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FOR PHYSICS

Local Analytic Sector subtraction: NNLO subtraction for any massless final state

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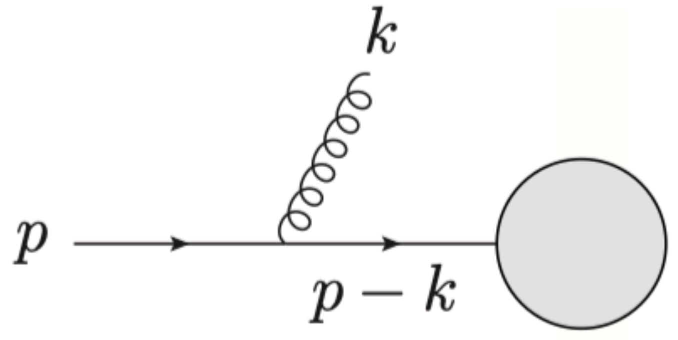
Based on: *JHEP 12(2018)107, JHEP 02(2021)037, JHEP 07(2023)140*

Real radiation



$$\sim \frac{1}{(p-k)^2} = \frac{1}{2E_pE_k(1-\cos\theta)} \xrightarrow[\substack{E_k \to 0 \\ \text{or} \\ \theta \to 0}]{} \infty \,. \qquad \int \frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}\,|M(\{p\},k)|^2 \underset{\substack{E_k \to 0 \\ \theta \to 0}}{\sim} \int \frac{dE_k}{E_k^{1+2\epsilon}}\,\frac{d\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{4\epsilon^2} \,.$$

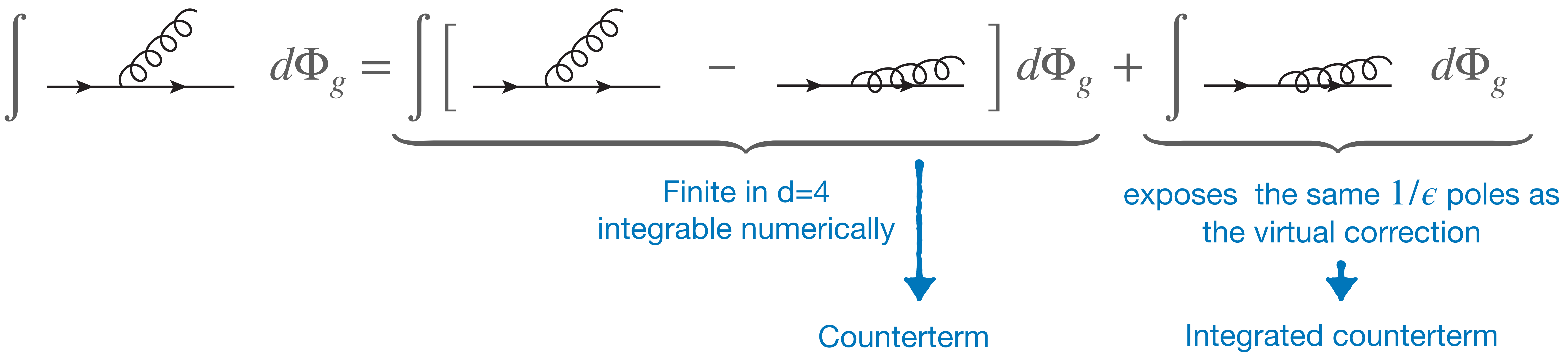
The problem

1. **Extract** infrared singularities affecting the real radiation in d-dimension **without integrating over the resolved phase space**

   → **fully differential** predictions for IR-safe observables

2. **Cancel** the $1/\epsilon$ poles stemming from the unresolved phase space integration against the poles of the **virtual contributions**

→ **Unresolved limits are universal and known** (even at N3LO) → a **general procedure is in principle feasible**



Finite in d=4
integrable numerically

exposes the same $1/\epsilon$ poles as
the virtual correction

Counterterm

Integrated counterterm

Subtraction: conceptually non-trivial, but if local and analytic then extremely versatile and numerically stable

Subtractions: status

NLO:

solved conceptually in the 90s and now implemented in automatic frameworks

NNLO:

still **looking for the optimal scheme** → **the problem is highly non-trivial and a simple generalisation of NLO not doable due to overlapping singularities**

