A+AMFlow for master integrals in singular kinematics

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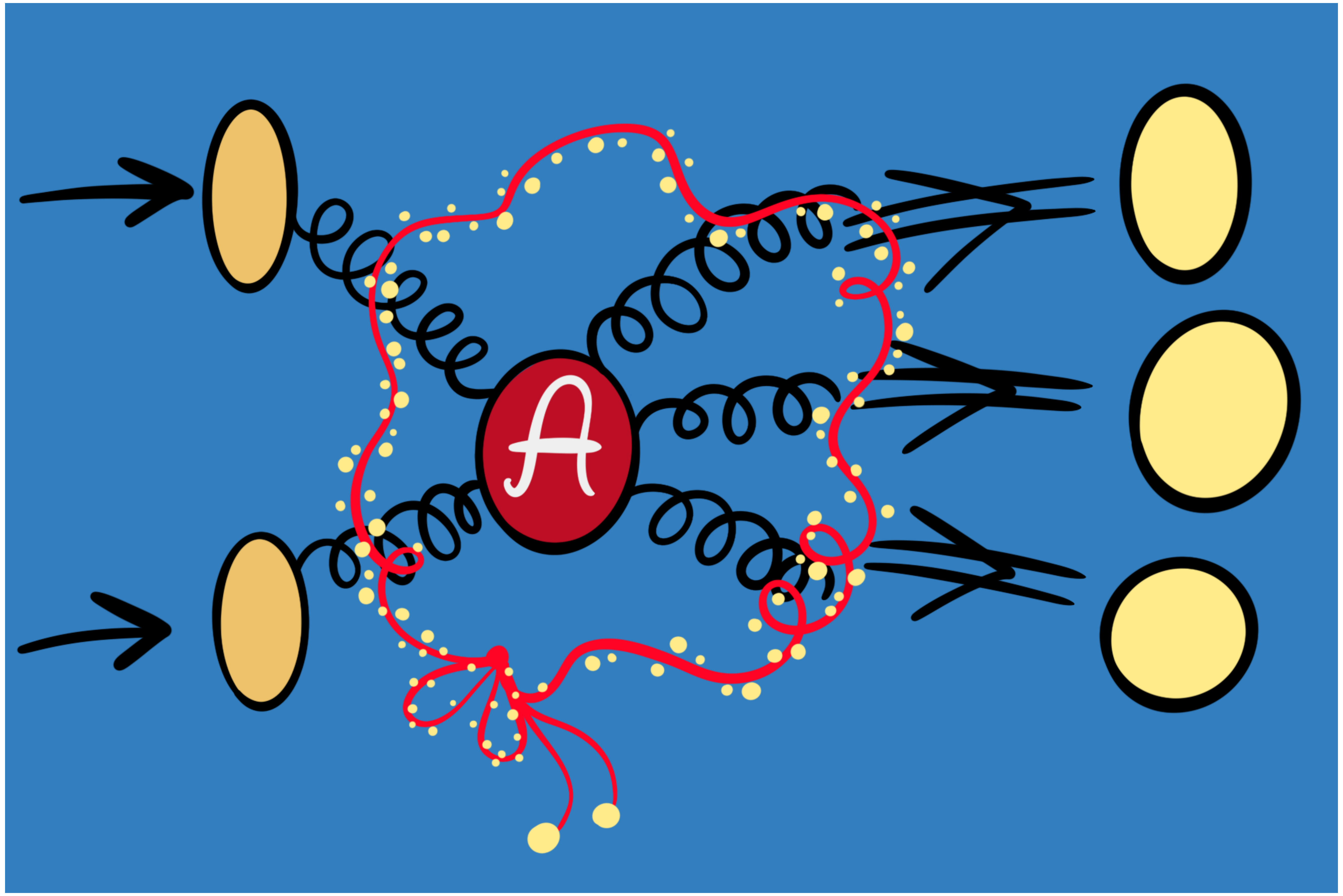
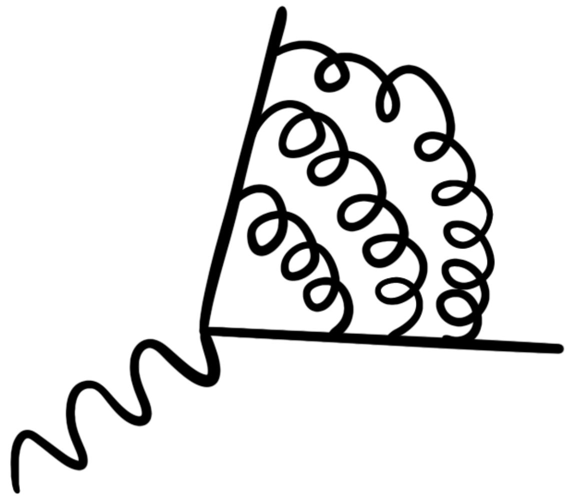
Recipe for a theoretical prediction

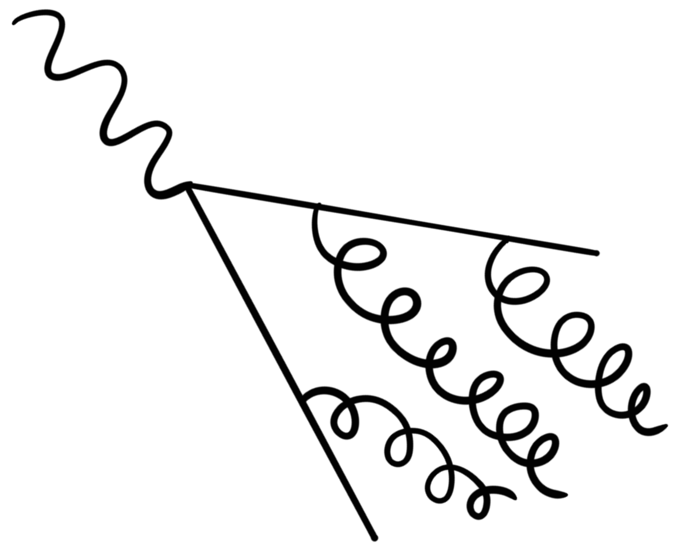


- PDFs to describe the proton structure
- hard scattering
- radiation and evolution to hadronic states

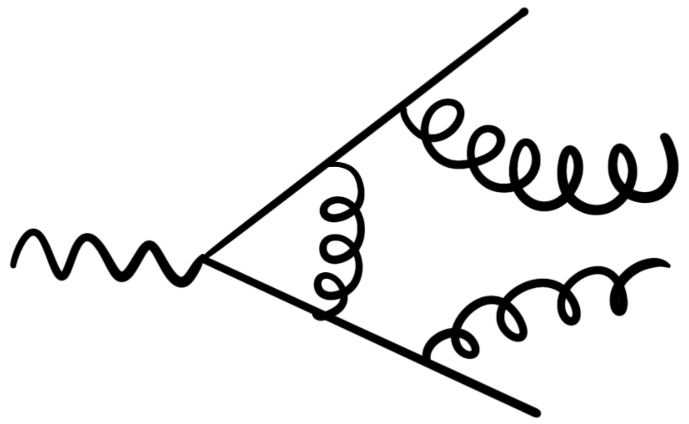
BUILDING A N^3LO, n—PARTONS FINAL STATE



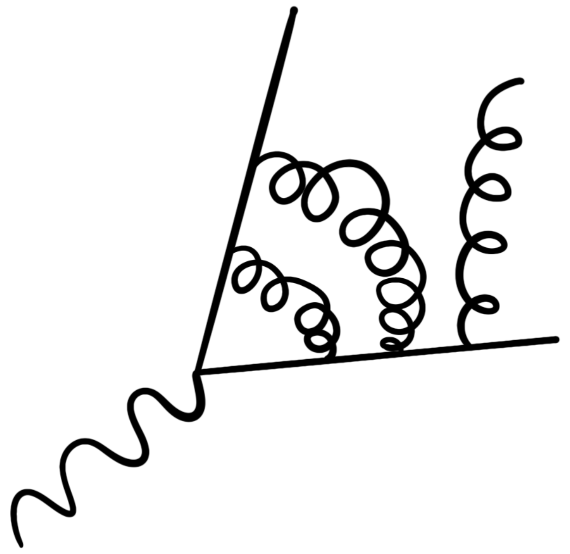
- perturbation theory: series expansion in power of α_S
- n-partons final state
- contribution from all matrix elements of the same order in α_S

