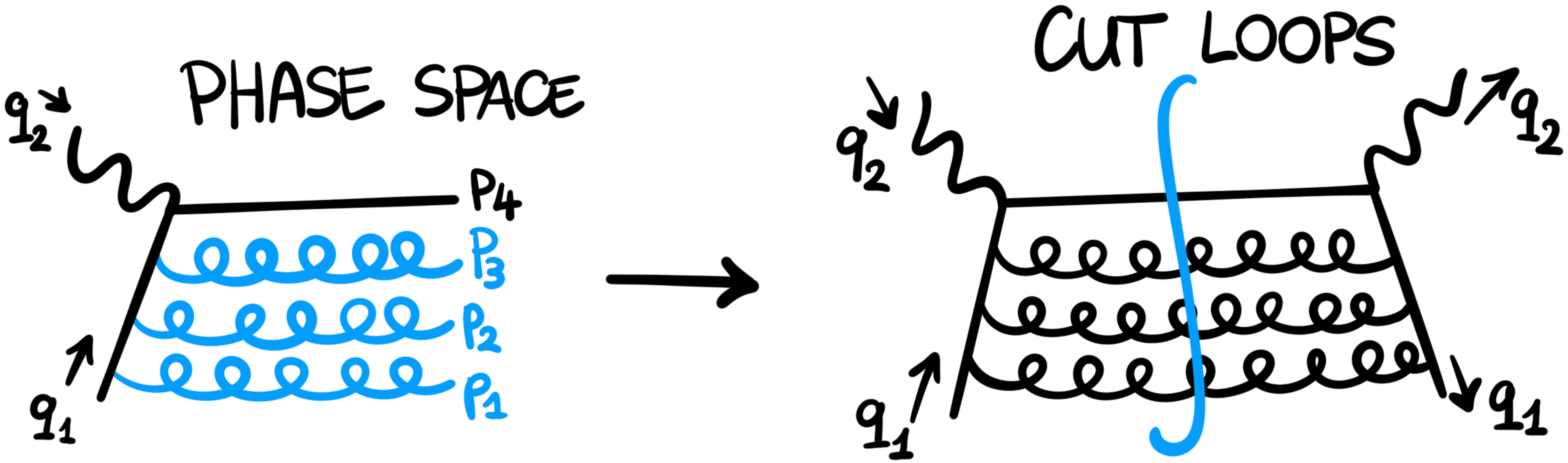
$$I_{RRR} = \int \frac{d^D p_1}{(2\pi)^D}\int \frac{d^D p_2}{(2\pi)^D}\int \frac{d^D p_3}{(2\pi)^D}\frac{1}{[p_1^2]_{cut}}\frac{1}{[p_2^2]_{cut}}\frac{1}{[p_3^2]_{cut}}\frac{1}{[p_4^2]_{cut}}\prod_j \frac{1}{D_j^{\alpha_j}}$$