Example: di-jet two-loop amplitudes ~ 20 years ago *[Anastasiou et al. '01]*

di-jet production at NNLO ~ 5 years ago *[Currie et al. '17]*

- many different proposal available

Antenna *[Gehrmann-De Ridder et al. '05]*

ColorfullNNLO *[Del Duca et al. '16]*

STRIPPER *[Czakon '10]*

Nested soft-collinear *[Caola et al. '17]*

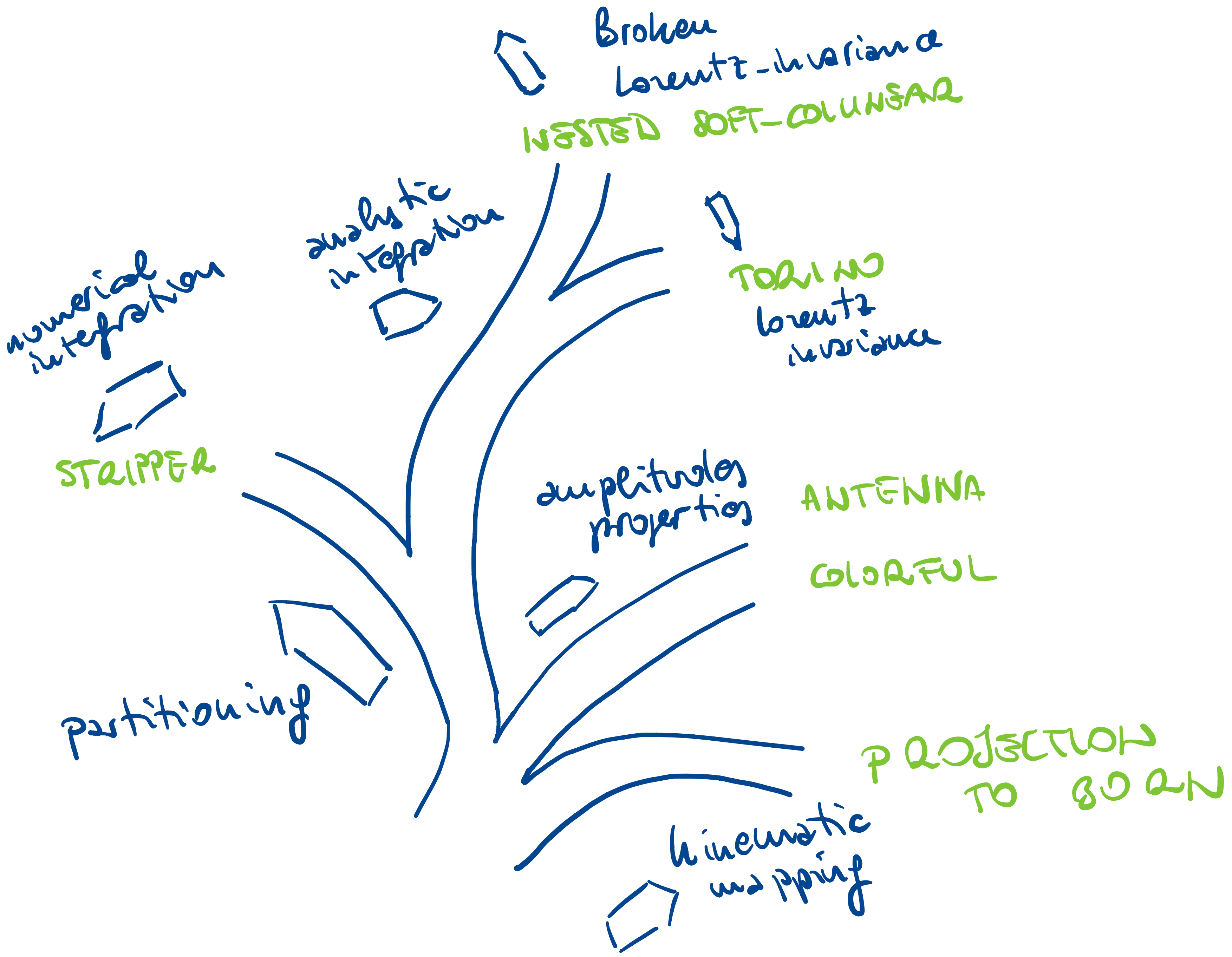
Local analytic sector *[Magnea, CSS et al. '18]*