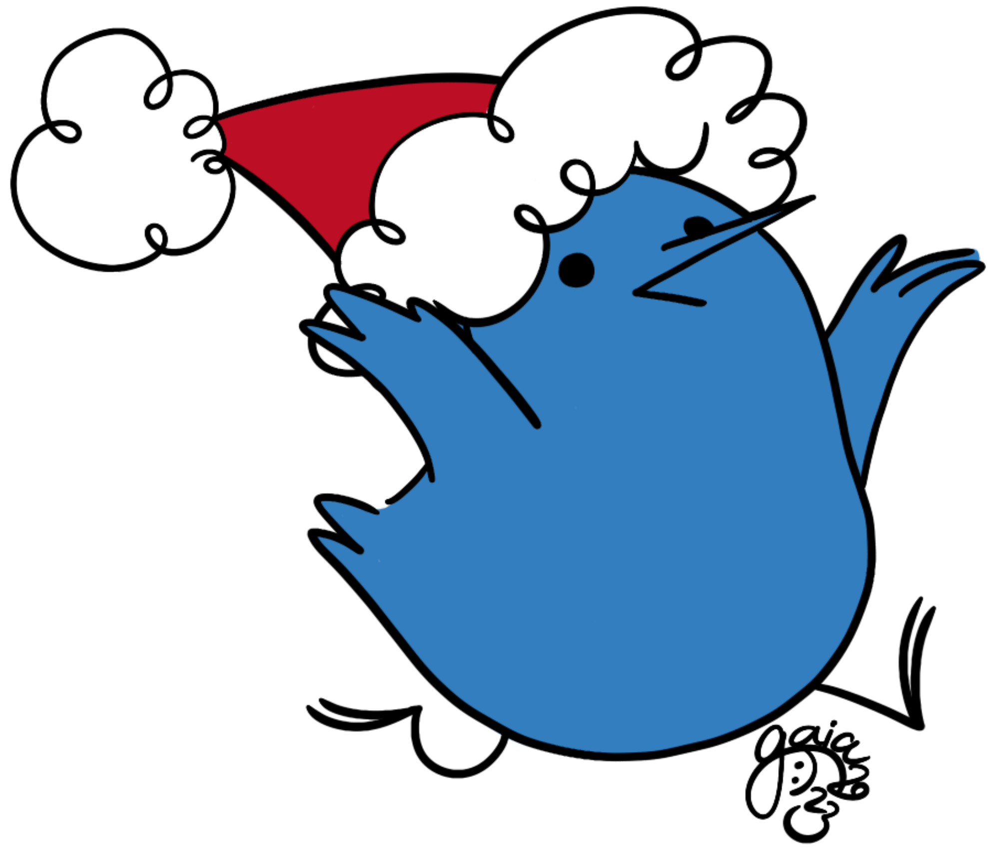


we need **several ingredients:**



- n—partons @ 3-loops → **V V V**
- $(n+1)$—partons @ 2-loops → **R V V**
- $(n+2)$—partons @ 1-loop → **R R V**
- $(n+3)$—partons @ tree lvl. → **R R R**





IR divergencies

- when partons become **soft** or **collinear**
- **subtraction schemes**

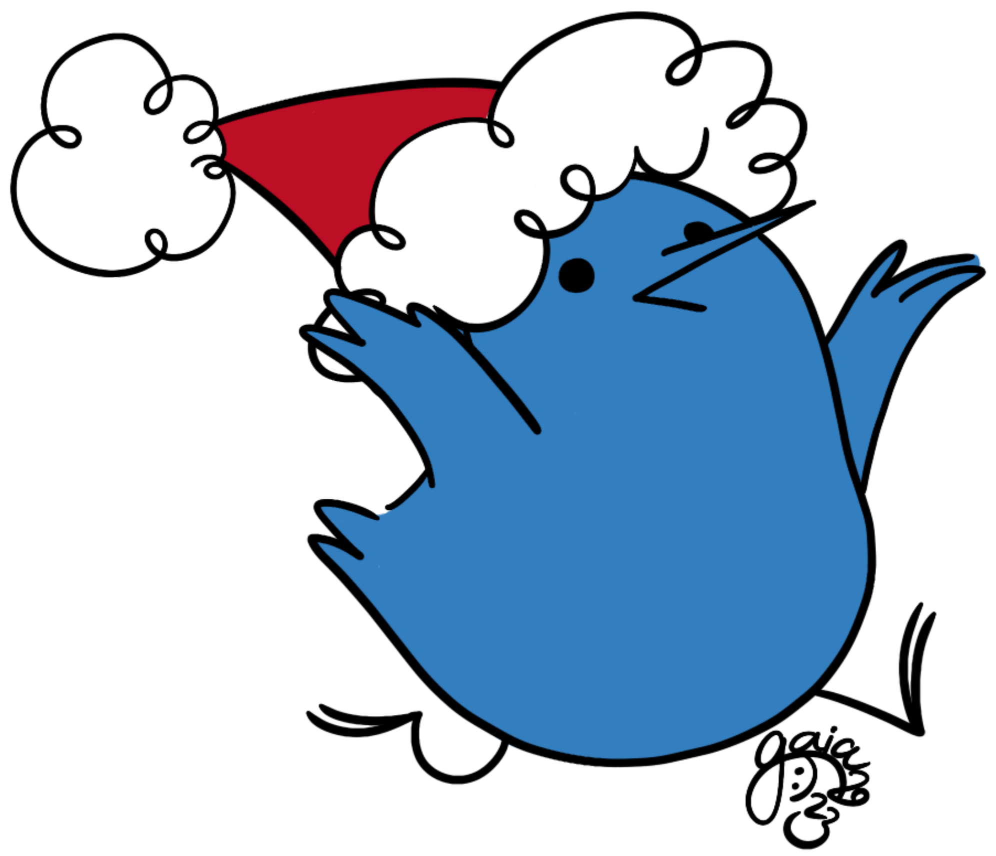
initial — final antennae

Hard radiators both in the **initial—state** and **final—state** partons

rederivation of NNLO $2 \to 3, 2 \to 2$ IF antennae

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (2009)

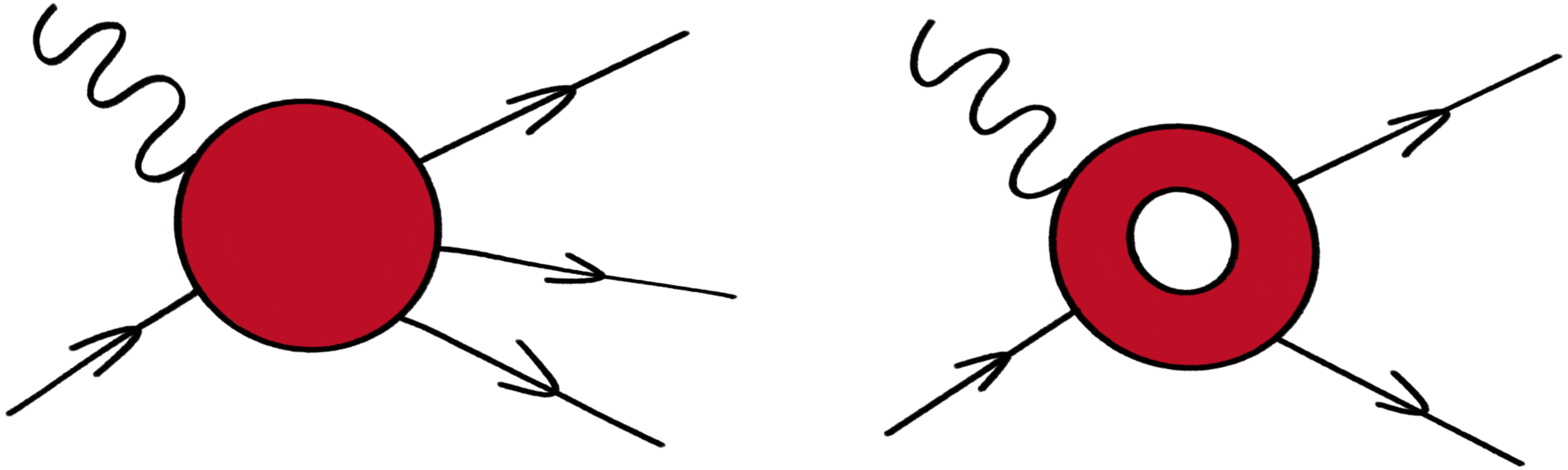
- known but required a lot of hands-on labour
- go higher in the transcendental weight [N3LO]
- develop a more **automated workflow**



IF antennae building blocks:
phase space integrals for DIS

NNLO DIS kinematics

$$q_1 + q_2 \to p_1 + p_2 + (p_3)$$



$2 \to 3$ $\quad$ $2 \to 2$

kinematics

- $q_2^2 = -Q^2 < 0$
- $q_1^2 = 0$
- $p_i^2 = 0, \quad i = 1,2,3$

$$Q^2 = 1$$

invariants

- $s = (q_1 + q_2)^2$
- $z = \dfrac{1}{2q_1q_2} \longrightarrow s = \dfrac{(1-z)}{z}$

We're interested in:
phase space ints for $2 \to 2$ and $2 \to 3$ DIS

Reverse Unitarity

- integration over n-particles phase space

$$d\Pi_n = \prod_{i=1}^{n} \frac{d^d p_i}{(2\pi)^d} \delta^+(p_i^2)\, \delta^d\left(q_1 + q_2 - \sum_i^n p_i\right)$$

- reverse unitarity: Anastasiou, Melnikov (2002)

