Subleading power corrections for event shape variables in e^+e^- annihilation

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Event shape variables

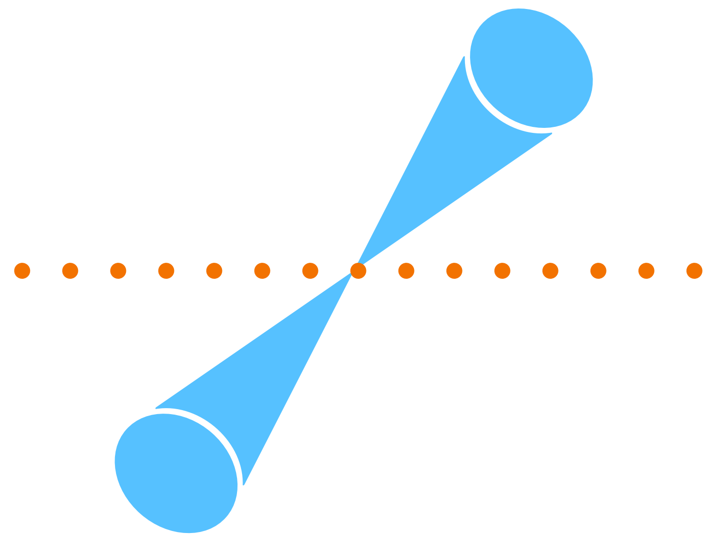
- **Event shape variables** play an important role in the study of QCD in e^+e^- annihilation. They characterize the **geometrical properties** of the hadronic final states. Some examples are **Thrust T and C-parameter**. Event shapes are defined to be infrared safe.

- The value of a given event shape variable encodes smoothly, for example, the **transition between pencil-like two-jet event to planar three-jet event**.

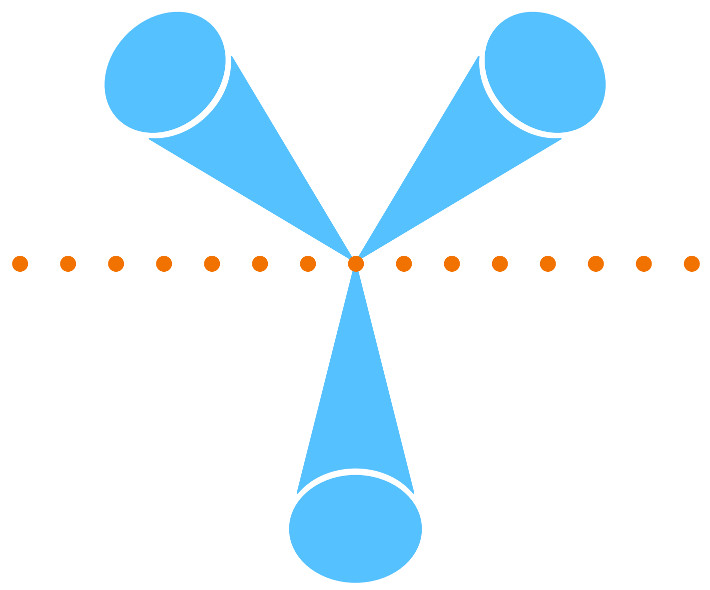
- In this talk I will focus on variables that are non-zero in three-jet configurations. If r is a generic variable, the two-jet limit corresponds to $r \to 0$.

	$1-T$	C-Parameter
Pencil-like event	0	0
Spherical event	1/2	1

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_i |\vec{p}_i|} \qquad C = 3 - \frac{3}{2} \sum_{i,j} \frac{(p_i \cdot p_j)^2}{(p_i \cdot q)(p_j \cdot q)}$$



Pencil-like event



Three-jet event

Subleading power corrections

- When $r \to 0$, the differential cross section in the event shape variable **develops large logarithmic contributions** that need to be resummed. Leading power resummation of different variables has been extensively studied.

[Banfi, Becher, Bonciani, Catani, Dissertori, Gehrmann, Luisoni, Mangano, Marchesini, Monni, Nason, Rodrigo, Salam, Schmelling, Trentadue, Turnock, Webber, Zanderighi, …]

- However, only recently a systematic study of **subleading contributions in the $r \to 0$ limit** started. Subleading contributions can be used to improve the performances of slicing schemes when the observable is used as slicing variable.

