Precision prediction on heavy quark decays

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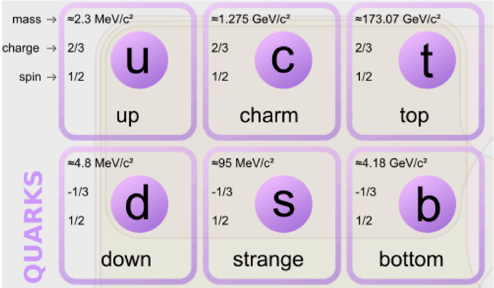
Motivation

The masses of quarks range from 2.3×10^{-3} to 1.73×10^{2} GeV.

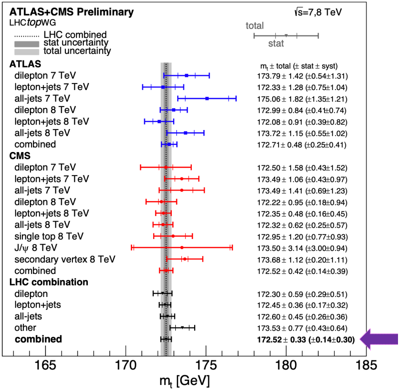
Heavy quarks are mystery.



Motivation

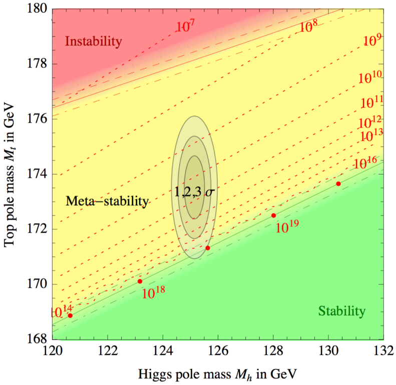
Top-quark mass is the one of the fundamental parameters in Standard Model.

Summary of the top-mass analyses at the LHC.



Motivation

Top-quark plays a special role in determining the vacuum stability [1307.3536].



Motivation

The Cabibbo-Kobayashi-Maskawa matrix [PDG 2014]

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.974 & 0.225 & 0.0036 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.041 & 0.999 \end{pmatrix}$$

The third line is derived from unitarity of the CKM matrix.

The direct measurement of $R = Br(t \to Wb)/Br(t \to Wq)$ yields $|V_{tb}| = 1.011^{+0.018}_{-0.017}$ under the assumption of CKM unitarity and existence of three generations.

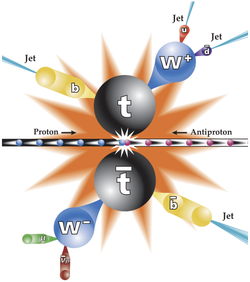
The single top production rate is proportional to $|V_{tb}|^2$. It is measured to be $|V_{tb}| = 1.02 \pm 0.05$.

Motivation

Top decay width Γ_t is one of the fundamental properties of top-quark.

Due to its large mass, Γ_t is expected to be very large (about 1 GeV $> \Lambda_{\mathrm{QCD}}$).

The measurement of Γ_t could hint at new-physics.



[Denisov, Vellidis 2015]

Motivation

The top-quark decays almost exclusively to Wb. $\Gamma_t = \Gamma_t(t \to Wb)$.

At the LHC, indirect techniques are precise but model dependent.

The most precise measurement is $\Gamma_t = 1.36 \pm 0.02\ (\text{stat.})^{+0.14}_{-0.11}\ (\text{syst.})$ GeV by CMS [CMS, 2014].

Direct techniques are less precise but model independent.

Direct measurement by ATLAS gives $\Gamma_t = 1.9 \pm 0.5$ GeV [ATLAS, 2019].

In the future e^+e^- collider, Γ_t can be measured with an uncertainty of 26 MeV [Zhan Li, et al, 2022].

Theoretical predictions:

NLO QCD corrections [Jezabek, Kuhn 1989, Czarnecki 1990, Li, Oakes, Yuan 1991]

NLO EW corrections [Denner, Sack 1991, Eilam, Mendel, Migneron, Soni 1991]

Asymptotic expansions of NNLO QCD corrections near $m_W \to 0$ and $m_W \to m_t$ [Czarnecki, Melnikov 1999, Chetyrkin, Harlander, Seidensticker, Steinhauser 1999, Blokland, Czarnecki, Slusarczyk, Tkachov 2004 2005]

Numerical results of full NNLO QCD corrections [Gao, Li, Zhu 2013, Brucherseifer, Caola, Melnikov 2013]

The full analytical results of NNLO QCD corrections have been obtained recently [Chen, Li, JW, Wang, 2022].