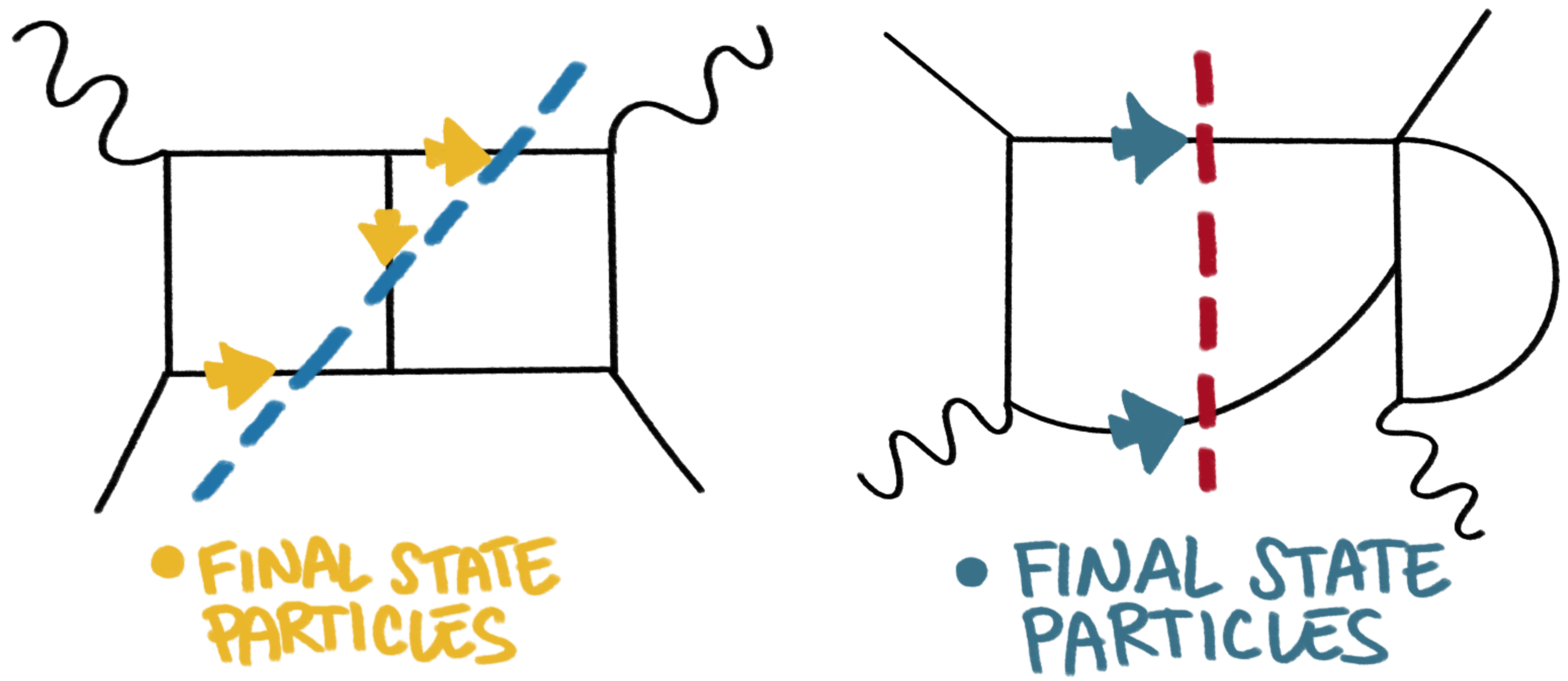
phase space → (cut) loops

via the following identification

$$-2\pi i \delta^+(p_i^+) = \frac{1}{p_i^2 + i0} - \frac{1}{p_i^2 - i0} = \frac{1}{[p^2]_{cut}}$$

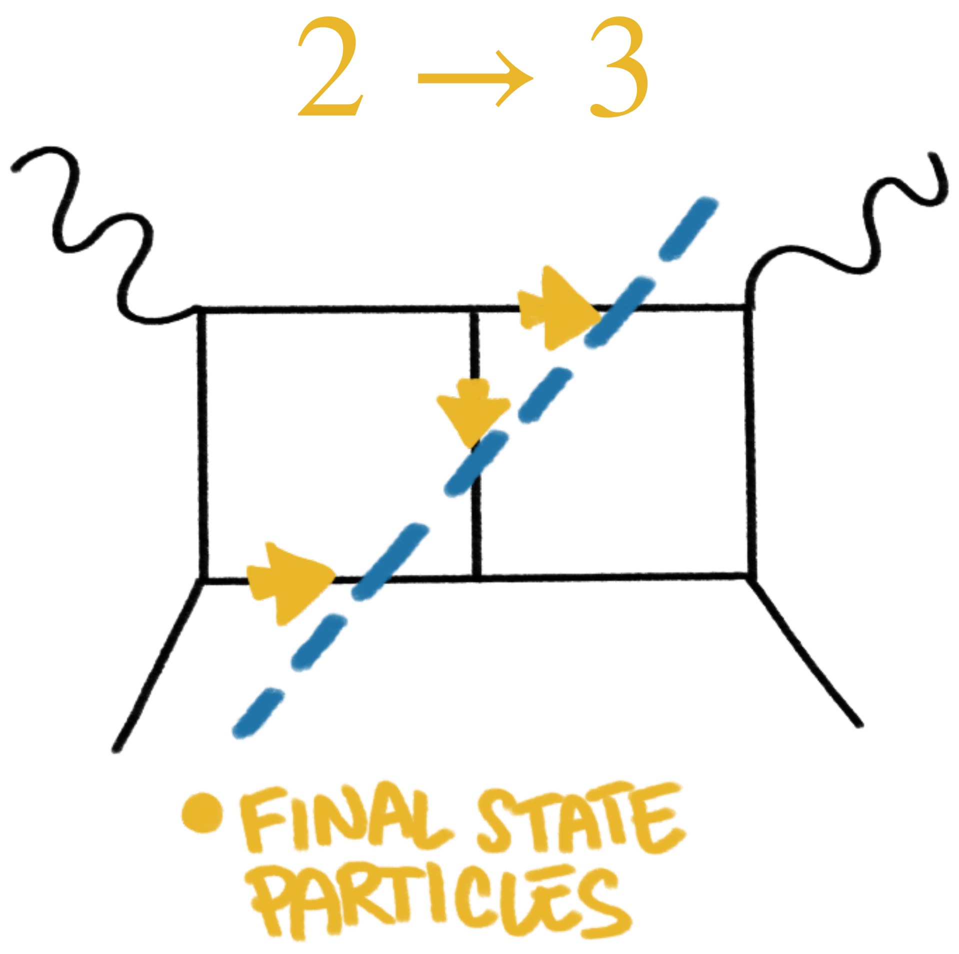
loop—calculations tools!

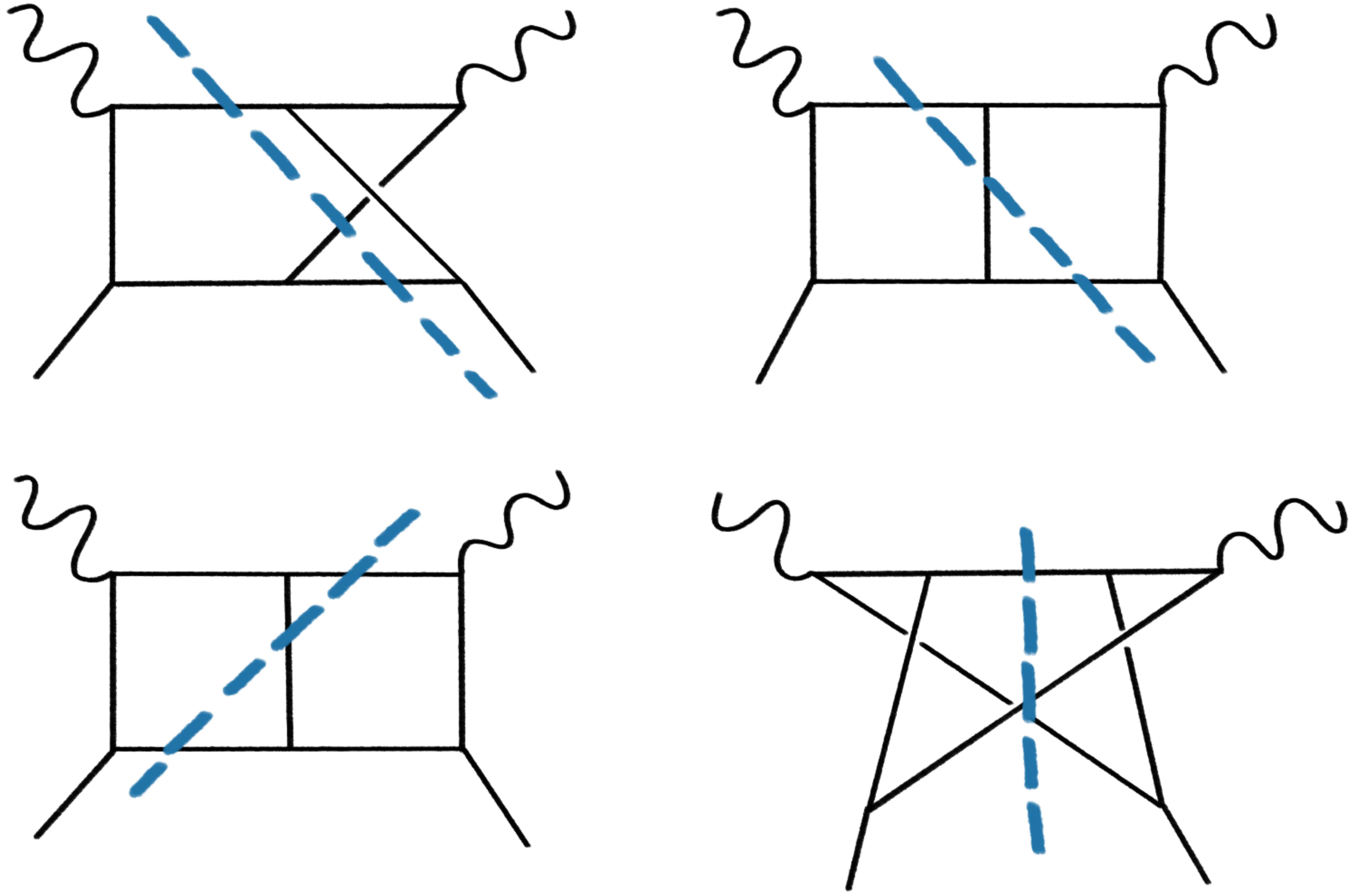
- Write down the forward DIS scattering process at NNLO
- Find physical cuts
    - 2 cuts → phase space $2 \to 2$ @ 1loop
    - 3 cuts → phase space $2 \to 3$ @ tree level