[Moult, Rothen, Stewart, Tackmann, Zhu (2017)] [Boughezal, Liu, Petriello (2017)] [Moult, Stewart, Vita, Zhu (2018)] [Moult, Stewart Vita (2019)] [Agarwal, van Beekveld, Laenen, Mishra, Mukhopadhyay, Tripathi (2023)] [Beneke, Garny, Jaskiewicz, Strohm, Szafron, Vernazza, Wang (2022)]

- At NLO the shape variable r can be used to set up a slicing scheme by splitting the real contribution into a contribution above and one below a small cut v:

$$\sigma_{\mathrm{NLO}} = \int d\sigma^R \theta(r - v) + \left(\int d\sigma^R \theta(v - r) + \int d\sigma^V + \int d\sigma^B \right)$$

- The term below the cut can be evaluated in the small v limit using suitable approximations for the phase space and the real matrix element. The IR poles from the real contribution will cancel with the ones in the virtual, and we obtain:

$$\int d\sigma^R \theta(v - r) + \int d\sigma^V + \int d\sigma^B = \int d\sigma^B \left[1 + \frac{\alpha_s}{\pi} (A_r \log^2(v) + B_r \log(v) + C_r + \mathcal{O}(v^p)) \right]$$

Subleading power corrections!

Definition of the observables

- We consider the observable 2-Jettiness τ_2 [Stewart, Tackmann, Waalewijn (2010)]. We choose the jet axis q_1 and q_2 using the JADE clustering algorithm. For an event with n final-state partons, τ_2 is defined as:

$$\tau_2 = \sum_{k=1}^{n} \min\left\{\frac{2p_k \cdot q_1}{Q^2}, \frac{2p_k \cdot q_2}{Q^2}\right\}$$

- The variable y_{23} is defined using the distance measure d_{ij} of the k_T clustering algorithm. At NLO, y_{23} is the minimum among all the distances d_{ij}.

$$d_{ij} = \frac{2\min\{E_i^2, E_j^2\}(1-\cos\theta_{ij})}{Q^2} \qquad y_{23} = \min\{d_{12}, d_{13}, d_{23}\}$$

- By limiting ourselves at NLO, we can also consider the variable:

$$k_T^{\mathrm{FSR}} = \sqrt{\frac{2(p_1 \cdot p_3)(p_2 \cdot p_3)}{p_1 \cdot p_2}}$$

Definition of the observables

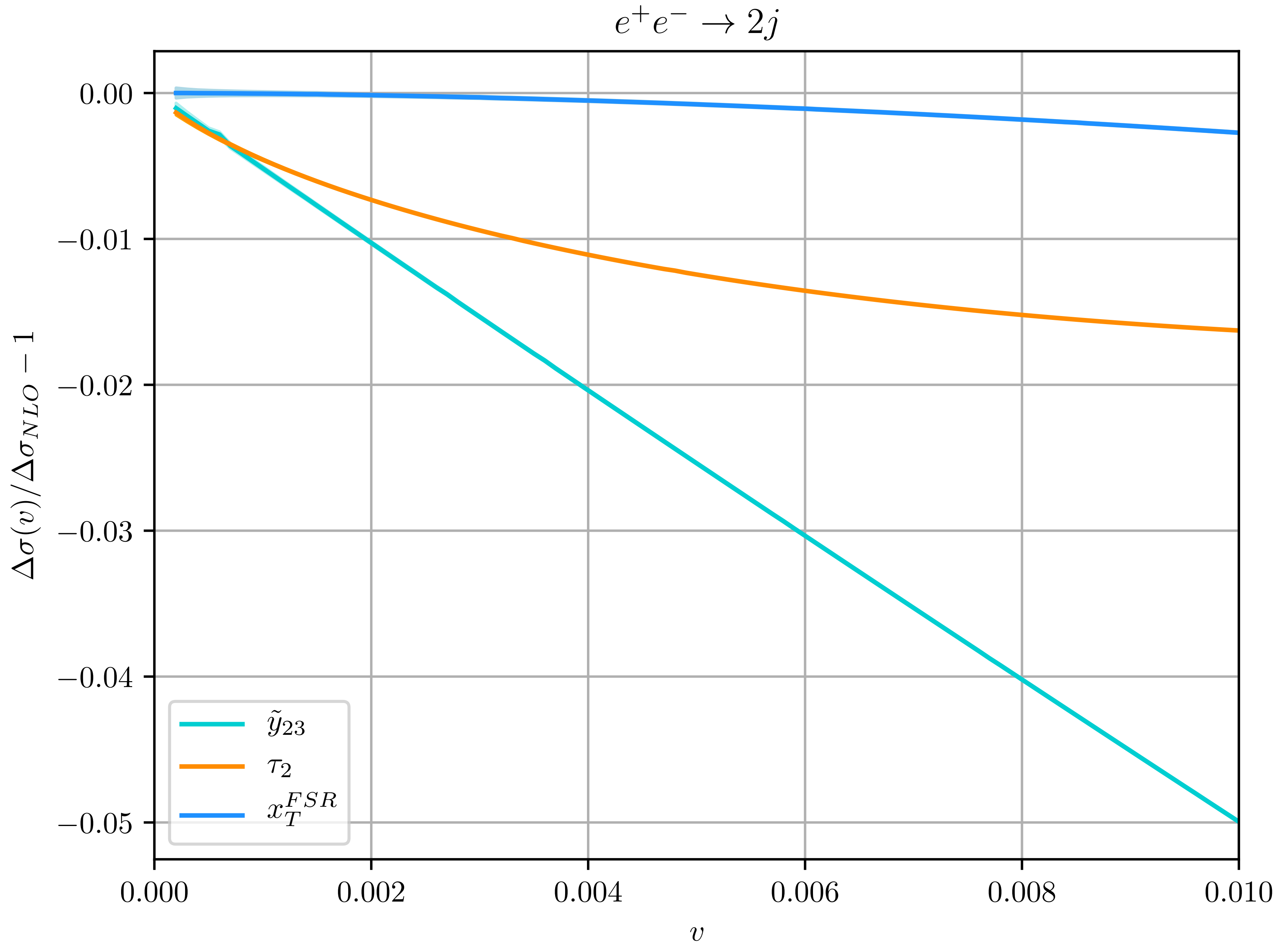
- We are considering variables whose dependence on the momentum k of a single soft emission, collinear to one of the hard legs of the Born event, can be parametrized as