NNNLO QCD results are calculated by two groups. [Chen, Li, Li, JW, Wang, Wu, 2023, Chen, Chen, Guan, Ma, 2023]

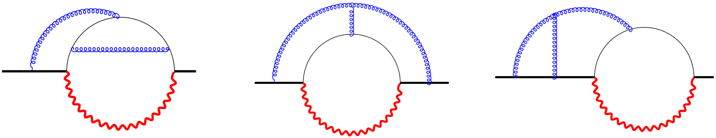
Bottom quark semileptonic decay $b \to X_u l v_l$ is intimately related to top quark decay.

Optical Theorem

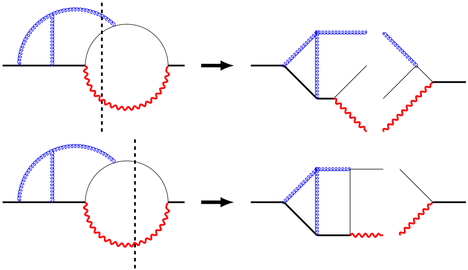
Consider the three-loop self-energy diagrams Σ for $t \to Wb \to t$

$$\Gamma_t = \frac{\mathrm{Im}(\Sigma)}{m_t} \tag{1}$$

Some typical three-loop diagrams in Σ



The imaginary part comes from cut diagrams. For example,



The separate virtual and real corrections are combined.

The complicated phase space integration can be avoided.

For $t \to Wb \to t$, b quark is assumed massless. Kinematic variable is $w = m_W^2/m_t^2$

After spin summation

$$\sum_{\text{spin}} u(k, m_t)\bar{u}(k, m_t) = \not{k} - m_t \tag{2}$$

the numerator of the amplitude is just a scalar product, such as

$$\int \mathcal{D}^D q_1 \ \mathcal{D}^D q_2 \ \mathcal{D}^D q_3 \frac{(k \cdot q_1)\,(q_1 \cdot q_2)\, q_3^2}{D_1 \ D_2 \ D_3 \ D_4 \ D_5 \ D_6 \ D_7 \ D_8 \ D_9}, \tag{3}$$

where q_1, q_2, q_3 are loop momenta, k is external momentum.

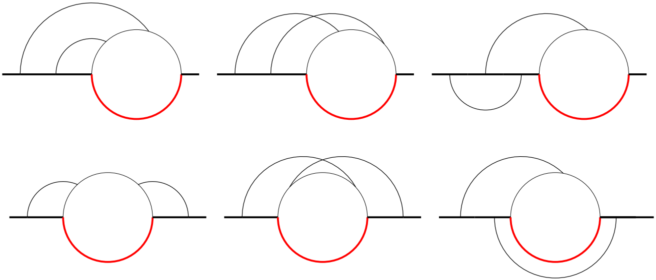
The amplitudes can be written as the linear combination of scalar integrals.

Master Integrals

After integral reduction, the scalar integrals can expressed by minimal set of integrals called master integrals.

In this step we used integration-by-parts (IBP) identities and package `FIRE` [Smirnov, Chuharev 2019].

The typologies of master integrals



The key is to analytically calculate the master integrals.

Canonical differential equation method [Henn 2013] is a powerful tool in analytical calculations. However, it is highly nontrivial to achieve such a form for processes involving massive propagators.

The differential equations of a canonical basis $\mathbf{F}$ can be written as

$$\frac{\partial \mathbf{F}(w,\epsilon)}{\partial w} = \epsilon \left[\sum_{i=1}^{4} \mathbf{R}_i \mathrm{dlog}(l_i)\right] \mathbf{F}(w,\epsilon), \quad w = \frac{m_W^2}{m_t^2}, \quad D = 4-2\epsilon \tag{4}$$

$l_i \in \{w-2, w-1, w, w+1\}$ and $\mathbf{R}_i$ being rational matrices. For example,

$$\frac{\partial F_4(w,\epsilon)}{\partial w} = \frac{\epsilon\,(F_5 - 2F_4)}{w-1} - \frac{\epsilon\,(F_4 + F_5)}{w} \tag{5}$$

By this canonical form, the differential equations can be solved recursively.