- write all integrals as a function of a minimal, linearly independent set of master integrals using IBP identities

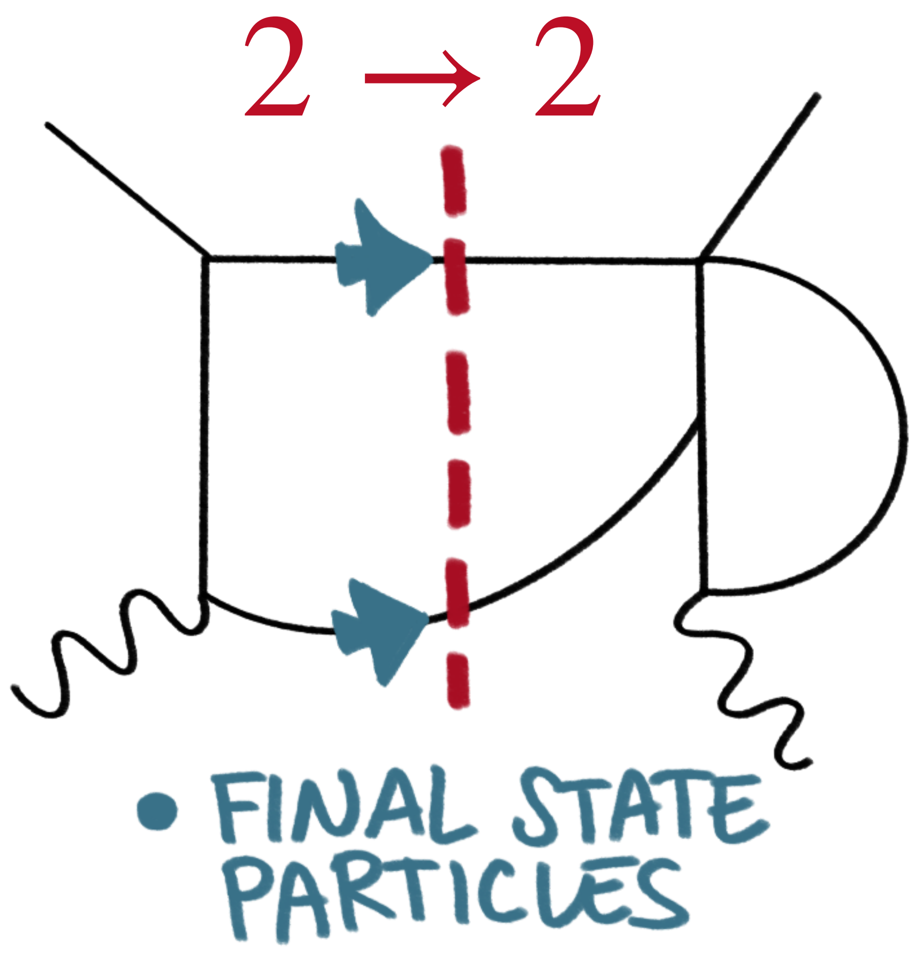
RR master integrals families

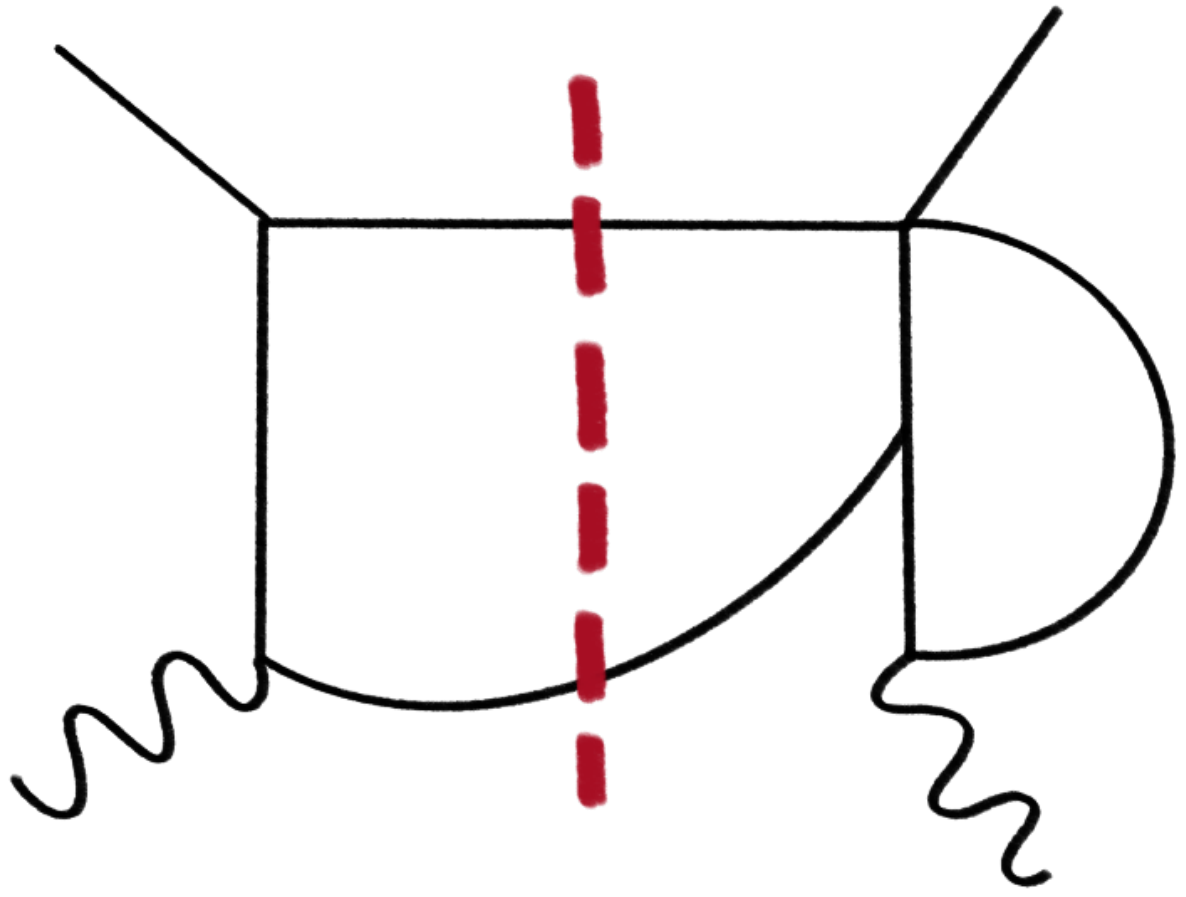


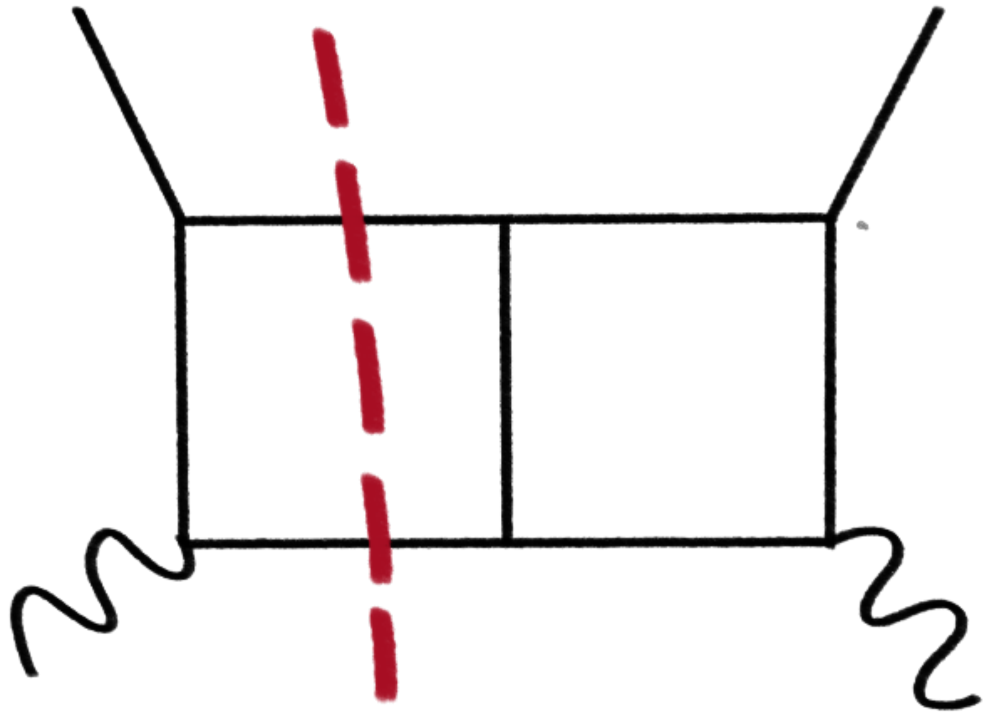


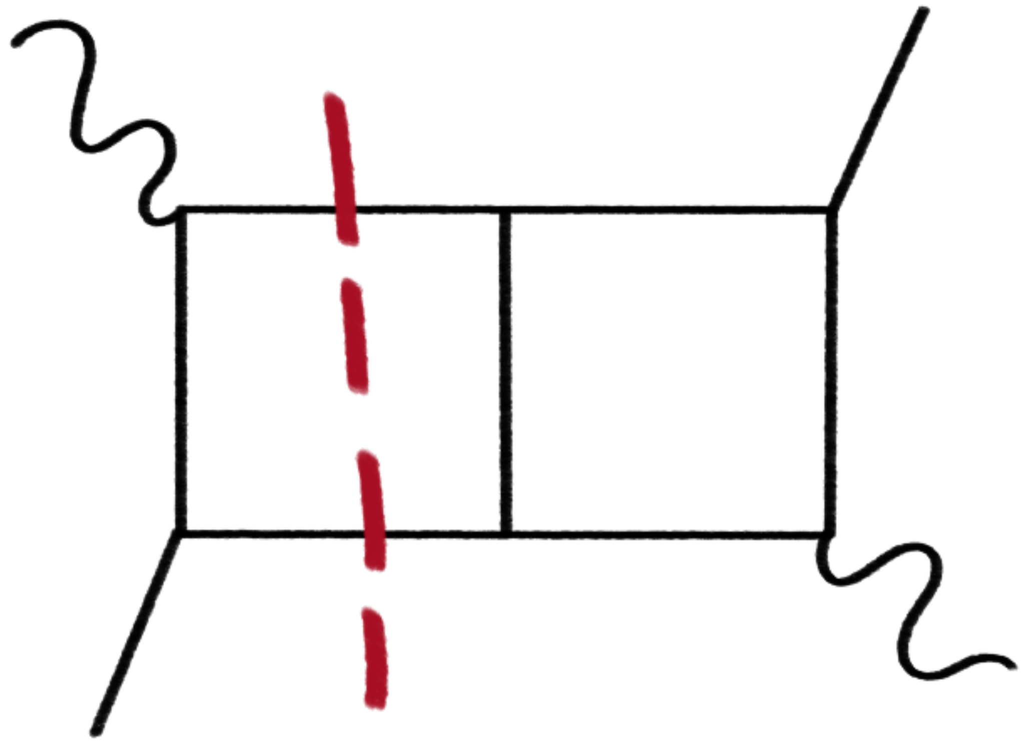
Canonical DE & solution in terms of HPLs

RV master integrals families









Canonical DE & solution in terms of HPLs

Computation of MIs

Can be done

- analytically in terms of special functions (MPLs, elliptic functions, …)
- numerically (Sector decomposition, AMFlow)

most effective method is **Differential Equations (DE)**

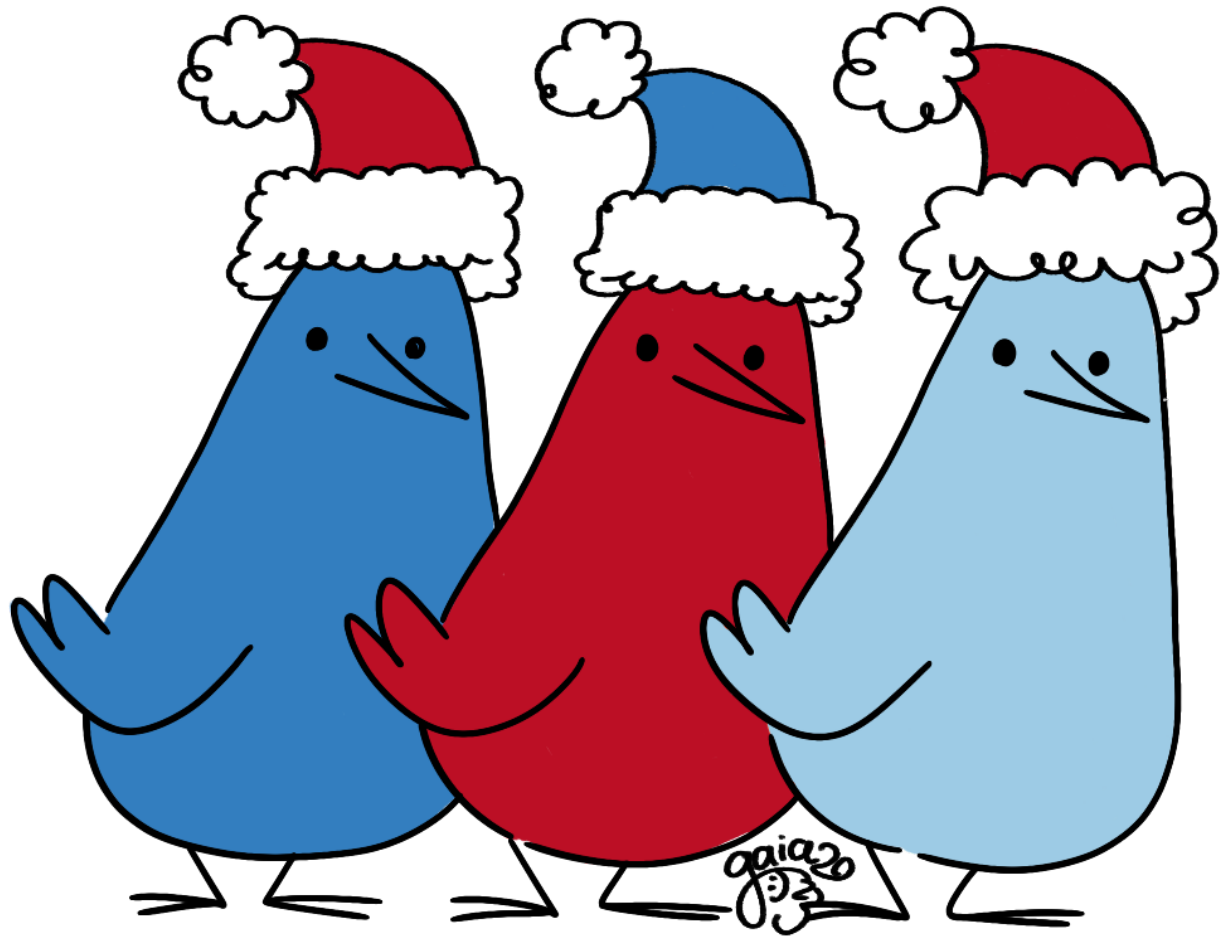
- derivative of MIs with respect to external invariants and/or internal masses
- reduce it again to MIs
- obtain a **system of DEs for the MIs**

$$\partial_z \vec{g} = M \cdot \vec{g}$$

DEs for master integrals

How to solve a differential equation:

- Generic solution
- Boundary condition



- System of DEs for the master integrals
- Can be put in canonical form: $\partial_z \vec{g} = \epsilon A \cdot \vec{g}$ Henn (2013)
- Generic solution in terms of iterated integrals
- In our calculations: only HPLs!

ϵ-dependence is factored out

Boundary conditions

we look at the kinematic limit $z \to 1 \Rightarrow s \to 0$ **(soft limit)**

RR

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

- extract the **leading behavior** of the MIs
- rescaling the integrals w.r.t. their leading behavior → regularity
- imposing that in this limit the terms $\log(1-z)$ and poles in $(1-z)$ vanish
- **relations between boundaries** of different MIs

RV