$$r(\{p_i\}, k) = \left(\frac{k_t^{(\ell)}}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}}$$

- Here, $\{p_i\}$ are the Born momenta and $k_t^{(\ell)}, \eta^{(\ell)} \geq 0$ denote the transverse momentum and the rapidity of k with respect to the leg ℓ.

- The observable τ_2 corresponds to $a = 1, b = 1$ while y_{23} corresponds to $a = 2, b = 0$. We will use the variable $\tilde{y}_{23} = \sqrt{y_{23}}$ in order to have an homogeneous scaling in $k_t^{(\ell)}$.

- We can test the size of the power corrections by plotting the relative deviation for the NLO correction $\Delta\sigma_{\text{NLO}}$ from its exact result as a function of the cut v. We can already see from a numerical calculation that different variables have a different scaling in the $v \to 0$ limit.



Setup of the calculation

- We now focus on the real contribution $d\sigma^R$. We can compute the complete tower of power corrections by integrating the real matrix element above the cut, and the subleading contribution can be obtained by expanding in the cut v.

- The calculation is performed using a phase space parametrization in terms of energy fractions x_i.

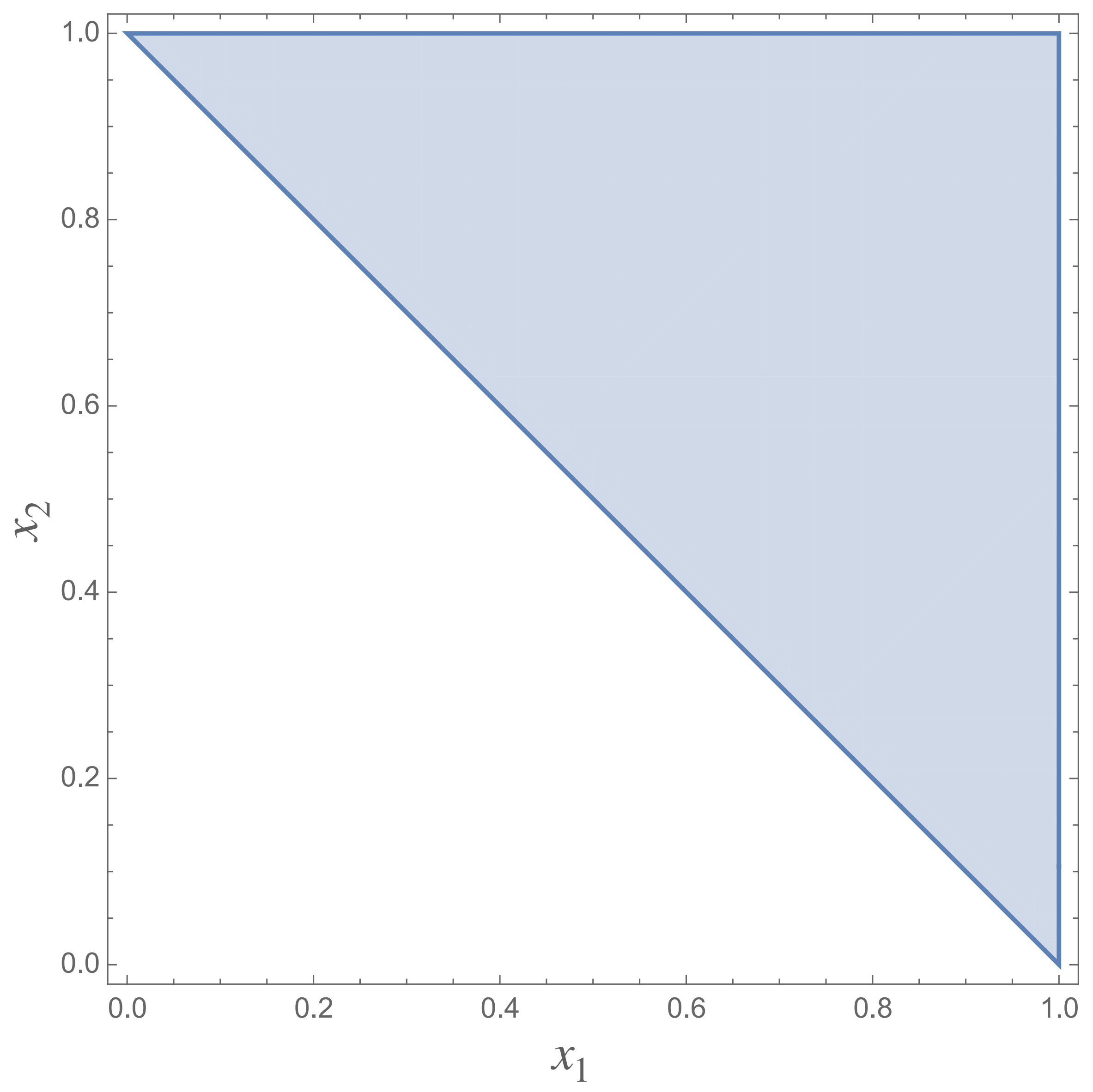
$$x_i = \frac{2p_i \cdot Q}{Q^2} \qquad x_1 + x_2 + x_3 = 2 \qquad \sigma_r^R(v) = \int d\sigma^R \theta(r - v) \equiv \sigma_0 \frac{\alpha_s}{2\pi} C_F R_r(v)$$

$$f(x_1, x_2) = \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \qquad R_r(v) = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, f(x_1, x_2)\theta(r(x_1, x_2) - v)$$

We want to obtain the subleading power corrections by computing this integral!

- The function $f(x_1, x_2)$ represents, up to a normalization factor, the squared matrix element for the process $e^+e^- \to \gamma^* \to q\bar{q}g$. The phase space is the in the (x_1, x_2) plane is the triangle $0 \leq x_1 \leq 1, 1 - x_1 \leq x_2 \leq 1$.

- The limits in which the gluon is collinear to one of the quarks are reached for $x_1 \to 1, x_2 \to 1$. The soft limit occurs in the point $(x_1, x_2) \to (1,1)$