Reverse Unitarity

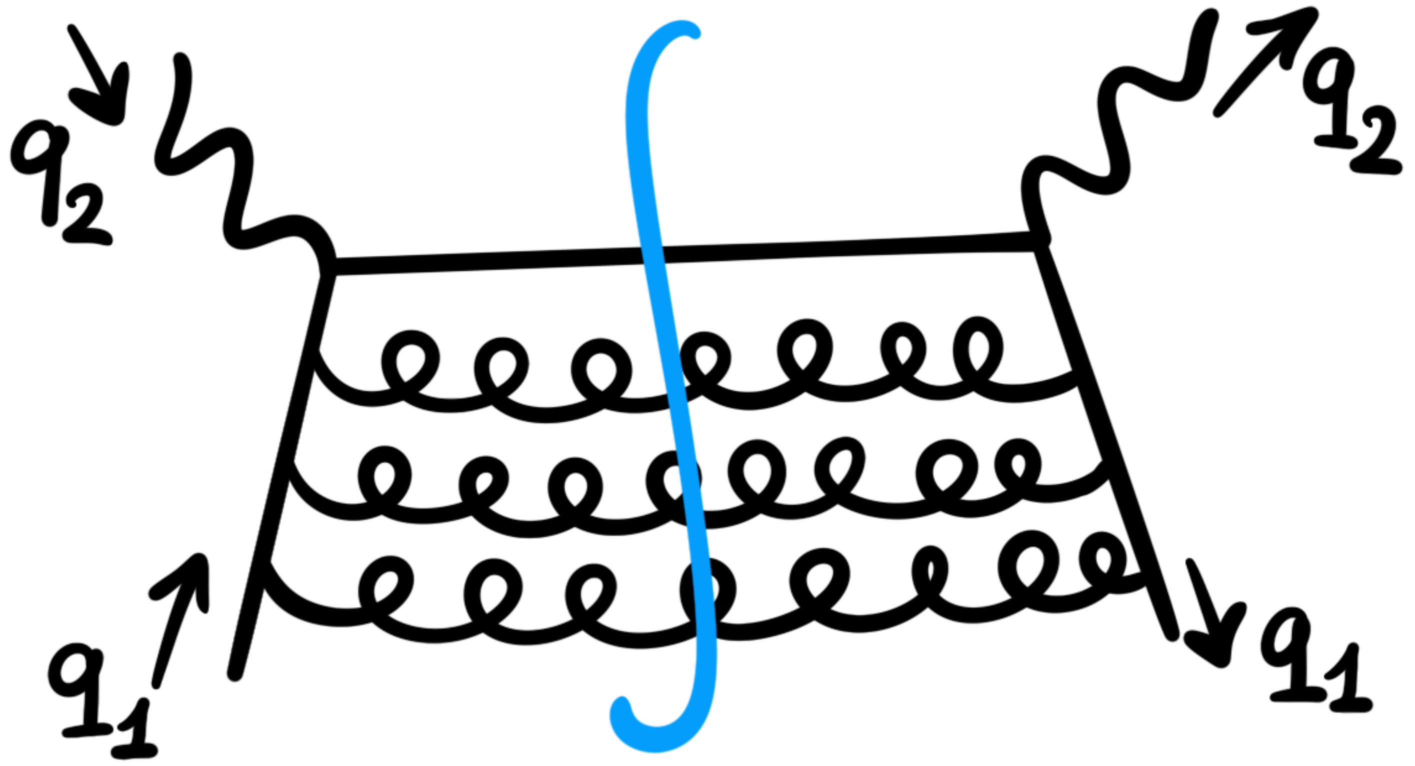
phase space → (cut) loops

$$-2\pi i\delta^+(p_i^+) = \frac{1}{p_i^2 + i0^+} - \frac{1}{p_i^2 - i0^+} = \frac{1}{[p_i^2]_{cut}}$$

Diagrammatically



$$\int d\Phi_4 (2\pi)^D \delta^D\Big(P+q-\sum_{i=1}^{4} p_i\Big) \prod_j \frac{1}{D_j^{a_j}} \rightarrow \int \prod_{i=1}^{3} \frac{d^D p_i}{(2\pi)^D} \frac{1}{\cancel{p_1^2}\,\cancel{p_2^2}\,\cancel{p_3^2}\,\cancel{p_4^2}} \prod_j \frac{1}{D_j^{a_j}}$$



This DIS amplitude contains a lot of integrals $\{I_j\}$

→ how to make things better?

Reduction to Master integrals

[Chetyrkin, Tkachov ’81; Laporta 2000]

Reduction into a basis of linearly independent **master integrals**

$$\{g_j\} \subset \{I_j\}$$

$$I_j = \sum_k c_{jk}\, g_k$$

rational coefficients (c_{jk})

master integrals (g_k)

modulo identities:

- Integration By Parts
- Lorentz Invariance
- symmetry relations

DE for Feynman integrals

[Barucchi, Ponzano ’73; Kotikov ’91; Bern, Dixon, Kosower ‘94; Gehrmann, Remiddi 2000]

Derivative of MI with respect to external invariants

$$\partial_z g_i = \sum_j a_{ij} I_j$$

Use IBP relations to rewrite RHS

$$I_j = \sum_k c_{jk}\, g_k$$

Obtain a system of first order DE for the MI!

$$\partial_z g_i = \sum_{jk} a_{ij} c_{jk}\, g_k$$

$$\partial_z \vec{g} = A(\epsilon, z) \cdot \vec{g}$$

How to solve a differential equation:

- Generic solution
- Boundary condition

Rewrite the DE in **canonical form** [Henn 2013]: solution in terms of iterated integrals