Geometric IR subtraction *[Herzog '18]*

Unsubtraction *[Sborlini et al. '16]*

FDR *[Pittau '12]*

Universal Factorisation *[Sterman et al. '20]*



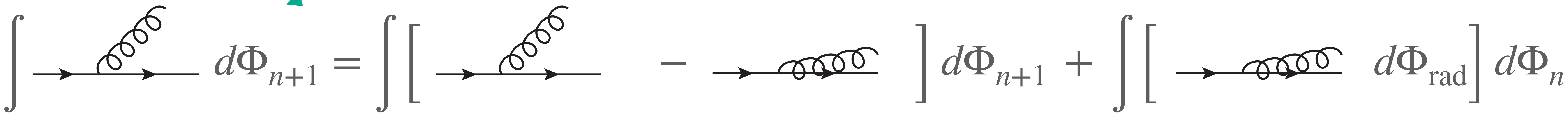
Details of the calculation: NLO as a playground

Local Analytic Sector Subtraction

Go **back to NLO** to implement a new scheme featuring **key properties** that can be **exported at NNLO**.

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d\to4}\left\{ \int d\Phi_{n+1}\, R\, \delta_{n+1}(X) + \int d\Phi_n\, V\, \delta_n(X) \right\}$$

X IR safe observable



$$\frac{d\sigma^{\rm NLO}_{ct}}{dX} = \int d\Phi_{n+1}\, K \qquad I = \int d\Phi_{\rm rad}\, K$$

$$\boxed{\frac{d\sigma^{\rm NLO}}{dX} = \int d\Phi_{n+1}\left(R\;\delta_{n+1}(X) - K\;\delta_n(X)\right) + \int d\Phi_n \left(V + I\right)\delta_n(X)}$$

Properties of the scheme:	**Minimal structure** and **simple integration**	**Analytically calculable** (possibly with standard techniques)
Requirements:	**Organise** all the overlapping **singularities** and choose an **appropriate kinematics**	Choose an **optimise parametrisation** of the phase space