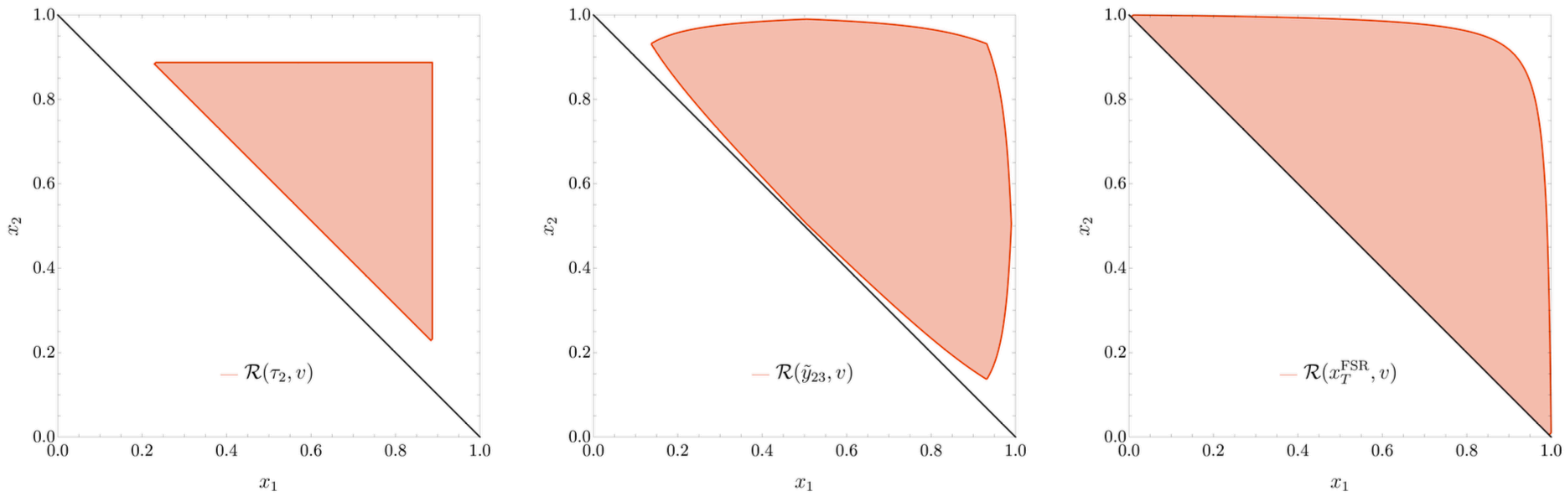


Observables in the (x_1, x_2) plane

- The observables have the following expressions in terms of the energy fractions:

$$d_{ij} = \frac{\min\{x_i^2, x_j^2\}}{x_i x_j}(1 - x_k) \qquad \tau_2 = x_k(1 - x_k) \text{ when } s_{ij} < s_{ik}, s_{jk} \qquad x_T^{\text{FSR}} = \frac{k_T^{\text{FSR}}}{Q} = \sqrt{\frac{(1 - x_1)(1 - x_2)}{x_1 + x_2 - 1}}$$

- It is interesting to look at the phase space regions in which $r > v$, where v is the fixed value of the cut, and $r \in \{\tau_2, \tilde{y}_{23}, x_T^{\text{FSR}}\}$



- We observe that the three variables cut in a different way the region $x_2 \sim 1 - x_1$, that corresponds to the region in which an hard gluon is emitted. Since the matrix element is not singular in this kinematic configuration, this region will give rise to pure power corrections.

Results for x_T^{FSR}, τ_2, y_{23}

- We analytically computed the full tower of power corrections for the variables x_T^{FSR}, τ_2, $\tilde{y}_{23}$.

$$R_{x_T^{\text{FSR}}}(v) = \frac{7}{2} + v^2 + (3 + 4v^2 + v^4)\log\left(\frac{v^2}{1+v^2}\right) - 2\text{Li}_2\left(-\frac{1}{v^2}\right) = 4\log^2 v + 6\log v + \frac{7}{2} + \frac{\pi^2}{3} + \underline{4(2\log v - 1)v^2} + \mathcal{O}(v^4)$$

Quadratic subleading power corrections

$$R_{\tau_2}(v) = \frac{5}{2} - \frac{\pi^2}{3} + 2\log^2\left(\frac{1-u}{u}\right) + (6u-3)\log\left(\frac{1-2u}{u}\right) - 6u - \frac{9u^2}{2} + 4\text{Li}_2\left(\frac{u}{1-u}\right) = 2\log^2 v + 3\log v + \frac{5}{2} - \frac{\pi^2}{3} + \underline{v(7 + 2\log v)} + \mathcal{O}(v^2)$$

$$u = \frac{1}{2}(1 - \sqrt{1-4v})$$

Linear log-enhanced subleading power corrections

$$R_{\tilde{y}_{23}}(v) = 4\log^2 v + 6\log v + \frac{5}{2} - \frac{\pi^2}{6} + 6\log 2 + \underline{(4\log(1+\sqrt{2}) - 8\sqrt{2})v} + \mathcal{O}(v^2)$$