Boundary Conditions

Setting $m_W = 0$ does not bring new divergences in the amplitude.

Most of the basis integrals are regular at $w = 0$. For example,

$$\frac{\partial F_4(w,\epsilon)}{\partial w} = \frac{\epsilon\,(F_5 - 2F_4)}{w-1} - \frac{\epsilon\,(F_4 + F_5)}{w} \tag{6}$$

$$\implies \quad F_4|_{w=0} + F_5|_{w=0} = 0 \tag{7}$$

The analytical results of some master integrals in $w = 0$ can be found in [Blokland, Czarnecki, Slusarczyk, Tkachov 2005, Ritbergen, Stuart 2000].

Boundary expressions can be reconstructed by numerical results using PSLQ algorithm with the package `AMFlow` [Liu, Ma 2022].

The analytical results of master integrals can be written as multiple polylogarithms (GPLs)

$$G_{a_1,a_2,\ldots,a_n}(x) \equiv \int_0^x \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(t)\,, \tag{8}$$

$$G_{\vec{0}_n}(x) \equiv \frac{1}{n!}\ln^n x\,. \tag{9}$$

In our problem, we only need harmonic polylogarithms (HPLs).

$$H_{a_1,a_2,\ldots,a_n}(x) = G_{a_1,a_2,\ldots,a_n}(x)|_{a_i\in\{-1,0,1\}}. \tag{10}$$

For example,

$$H_0(x) = \ln x, \quad H_{1,0}(x) = \int_0^x \frac{\mathrm{d}t}{1-t}\ln t, \quad H_{-1,1,0}(x) = \int_0^x \frac{\mathrm{d}t}{t+1} H_{1,0}(t). \tag{11}$$

HPLs have good mathematical properties.

Analytical Results

Combing analytical results of master integrals and IBP relations, we obtain the results for the bare amplitudes, which contain UV divergences. Using the standard procedure of renomarization, we checked that these divergences indeed cancel.

QCD corrections of Γ_t up to NNNLO:

$$\Gamma(t \to Wb) = \Gamma_0 \left[X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \left(\frac{\alpha_s}{\pi}\right)^3 X_3 \right], \tag{12}$$

$$\Gamma_0 = \frac{G_F m_t^3 |V_{tb}|^2}{8\sqrt{2}\pi}. \tag{13}$$

The LO and NLO corrections are

$$\begin{aligned} X_0 &= (2w+1)(w-1)^2, \\ X_1 &= C_F \Bigg(X_0 \left(-2H_{0,1}(w) + H_0(w) H_1(w) - \frac{\pi^2}{3} \right) + \frac{1}{2}(4w+5)(w-1)^2 H_1(w) \\ &\quad + w(2w^2+w-1) H_0(w) + \frac{1}{4}(6w^3 - 15w^2 + 4w + 5) \Bigg) \end{aligned} \tag{14}$$

Analytical Results

According to color structure,

$$\Gamma(t \to Wb) = \Gamma_0 \left[X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \left(\frac{\alpha_s}{\pi}\right)^3 X_3 \right], \tag{15}$$

$$X_2 = C_F (T_R n_l X_l + T_R n_h X_h + C_F X_F + C_A X_A) \tag{16}$$

$$X_l = -\frac{X_0}{3} \left[H_{0,1,0}(w) - H_{0,0,1}(w) - 2H_{0,1,1}(w) + 2H_{1,1,0}(w) - \pi^2 H_1(w) - 3\zeta(3) \right] + g_l(w),$$

$$\begin{aligned} X_F = {} & \frac{1}{12} X_0 [-6 \left(2H_{0,1,0,1}(w) + 6H_{1,0,0,1}(w) - 3H_{1,0,1,0}(w) - 12\zeta(3) H_1(w) \right) - \pi^2 H_{1,0}(w)] \\ & + (X_0 + 4w) \left(-\frac{1}{6} \pi^2 H_{0,-1}(w) - 2H_{0,-1,0,1}(w) \right) \\ & + \frac{1}{12} \left(18w^3 - 3w^2 + 76w + 15 \right) \pi^2 H_{0,1}(w) - \frac{1}{2} \left(4w^3 - 2w^2 + 4w + 3 \right) H_{0,0,0,1}(w) \\ & + \frac{1}{2} \left(4w^3 - 2w^2 + 16w + 3 \right) H_{0,0,1,0}(w) + w \left(2w^2 - 7w - 16 \right) H_{0,0,1,1}(w) \\ & - \frac{1}{2} \left(2w^3 - 11w^2 - 28w - 1 \right) H_{0,1,1,0}(w) + \frac{1}{720} \pi^4 \left(42w^3 - 191w^2 - 328w - 11 \right) + g_F(w). \end{aligned}$$