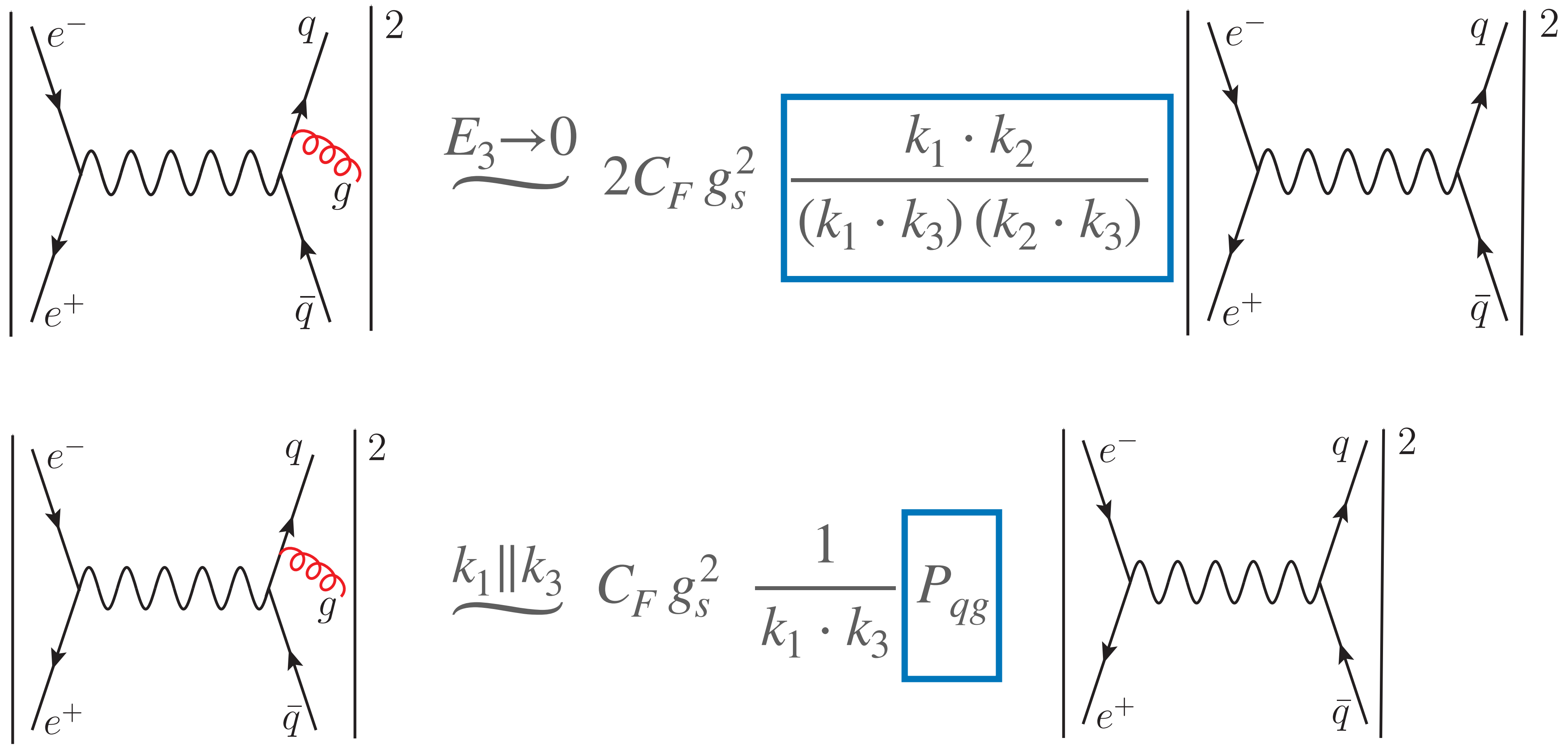
Ingredients of the subtraction

- *Projection operators: extract from the real-radiation matrix element its leading soft and collinear limits [Altarelli, Parisi '77]*

$$S_i\, R\big(\{k\}\big) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}}\, B\left(\{k\}_{\not i}\right)$$

$$C_{ij}\, R\big(\{k\}\big) \propto \frac{1}{s_{ij}}\, P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\; B^{\mu\nu}\big(\{k\}_{\not i \not j}, k_{ij}\big)$$

$$S_i\, C_{ij}\, R\big(\{k\}\big) \propto \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\left(\{k\}_{\not i}\right)$$



$$\underset{\sim}{E_3\to 0}\quad 2C_F\, g_s^2\, \boxed{\frac{k_1\cdot k_2}{(k_1\cdot k_3)\,(k_2\cdot k_3)}}$$

$$\underset{\sim}{k_1 \| k_3}\quad C_F\, g_s^2\, \frac{1}{k_1\cdot k_3}\, \boxed{P_{qg}}$$

- *Phase space partitioning (FKS): multiple singular configuration that overlap*

  - **Unitary partition.**
  - Select a **minimum number of singularities** in each sector.
  - Do **not affect** the **analytic integration** of the counterterms.

$$R = \sum_{i,j} R\,\mathscr{W}_{ij} = R\,\mathscr{W}_{31} + R\,\mathscr{W}_{32} + \dots$$