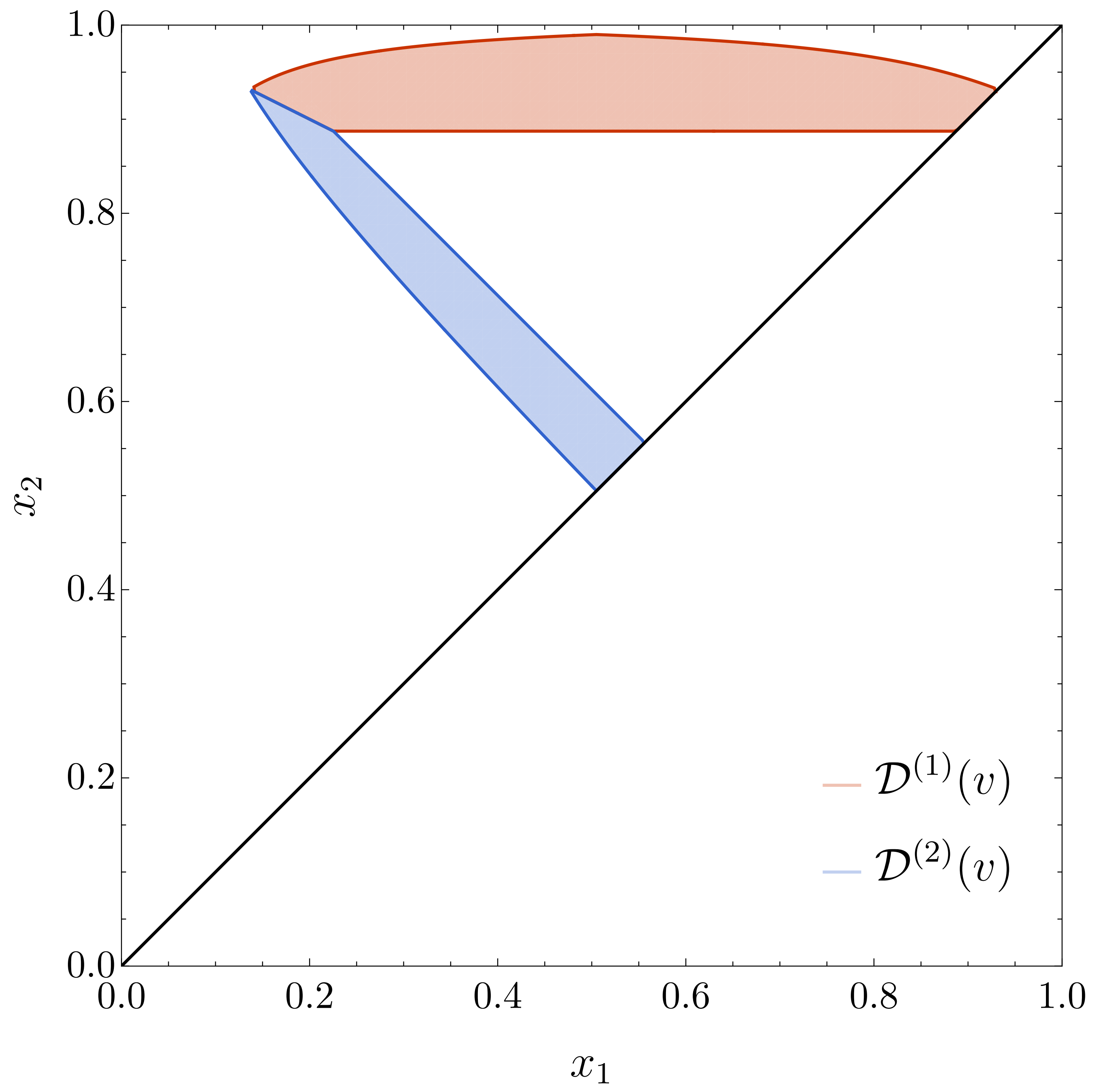
(The complete expression to all orders in v is reported in the paper)

Linear subleading power corrections

- However, this analysis **does not shed light on the physical origin** of the power corrections nor on the observed differences among the variables…

Comparison between τ_2 and $\tilde{y}_{23}$

- To gain further insight, we compare the phase space regions $\mathscr{R}(\tau_2; v)$ and $\mathscr{R}(\tilde{y}_{23}, v)$ that respectively correspond to $\tau_2 > v$ and $\tilde{y}_{23} > v$. The region that removes the logarithmically-enhanced linear power correction is then given by the difference $D = \mathscr{R}(\tilde{y}_{23}; v) \backslash \mathscr{R}(\tau_2; v)$



$$2\int_{D^{(1)}} dx_1 dx_2\, f(x_1, x_2) = 2\log^2 v + 3\log v + \frac{\pi^2}{6} + 6\log 2 + v(-7 - 8\sqrt{2} + 8\log(1+\sqrt{2})) + \mathcal{O}(v^2)$$

$$2\int_{D^{(2)}} dx_1 dx_2\, f(x_1, x_2) = -4v\log(1+\sqrt{2}) - \underline{2v\log v} + \mathcal{O}(v^2)$$

The log-enhanced contribution comes only from the region $D^{(2)}$!