Leading color dominates the higher order corrections by about 95%.

$$X_3 = C_F\Big[N_c^2 Y_A + \widetilde{Y}_A + \frac{\overline{Y}_A}{N_c^2} + n_l n_h Y_{lh} + n_l\left(N_c Y_l + \frac{\widetilde{Y}_l}{N_c}\right) + n_l^2 Y_{l2} + n_h\left(N_c Y_h + \frac{\widetilde{Y}_h}{N_c}\right) + n_h^2 Y_{h2}\Big]. \tag{17}$$

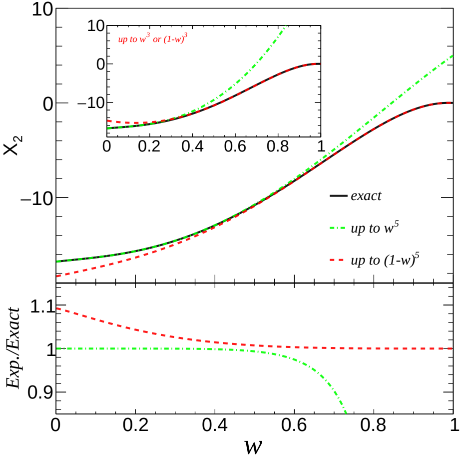
The leading color coefficients are expressed in terms of HPLs. In the $\omega \to 0$ limit,

$$\begin{aligned} Y_A = & \Big[\frac{203185}{41472} - \frac{12695\pi^2}{1944} - \frac{4525\zeta(3)}{576} - \frac{1109\pi^4}{25920} + \frac{37\pi^2\zeta(3)}{36} + \frac{1145\zeta(5)}{96} + \frac{47\pi^6}{2835} - \frac{3\zeta(3)^2}{4}\Big] \\ & + w\Big[-\frac{157939}{2304} + \frac{140863\pi^2}{20736} + \frac{5073\zeta(3)}{64} - \frac{14743\pi^4}{6480} - \frac{169\pi^2\zeta(3)}{72} - \frac{45\zeta(5)}{16} + \frac{3953\pi^6}{22680} - \frac{15\zeta(3)^2}{4} \\ & + w^2\Big[\ \ (w)\left(\frac{851099}{27648} - \frac{5875\pi^2}{2304} - \frac{33\zeta(3)}{8} + \frac{\pi^4}{10}\right) - \frac{82610233}{331776} + \frac{799511\pi^2}{27648} \\ & + \frac{4093\zeta(3)}{32} - \frac{5987\pi^4}{2880} - \frac{91\pi^2\zeta(3)}{16} - \frac{275\zeta(5)}{8} + \frac{347\pi^6}{3024} - \frac{9\zeta(3)^2}{8}\Big] + \mathcal{O}(w^3). \end{aligned} \tag{18}$$

Cross Check

Two different gauges for the W boson propagator have been used.

The result expanded in $w = 0$ and $w = 1$ $(w = m_W^2/m_t^2)$ coincides with [Blokland, Czarnecki, Slusarczyk, Tkachov 2004 2005].



Off-Shell W Boson effect

Including the W boson width of $\Gamma_W = 2.085$ GeV, Γ_t become [Jezabek, Kuhn 1989]

$$\tilde{\Gamma}_t \equiv \Gamma(t \to W^* b) = \frac{1}{\pi} \int_0^{m_t^2} dq^2 \frac{m_W \Gamma_W}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2} \Gamma_t(q^2/m_t^2), \tag{19}$$

In the narrow width limit, $\Gamma_W \to 0$, $\tilde{\Gamma}_t \to \Gamma_t$.