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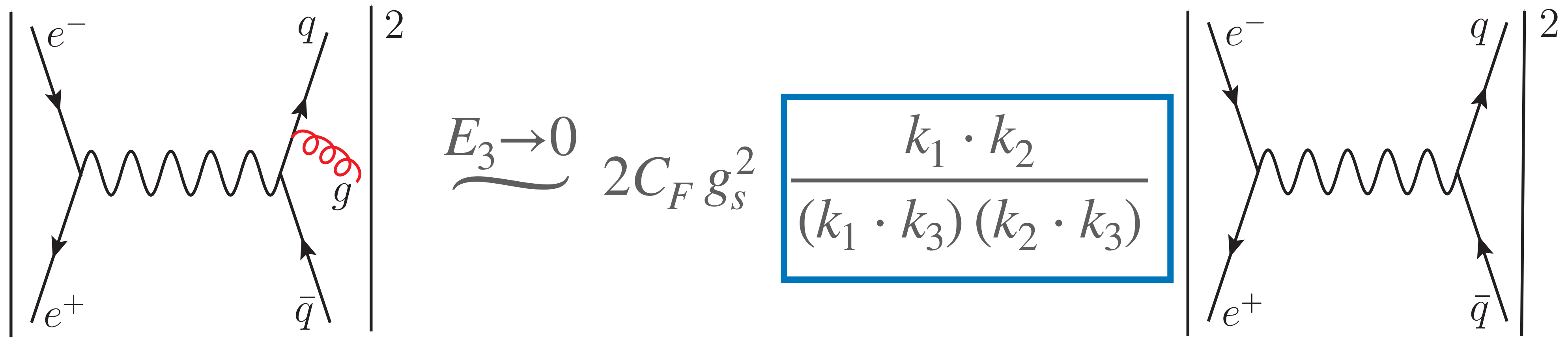
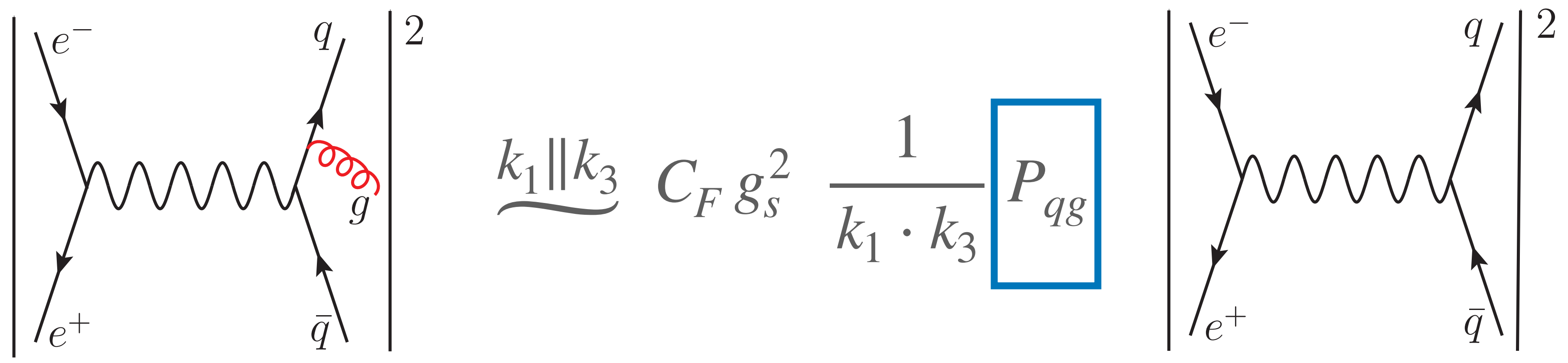


$$\Big|\cdots\Big|^2 \underset{\sim}{E_3\to 0}\; 2C_F\,g_s^2\,\boxed{\frac{k_1\cdot k_2}{(k_1\cdot k_3)\,(k_2\cdot k_3)}}\,\Big|\cdots\Big|^2$$



$$\Big|\cdots\Big|^2 \underset{\sim}{k_1\|k_3}\; C_F\,g_s^2\,\frac{1}{k_1\cdot k_3}\,\boxed{P_{qg}}\,\Big|\cdots\Big|^2$$

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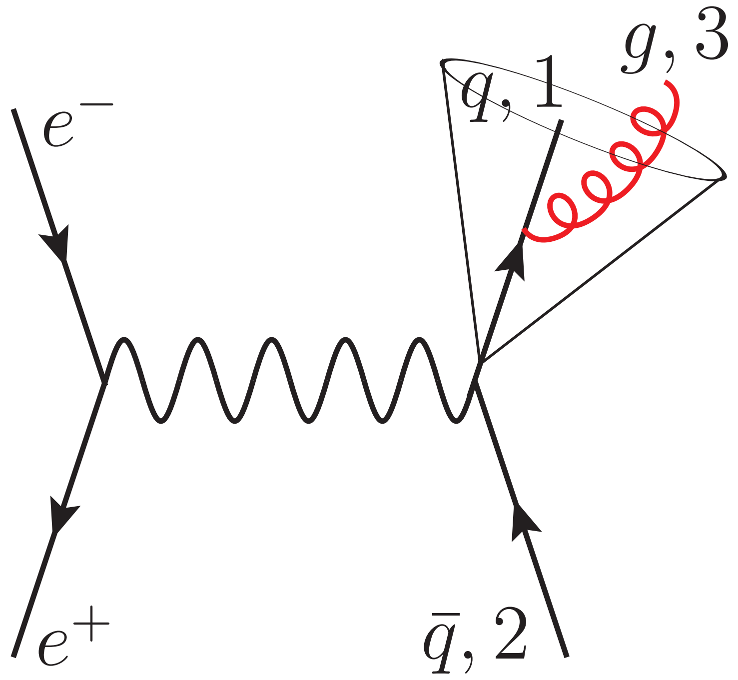