$$I_i^{RV} \sim (1-z)^{m_i - 2\epsilon} \sum_j d_j(\epsilon)(1-z)^j + (1-z)^{l_i - \epsilon} \sum_j e_j(\epsilon)(1-z)^j, \quad m_i, l_i \in \mathbb{Z}$$

- extract the **leading behavior** of the MIs
- rescaling the integrals w.r.t. their leading behavior → regularity
- imposing that in this limit the terms $\log(1-z)$ vanish
- **relations between boundaries** of different MIs

Now we need to fix the remaining boundaries!

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

We need $c_0(\epsilon)$

$$I_i^{RV} \sim (1-z)^{m_i - 2\epsilon} \sum_j d_j(\epsilon)(1-z)^j + (1-z)^{l_i - \epsilon} \sum_j e_j(\epsilon)(1-z)^j, \quad m_i, l_i \in \mathbb{Z}$$

We need $d_0(\epsilon)$, $e_0(\epsilon)$

Wishlist:

- Analytic boundaries
- General algorithm to obtain them

AMFlow framework Liu, Ma (2022)

- Fully **numerical**
- Evaluate FI at any loop order in a **non-singular** point

Outline:

- Add aux mass η^2 to some propagators $\rightarrow$ auxiliary family

$$I^{phys}(\epsilon, \vec{z}) \rightarrow I^{aux}(\epsilon, \vec{z}, \eta^2)$$

- Derive DE with respect to the mass

$$\partial_{\eta^2} \vec{I}^{aux} = A_\eta \cdot \vec{I}^{aux}$$

- "Flow" $\eta^2 \rightarrow 0$ for physical solution: $\lim_{\eta^2 \rightarrow 0} I^{aux} = I^{phys}$
- All implemented in a MATHEMATICA package


ANALYTIC
AUXILIARY
MASS
FLOW
gaia

AAMFlow

GF, Gehrmann, Schönwald (to appear)

- Fully **analytical** → can be used near **singular** points

Outline:

- Add aux mass η^2 to **chosen** propagators:
  - limits in kinematical variable and η^2 need to **commute**
- Derive DE with respect to η^2 & solve it
- Fix constants of integration in $\eta^2 \to \infty$ limit (easy!)
- "Flow" to $\eta^2 \to 0$ for physical solution:
  - **method of regions** to extract the physical solution