$$\tilde{\Gamma}_t = \Gamma_0 \left[\tilde{X}_0 + \frac{\alpha_s}{\pi} \tilde{X}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \tilde{X}_2 + \left(\frac{\alpha_s}{\pi}\right)^3 \tilde{X}_3 \right], \quad r = \frac{\Gamma_W}{m_W}, \quad w = \frac{m_W^2}{m_t^2}$$

$$\tilde{X}_0 = \frac{1}{2\pi}\big(- (2(r-i)w - i((r-i)w+i)^2 G(w+irw,1)) \\ - ((r+i)w-i)^2 2(r+i)w + iG(w-irw,1) - 4r(1-2w)w), \tag{20}$$

$$\tilde{X}_1 = \frac{1}{18\pi}((r+i)w-i)(2(4\pi^2-9)(r+i)^2w^2 + (4\pi^2-27)(1-ir)w + 4\pi^2 - 15)G(w-iw,1) \\ + (r-i)w-i)(2(4\pi^2-9)(r-i)^2w^2 + (4\pi^2-27)(1+ir)w + 4\pi^2 - 15)G(w+iw,1) \\ + \cdots) \tag{21}$$

Input parameters from [P.D.G 2022]

$$\begin{aligned} m_t &= 172.69\ \textit{GeV}, \quad m_b = 4.78\ \textit{GeV}, \\ m_W &= 80.377\ \textit{GeV}, \quad \Gamma_W = 2.085\ \textit{GeV}, \\ m_Z &= 91.1876\ \textit{GeV}, \quad G_F = 1.16638 \times 10^{-5}\ \textit{GeV}^{-2}, \\ |V_{tb}| &= 1, \quad \alpha_s(m_Z) = 0.1179. \end{aligned} \tag{22}$$

$\Gamma_t^{(0)} = 1.486$ GeV with $m_b = 0$ and on-shell W.

$$\begin{aligned} \Gamma_t = \Gamma_t^{(0)}[& (1 + \delta_b^{(0)} + \delta_W^{(0)}) \\ & + (\delta_b^{(1)} + \delta_W^{(1)} + \delta_{EW}^{(1)} + \delta_{QCD}^{(1)}) \\ & + (\delta_b^{(2)} + \delta_W^{(2)} + \delta_{EW}^{(2)} + \delta_{QCD}^{(2)} + \delta_{EW\times QCD}^{(2)}) \\ & + (\delta_b^{(3)} + \delta_W^{(3)} + \delta_{EW}^{(3)} + \delta_{QCD}^{(3)} + \delta_{EW\times QCD}^{(3)})] \end{aligned} \tag{23}$$

Numerical Results

Corrections in percentage ($\%$) normalized by the LO width $\Gamma_t^{(0)} = 1.486$ GeV with $m_b = 0$ and on-shell W.

	$\delta_b^{(i)}$	$\delta_W^{(i)}$	$\delta_{EW}^{(i)}$	$\delta_{QCD}^{(i)}$	Γ_t [GeV]
LO	-0.273	-1.544	—	—	1.459
NLO	0.126	0.132	1.683	-8.575	$1.361^{+0.0091}_{-0.0130}$
NNLO	$*$	0.030	$*$	-2.070	$1.331^{+0.0055}_{-0.0051}$
N^{3}LO	$*$	0.009	$*$	-0.667	$1.321^{+0.0025}_{-0.0021}$

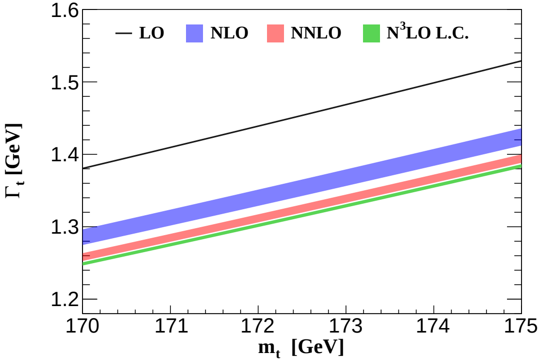
QCD corrections are dominant. The NLO EW correction is comparable to the NNLO QCD correction.

The off-shell W boson effect at NLO is $\sim 0.1\%$.

The b quark mass correction at NLO is not severely suppressed compared to the LO due to the large logarithms induced by soft quarks.

Theoretical Uncertainties

QCD renormalization scale $\mu \in [m_t/2, 2m_t]$, the variation is about $\pm 0.8\%$, $\pm 0.4\%$ and $\pm 0.2\%$ at NLO, NNLO and NNNLO, respectively.



$$\Gamma_t(m_t) = 0.027037 \times m_t - 3.34801 \;\, GeV. \tag{24}$$

Theoretical Uncertainties

The uncertainties at NNLO from $\alpha_s(m_Z) = 0.1179 \pm 0.0009$ and $m_W = 80.377 \pm 0.012$ GeV are 0.1% and 0.01%.