- This region corresponds to the physical configuration in which the **gluon is hard** and recoils against a **collinear and/or soft quark-antiquark pair**.

- We are far from the regions in which the real matrix element is singular, so the contribution from this region is a **pure power correction**.

Collinear expansion of the matrix element

- We want now to study the expansion of the matrix element in the singular limit $x_2 \to 1$, that corresponds to the configuration in which the momentum of the gluon becomes collinear to the one of the quark. This is the only singular limit that can be reached in the region $D^{(2)}$.

- The expansion of the matrix element in this limit is:

$$f(x_1, x_2) = \frac{1 + x_1^2}{(1 - x_1)(1 - x_2)} - \frac{2}{1 - x_1} + \mathcal{O}(1 - x_2) \equiv f^{(0)}_{\rm coll}(x_1, x_2) + f^{(1)}_{\rm coll}(x_1, x_2) + \mathcal{O}(1 - x_2)$$

$$2 \int_{D^{(2)}} dx_1 dx_2\, f^{(0)}_{\rm coll}(x_1, x_2) = v \left(1 + 2\log 2 - 4\log(1 + \sqrt{2}) - 2\log v\right) + \mathcal{O}(v^2)$$

- The **collinear approximation** of the matrix element is sufficient to capture the **logarithmically-enhanced linear power correction**. However, integrating down to the $\tilde{y}_{23}$ contour does not lead to logarithmically-enhanced power corrections.

- We associate this result to the fact that the phase space volume removed by a cut on $\tilde{y}_{23}$ scales quadratically with v, while it scales linearly for the case of τ_2.

- **In conclusion**, we shown that for τ_2 the **logarithmically-enhanced power correction is a pure phase-space effect!** This result is observable dependent.

The case of thrust

- We now focus on thrust and consider the variable $1-T$. In term of the energy fractions we have:

$$1-T=\min\{1-x_1,1-x_2,1-x_3\}$$

- We report here the result for the cumulative cross-section up to $\mathcal{O}(v)$. **The result for $1-T$ coincides with the one for τ_2 up to leading power**, including the constant term. This is due to the fact that τ_2 coincides with $1-T$ for an appropriate choice of the jet axes q_1 and q_2.

$$R_{1-T}(v)=2\log^2 v+3\log v+\frac{5}{2}-\frac{\pi^2}{3}+2v(2-\log v)+\mathcal{O}(v^2)$$

- However, contrary to what happens for τ_2, the **subleading logarithmically-enhanced term** does not originate only from the the hard gluon region $D^{(2)}$, **but also from the region $D^{(1)}$** in which the gluon is collinear to the quark.

- As for τ_2, the subleading power correction stemming from the region $D^{(2)}$ is fully captured by the leading power expansion of the matrix element $f^{(0)}_{\rm coll}$ in the collinear limit $x_2\to 1$. However, t**o fully capture the subleading power correction rising from $D^{(1)}$**, it is necessary to include the **next-to-leading power expansion of the matrix element**.

- Our analysis is in perfect agreement with the one performed in the SCET framework [Moult, Rothen, Stewart, Tackmann, Zhu (2017)]

C-parameter

- We now consider the C-parameter. For massless particles in the final state we have:

$$C = 3 - \frac{3}{2}\sum_{i,j} \frac{(p_i \cdot p_j)^2}{(p_i \cdot q)(p_j \cdot q)}$$

- In the 2-jet limit $C \to 0$, thrust and C-parameter are related by the relation $C = 6(1-T)$ that is valid up to next-to-leading logarithmic accuracy. We can consider the variable $c = C/6$ that has the following expression in terms of the energy fractions:

$$c = \frac{(1-x_1)(1-x_2)(1-x_3)}{x_1 x_2 x_3}$$

- The evaluation of the cross-section in this case is more complicated and involves elliptic integrals [Gardi, Magnea (2003)] [Agarwal, van Beekveld, Laenen, Mishra, Mukhopadhyay (2023)]. We find:

$$R_c(v) = 2\log^2 v + 3\log v + \frac{5}{2} - \frac{2}{3}\pi^2 + v(7 - 4\log v) + \mathcal{O}(v^2)$$

Summary for thrust and C-parameter

- Repeating the analysis for the C-parameter in the regions $D^{(1)}$ and $D^{(2)}$ we observe the same pattern of thrust.