We look at the boundaries in $z \to 1$: kinematical endpoint singularity

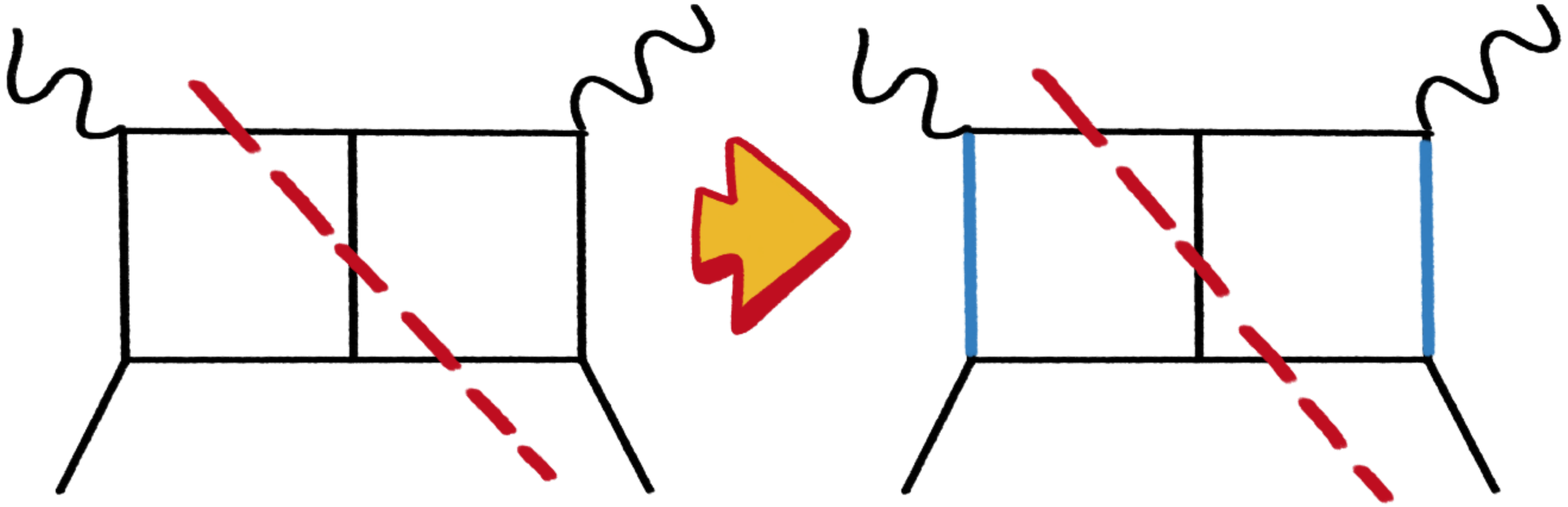
RECIPE:

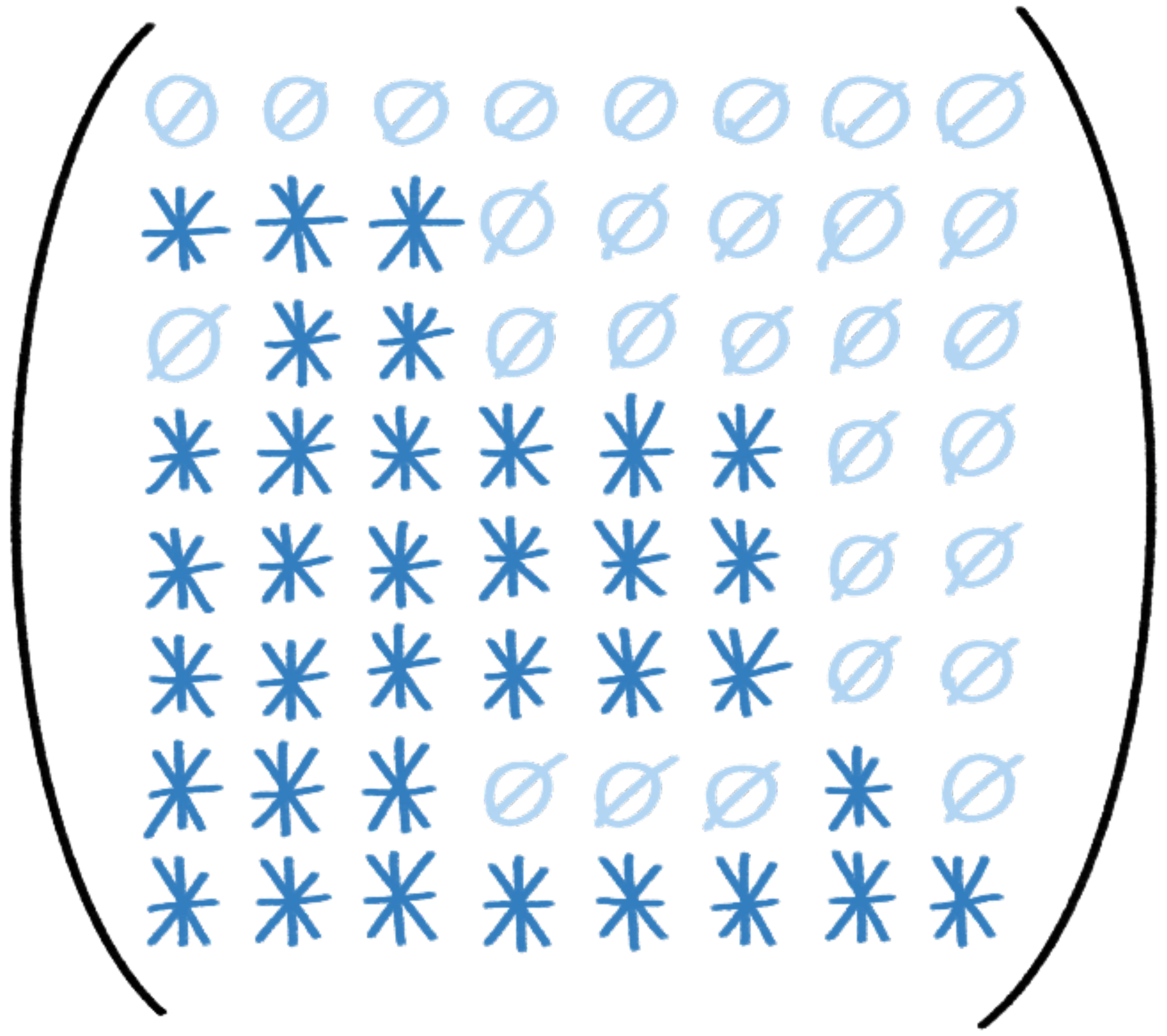
- choose a family for which to calculate the boundaries
- choose propagators to which add an auxiliary mass
- derive DE with respect to $u = 1/\eta^2$
- fix constants of integration in $\lim u \to 0$
- limit $\eta^2 \to 0$ & disentangle regions
- extract physical region

Proof of concept

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

- We need $c_0(\epsilon)$ of this top sector:





8 master integrals

- Add auxiliary mass → auxiliary topology
- Differential equation wrt $u = 1/\eta^2$ for the $c_0(\epsilon)$

Intermezzo: large mass limit

Beneke, Smirnov (1997)

- Depends on scaling of loop moms

soft $k \sim \mathcal{O}(1)$ or large $k \sim \mathcal{O}(\eta)$

- **SOFT propagators:**

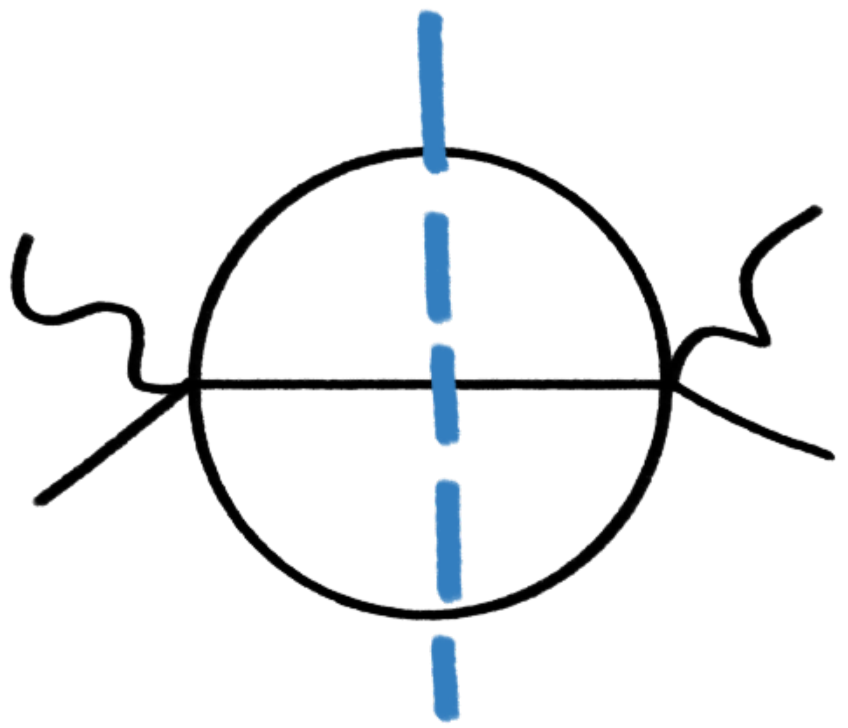
$$\frac{1}{(k+p)^2 - \eta^2} \sim -\frac{1}{\eta^2}$$

- **LARGE propagators:**

$$\frac{1}{(k+p)^2 - \kappa\eta^2} \sim -\frac{1}{k^2 - \eta^2}, \quad \kappa \in \{0,1\}$$

Large mass limit: RR ints

- Loop momentum scales **only soft**
  - **“Pinch” propagators with auxiliary mass**
- Example of boundaries



+ topologies reducible to it!