The deviation between the α and G_F scheme in the EW correction is 0.1% at NLO.

The missing NNLO EW as well as the mixed *EW* $\times$ *QCD* corrections are estimated to be 0.1%.

Considering all the possible uncertainties, the uncertainty at NNLO is less than 0.3%.

Mathematica program TopWidth can be downloaded from

https://github.com/haitaoli1/TopWidth.

```
<< TopWidth`

        (****** TopWidth-1.0 ******)
    Authors: Long-Bin Chen, Hai Tao Li, Jian Wang, YeFan Wang
    TopWidth[QCDorder, mbCorr, WwidthCorr, EWcorr, mu] is provided for top width calculations
    Please cite the paper for reference: arXiv:2212.06341

*-*-*-*-*-* HPL 2.0 *-*-*-*-*-*

Author: Daniel Maitre, University of Zurich
Rules for minimal set loaded for weights: 2, 3, 4, 5, 6.
Rules for minimal set for + - weights loaded for weights: 2, 3, 4, 5, 6.
Table of MZVs loaded up to weight 6
Table of values at I loaded up to weight 6
$HPLFunctions gives a list of the functions of the package.
$HPLOptions gives a list of the options of the package.
More info in hep-ph/0507152, hep-ph/0703052 and at
 http://krone.physik.unizh.ch/~maitreda/HPL/

(* SetParameters[mt, mb, mw, Wwidth, mz, GF] *)
(* If the parameters are not set by the users the code will use the default ones *)
```

SetParameters$\left[\frac{17269}{100}, \frac{478}{100}, 80377/1000, 2085/1000, 911876/10000, 11663788\times 10^{-12}\right]$

```
(* NNLO decay width *)
```

TopWidth$\left[2, 1 \text{ (* with mb effects *)}, 1 \text{ (* with Γw effects*)}, 1 \text{ (* with NLO EW effects *)}, \frac{17269}{100}\right]$

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1.33051
```

Semileptonic $b \to u$ decays: dilepton invariant mass spectrum

$$\frac{d\Gamma(b \to X_u e \bar{\nu}_e)}{dq^2} = \Gamma_b^{(0)} \sum_{i=0}^{3} \left(\frac{\alpha_s}{\pi}\right)^i X_i\left(\frac{q^2}{m_b^2}\right). \quad (25)$$

with $\Gamma_b^{(0)} = G_F^2 |V_{ub}|^2 m_b^3 / 96\pi^3$.

The b-quark semileptonic decay width can be expanded in α_s,

$$\Gamma(b \to X_u e \bar{\nu}_e) = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192\pi^3}\left[1 + \sum_{i=1} \left(\frac{\alpha_s}{\pi}\right)^i b_i\right]. \quad (26)$$

b_1 and b_2: [Ritbergen,1999]. $b_3 = (-202 \pm 20) C_F$: expanding in $\delta = 1 - m_u/m_b$ [Fael, Schonwald, Steinhauser, 2020]. Fermionic contribution: [Fael, Usovitsch, 2023]

$$\begin{aligned} b_3 = C_F \Big[& N_c^2\left(\frac{9651283}{82944} - \frac{1051339\pi^2}{62208} - \frac{67189\zeta(3)}{864} + \frac{4363\pi^4}{6480} + \frac{59\pi^2\zeta(3)}{32} + \frac{3655\zeta(5)}{96} - \frac{109\pi^6}{3780}\right) \\ & + n_l N_c\left(-\frac{729695}{27648} + \frac{48403\pi^2}{15552} + \frac{1373\zeta(3)}{108} + \frac{133\pi^4}{1728} - \frac{13\pi^2\zeta(3)}{72} - \frac{125\zeta(5)}{24}\right) \\ & + n_l^2\left(\frac{24763}{20736} - \frac{1417\pi^2}{15552} - \frac{37\zeta(3)}{216} - \frac{121\pi^4}{6480}\right) + \textit{subleading color}\Big] \\ = & (-195.3 \pm 9.8) C_F . \end{aligned}$$

Summary

We provide the analytical result of top-quark width at NNNLO in QCD.

The analytical result can be used to perform both fast and accurate evaluations.

The most precise top-quark width is predicted to be 1.321 GeV for mt = 172.69 GeV with a total theoretical uncertainty less than 0.3%.

The dilepton invariant mass spectrum in semileptonic $b \to u$ decays is also obtained at NNNLO in QCD.

Backup