$$\partial_z \vec{g} = \epsilon A^{\star}(z) \cdot \vec{g}$$

Boundary Conditions

- Consistency conditions

  Finding relations between boundaries

- Evaluation in some kinematic limit

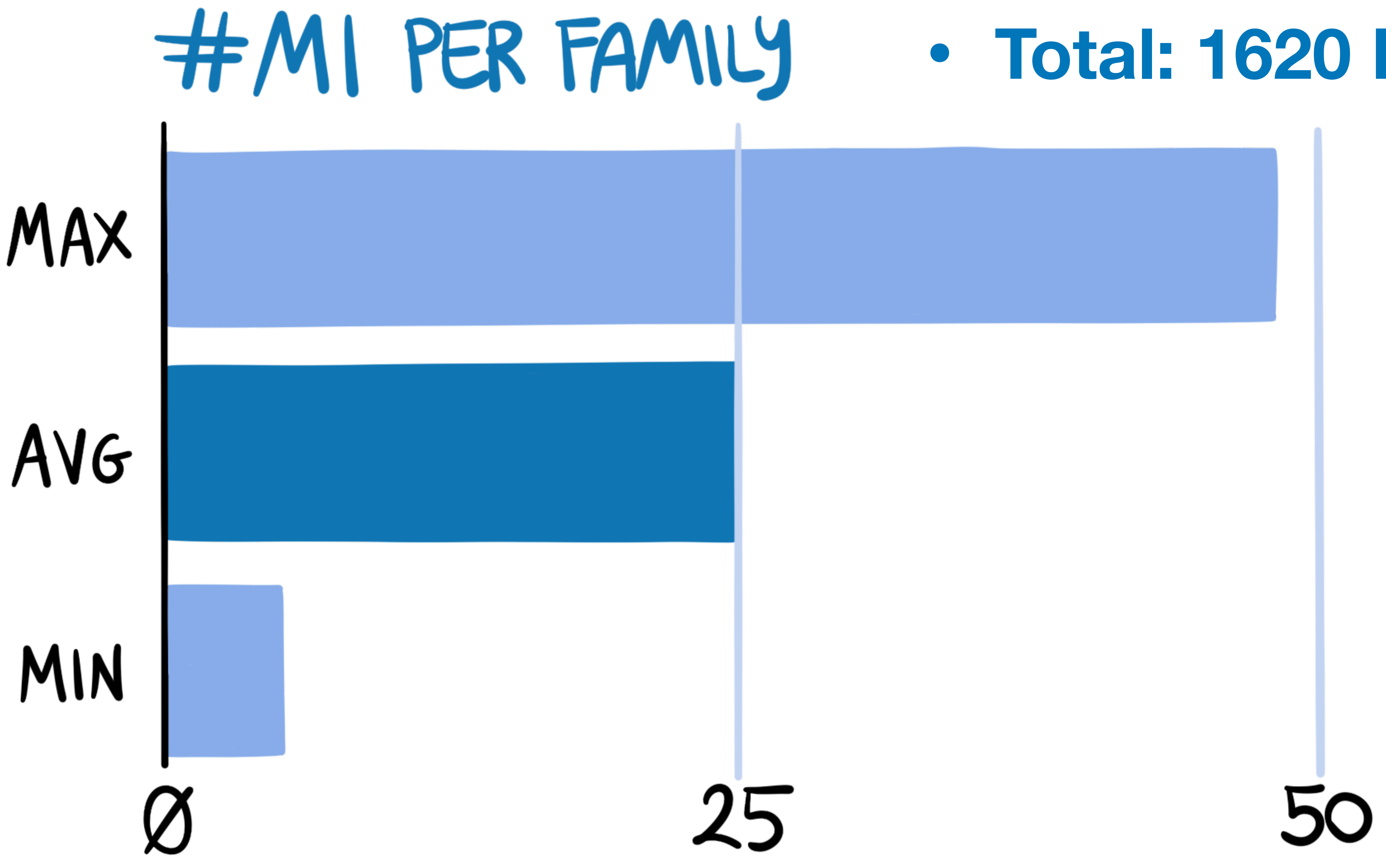
  Fix the remaining ones

Getting to know the RRR families

Physical 4-cuts of the 3 loop inclusive DIS amplitude

$$I_{RRR} = \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \frac{1}{[p_1^2]_{cut}} \frac{1}{[p_2^2]_{cut}} \frac{1}{[p_3^2]_{cut}} \frac{1}{[p_4^2]_{cut}} \prod_j \frac{1}{D_j^{\alpha_j}}$$

- 65 families
- Few number of MI for each family →**4 cuts**
- **Total: 1620 MIs** (**No symmetries** between families included)



- DE matrix M is a function of $M(z, \epsilon)$ [Lee 2020]

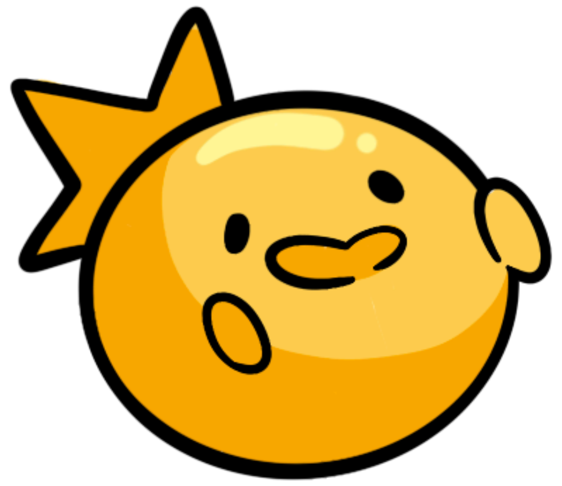
⇒ **playground for automatic tools! eg** `LIBRA`

- Analysis of their parametric representation to simplify the DE & get to a canonical form [Henn 2013]

Results

Canonical DE for all the families ✔

We can find a generic solution



Boundary conditions 🔜

[Liu, Ma 2022]

- Numerical evaluation with `AMFlow` **@ 200 digits** (~80% done…) **& PSLQ**
- Constraints from symmetry relations between the families
- Calculation of the amplitude → which boundaries are actually needed

Outlook:

- Extend calculation to RVV and RRV layers
- Ultimate goal: obtaining the full set of integrated initial-final antennae

Thank you for your attention!