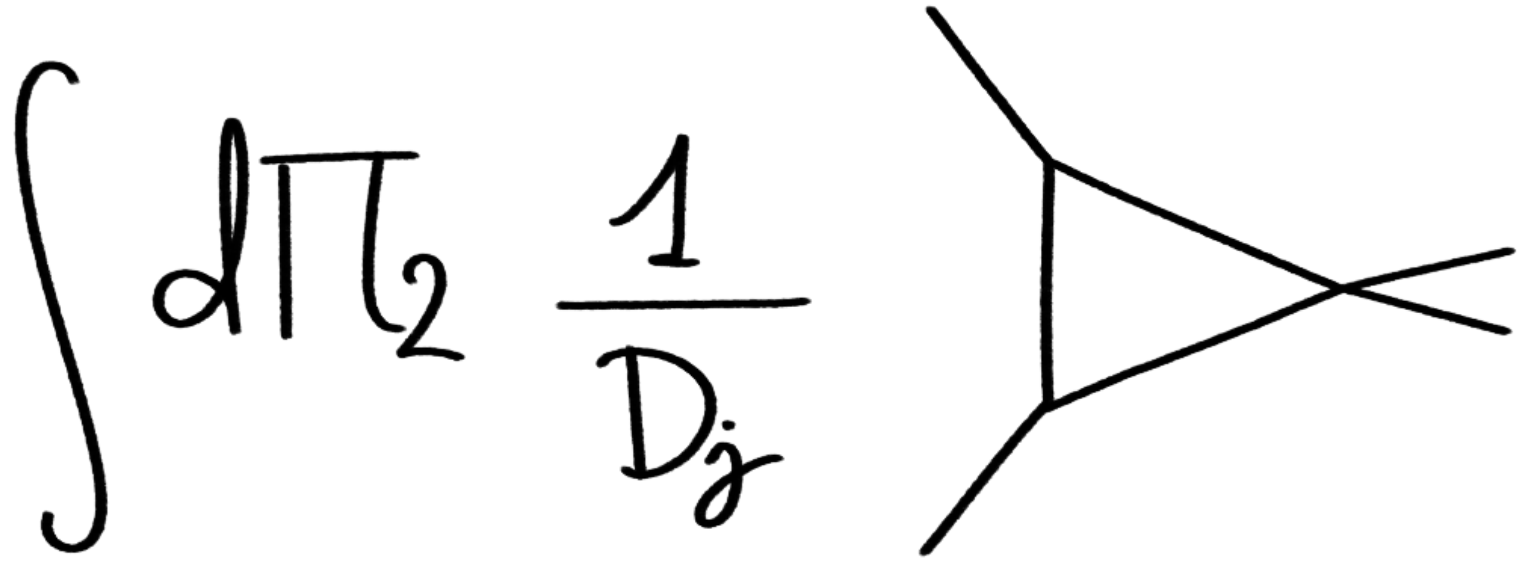
Integrated $2 \to 3$ phase space

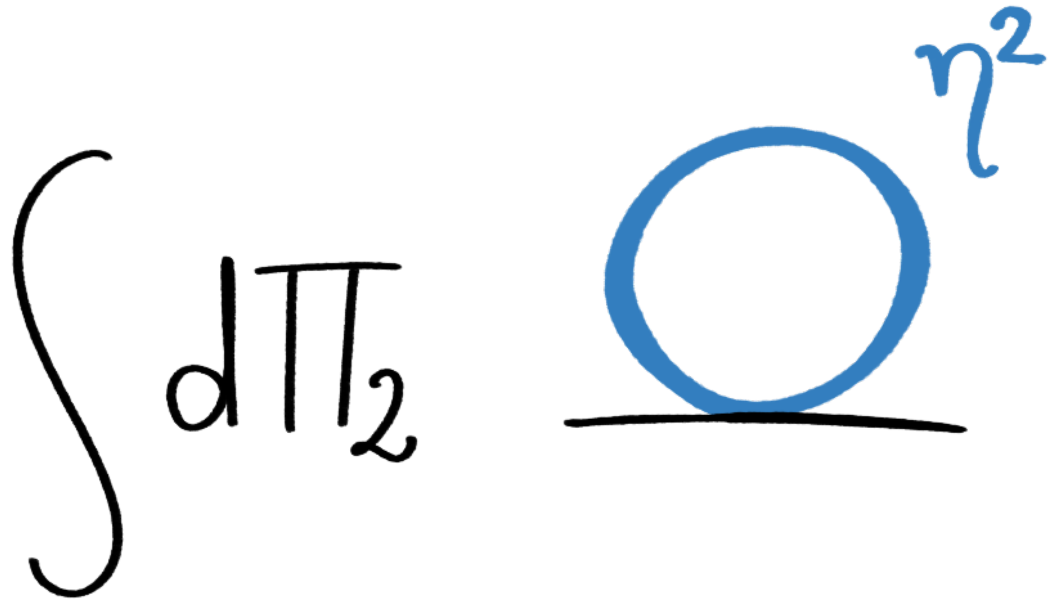
Large mass limit: RV ints

- Loop momentum scales **soft** or **large**
- We have two regions

$$\lim_{\eta^2\to\infty} \vec{I}^{aux}_{RV} = \lim_{\eta^2\to\infty, k\sim SOFT} \vec{I}^{aux}_{RV} + \lim_{\eta^2\to\infty, k\sim LARGE} \vec{I}^{aux}_{RV}$$

$k \sim SOFT$ $\qquad$ $k \sim LARGE$





- Most complicated soft region
- D_j depends only on kinematics

- All large regions are massive tadpoles

Flow to vanishing auxiliary mass

We have the solution of $I^{aux}(\eta^2)$ $\quad \lim_{\eta^2 \to 0} I^{aux} = I^{phys}$

We can take naively the limit $\eta^2 \to 0$ in our solution and obtain this expansion:

$$c_0(\eta, \epsilon) = \sum_{k=min}^{\infty} \epsilon^k \left[r_{k,0} + \sum_{m=1}^{k} r_{k,m} \log^m(\eta) \right]$$

$r_{k,m}$ known!

But we also know the analytic structure of the limit

$$c_0(\eta, \epsilon) = d_0(\epsilon) + \eta^{-\epsilon} d_1(\epsilon) + \eta^{-2\epsilon} d_2(\epsilon) + \mathcal{O}(\eta)$$

Hard region = physical region

$$c_0(\eta,\epsilon) = d_0(\epsilon) + \eta^{-\epsilon} d_1(\epsilon) + \eta^{-2\epsilon} d_2(\epsilon) + \mathcal{O}(\eta)$$

$\epsilon-$expansion gives:

$$\left(d_R = \sum_{k=min}^{\infty} \epsilon^k d_R^{(k)}, \quad R = 0,1,2 \right)$$

$$\begin{aligned} c_{ij}^{(l)} = &\ d_0^{(0)} + d_1^{(0)} + d_2^{(0)} \\ &+ \epsilon \left(d_0^{(1)} + \left(-d_1^{(0)} - 2d_2^{(0)}\right) \log(\eta) + d_1^{(1)} + d_2^{(1)} \right) \\ &+ \epsilon^2 \left(d_0^{(2)} + d_1^{(2)} + d_2^{(2)} + \frac{1}{2}\left(d_1^{(0)} + 4d_2^{(0)}\right) \log^2(\eta) + \left(-d_1^{(1)} - 2d_2^{(1)}\right) \log(\eta) \right) \\ &+ \mathcal{O}(\epsilon^3) \end{aligned}$$

Compare this with $$c_0(\eta,\epsilon) = \sum_{k=min}^{\infty} \epsilon^k \left[r_{k,0} + \sum_{m=1}^{k} r_{k,m} \log^m(\eta) \right]$$

⇒ extract hard region: all the $d_0^{(k)}$

We can obtain e.g. $d_0^{(0)}$ by comparing the two limits

$$c_0(\eta,\epsilon) = r_{0,0} + \ldots + \epsilon r_{1,1}\log(\eta) + \ldots + \epsilon^2 r_{2,2}\log^2(\eta)$$

$r_{0,0}, r_{1,1}, r_{2,2}$ known!