- We report a summary of the results for thrust and C-parameter. The symbol $\sim$ means that we are restricting the result to the logarithmically-enhanced power correction.

$$2\int_{D^{(2)}(v)} dx_1 dx_2\, f(x_1, x_2) \sim 2\int_{D^{(2)}(v)} dx_1 dx_2\, f^{(0)}_{\mathrm{coll}}(x_1, x_2) \sim -2v\log v$$

$$2\int_{D^{(1)}(v)} dx_1 dx_2 f(x_1, x_2) \sim \begin{cases} +4v\log v & \text{for } 1-T \\ +6v\log v & \text{for } c \end{cases}$$

The observable r_b

- As mentioned before, we consider variables $r(\{p_i\}, k)$ that in the infrared limits can be parametrized as:

$$r(\{p_i\}, k) = \left(\frac{k_t^{(\ell)}}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}}$$

- We would like to study the power corrections for a variable with a generic b exponent. Variables of this type have been studied in order to assess the logarithmic accuracy of Monte Carlo parton showers [Banfi, Salam, Zanderighi (2005)] [Dasgupta, Dreyer, Hamilton, Monni, Salam, Soyez (2020)].

- We define the class of observables depending on b as:

$$r_b = (1 - T)^b \tilde{y}_{23}^{1-b}$$

Subleading power corrections of r_b

- We find the following subleading power result for the cumulative cross-section for r_b,

$$R_{r_b}(v) = \frac{2}{1+b}(2\log^2 v + 3\log v) + \frac{5}{2} - (1+b)\frac{\pi^2}{6} + 6\frac{1-b}{1+b}\log 2 + F_1(b)v + F_2(b)v^{\frac{2}{1+b}} + \mathcal{O}(v^2)$$

- Here $F_1(b)$ and $F_2(b)$ are functions of b involving gamma functions, polygamma functions, incomplete beta functions. For $b \neq 0$ we find that **there are not explicit logarithmically-enhanced** contributions, **but there are both linear and fractional subleading power corrections**.

- For $b = 0$ and $b = 1$ we reproduce the results for $\tilde{y}_{23}$ and thrust respectively.

$$\lim_{b\to 0} R_{r_b}(v) = R_{\tilde{y}_{23}}(v) \qquad \lim_{b\to 1} R_{r_b}(v) = R_{1-T}(v)$$

Expressions for $F_1(b), F_2(b)$

$$F_1(b) = \frac{2^{\frac{5+b}{2}} b}{1+b} + 4B_{1/2}\left(-\frac{1+b}{2}, 0\right) - 2B_{1/2}\left(\frac{1-b}{2}, 0\right)$$

$$F_2(b) = 4B_{1/2}\left(\frac{b-1}{b+1}, 0\right) - 4B_{1/2}\left(\frac{2b}{1+b}, 0\right) + \frac{\Gamma\left(\frac{b-1}{b+1}\right)\left(4\left(b^4 + 3b^3 + 6b^2 + b + 1 + \frac{b(b^3 - 7b^2 + 3b + 3)}{b+1} B_{\frac{1}{2}}\left(\frac{b-1}{b+1}, \frac{2}{b+1}\right)\right) - 4b^{\frac{2+b}{1+b}}(b+1)^2\right)}{(b+1)^3\,\Gamma\left(\frac{2b}{b+1} + 1\right)} + \frac{5b^2 + 6b - 3}{(1+b)^2}\left(\psi\left(\frac{b}{1+b}\right) - \psi\left(\frac{1+3b}{2(1+b)}\right)\right)$$

- $B_z(a, b)$ is the incomplete beta function and $\psi(z)$ is the polygamma function of order 0.

$$B_z(a, b) = \int_0^z dt\, t^{a-1}(1-t)^{b-1}$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

Summary

- We have studied subleading power corrections to event shape variables in e^+e^- collisions, starting from τ_2 and y_{23}. Both variables have linear power correction: for τ_2 the linear term is logarithmically enhanced, while for y_{23} it is not.

- After computing the cumulative cross-section for these observables, we discussed the origin of the different power corrections. Our main observation is that these variables cover the phase space in different ways, and the different power correction can be traced to the different cuts of the singular region in (x_1, x_2) plane.

- The logarithmically-enhanced power correction for τ_2 can be obtained with a collinear approximation of the matrix element

- For thrust and C-parameter, the logarithmically-enhanced power correction does not originate only from non-singular regions of the phase-space, but also from subleading power expansion of the matrix element in the collinear limit.

- We finally considered a class of variables r_b that depends on a parameter b that gives different weights to central and forward emissions. These variables have a non-trivial structure of subleading power corrections.

- Recent studies of subleading power corrections were mostly carried out in the SCET framework, for thrust and jettiness. Our results extend these findings to other observables, offering a different perspective on the structure of power corrections.