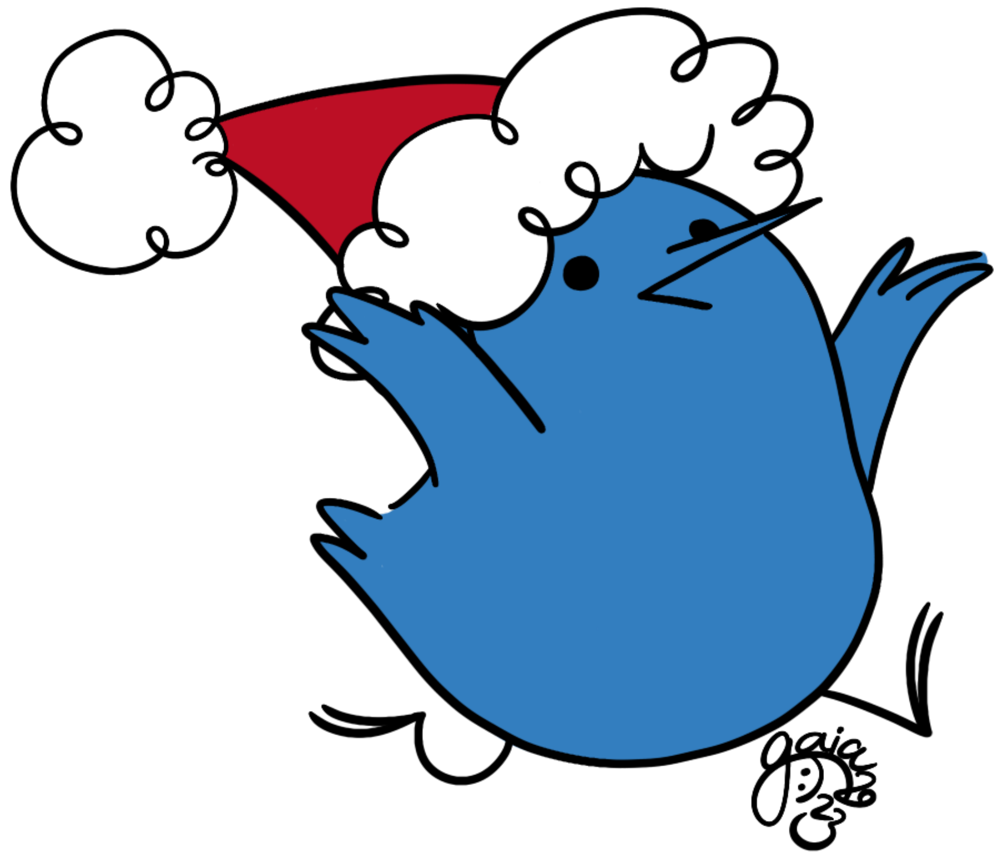
$$\begin{aligned} c_0(\eta,\epsilon) = \; & d_0^{(0)} + d_1^{(0)} + d_2^{(0)} \\ & + \epsilon \left(d_0^{(1)} + \left(-d_1^{(0)} - 2d_2^{(0)}\right)\log(\eta) + d_1^{(1)} + d_2^{(1)} \right) \\ & + \epsilon^2 \left(d_0^{(2)} + d_1^{(2)} + d_2^{(2)} + \frac{1}{2}\left(d_1^{(0)} + 4d_2^{(0)}\right)\log^2(\eta) + \left(-d_1^{(1)} - 2d_2^{(1)}\right)\log(\eta) \right) \\ & + \mathcal{O}(\epsilon^3) \end{aligned}$$

Set up this system of eq.s to obtain $d_0^{(0)}$

$$\begin{cases} -d_1^{(0)} - 2d_2^{(0)} = r_{1,1}, \\ d_1^{(0)}/2 + 2d_2^{(0)} = r_{2,2}, \\ d_0^{(0)} + d_1^{(0)} + d_2^{(0)} = r_{0,0} \end{cases}$$

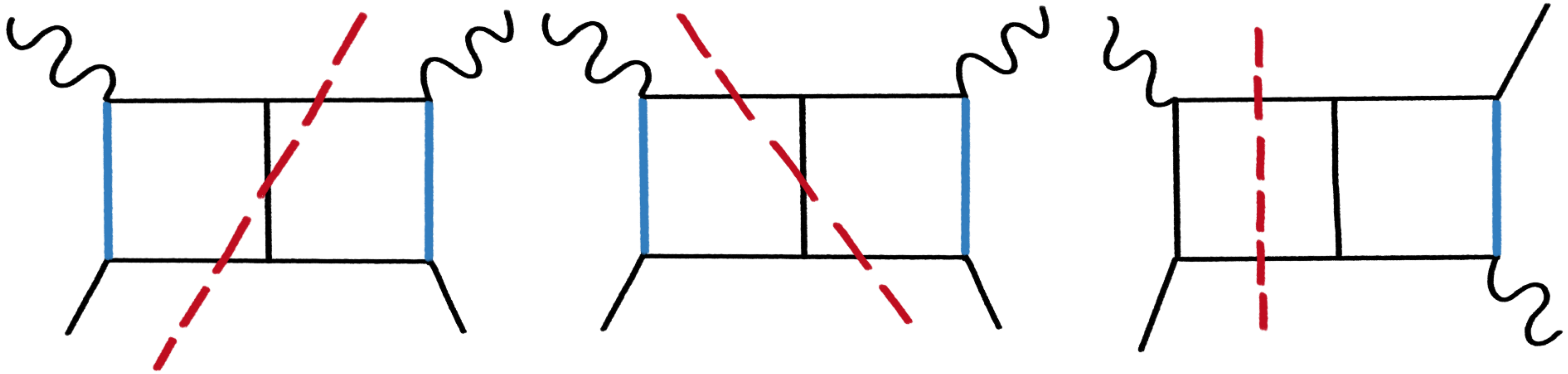
- Analogous system for all $d_0^{(k)}$
- Fixed all $\epsilon-$expansion of hard region: ✓

$$\lim_{z\to 1} I = (1-z)^{-1+2\epsilon}\Bigg\{ -\frac{1}{\epsilon^3} + \frac{5\pi^2}{6\epsilon} + \frac{38\zeta_3}{3} + \frac{7\pi^4}{72}\epsilon + \left(\frac{562\zeta_5}{5} - \frac{74\pi^2\zeta_3}{9}\right)\epsilon^2 + \left(\frac{155\pi^6}{1008} - \frac{191\zeta_3^2}{9}\right)\epsilon^3 + \mathcal{O}(\epsilon^4)\Bigg\} + \mathcal{O}\Big((1-z)^0\Big)$$



Results

- Procedure applied to fix all nontrivial RR and RV boundaries
- Required the following auxiliary topologies:



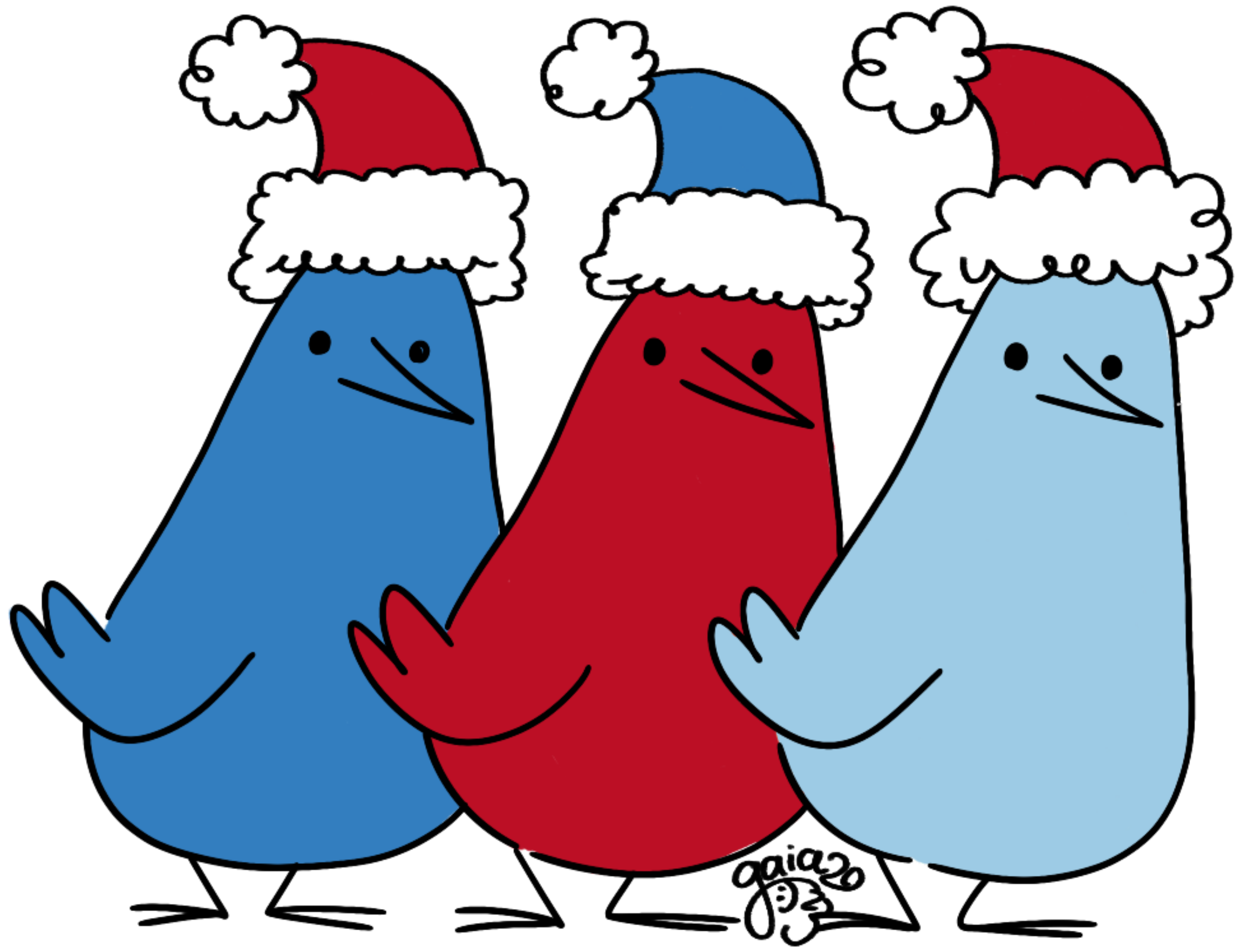
- Results used to derive IF antennae functions at higher epsilon order

Conclusion

- Analytical extension of auxiliary-mass-flow method
- Feasible to study integrals near singular kinematical points
- Automated procedure

& Outlook

- Extension to 3 loop integrals



Thank you for your attention!

Fig.1 Practical way to add auxiliary mass