←

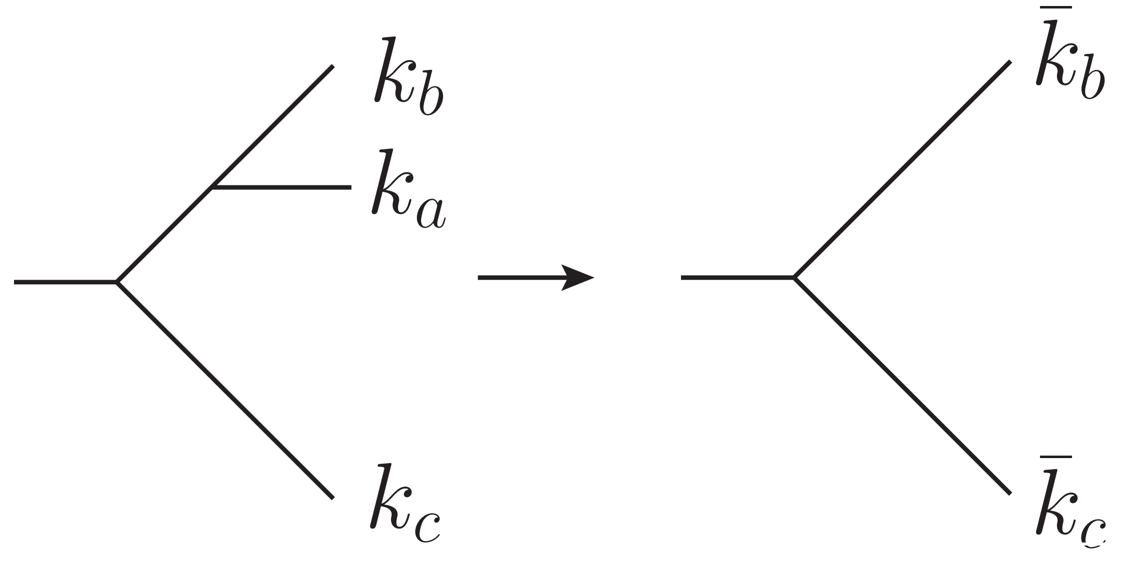
Damp: $\vec n_2 \parallel \vec n_3$

Enhance: $\vec n_1 \parallel \vec n_3$

$$\mathscr{W}_{31} \sim \frac{1}{s_{31}}$$

Ingredients of the subtraction

- *Momentum mapping (CS):*
  - **Factorisation of the phase space** $d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{\text{rad}}\left(s_{bc}^{(abc)}; y, z, \phi\right)$
  - **On-shell** particle **conserving momentum** in the entire PS



Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{\{k\}_{\cancel{a}\cancel{b}\cancel{c}}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)}\right\}$

$$k_a + k_b + k_c = \bar{k}_b^{(abc)} + \bar{k}_c^{(abc)}$$

Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

- *Analytic integration:*

Freedom to **adapt** the **parametrisation to the kernel:** $\bar{\mathbf{S}}_i R\left(\{k\}\right) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right)$ → Exact analytic integration

$$I^s \propto \sum_{c,d\neq i} \int d\Phi_{\text{rad}}^{(icd)} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) = \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon} \frac{1-z}{z} B_{cd}\left(\{k\}^{(icd)}\right)$$

$$= \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)} B_{cd}\left(\{k\}^{(icd)}\right)$$

Lesson from NLO

- **Unitary partition** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$
- Collection of relevant IRC limits for a given sector
- **Catani-Seymour** final-state **dipole mapping**

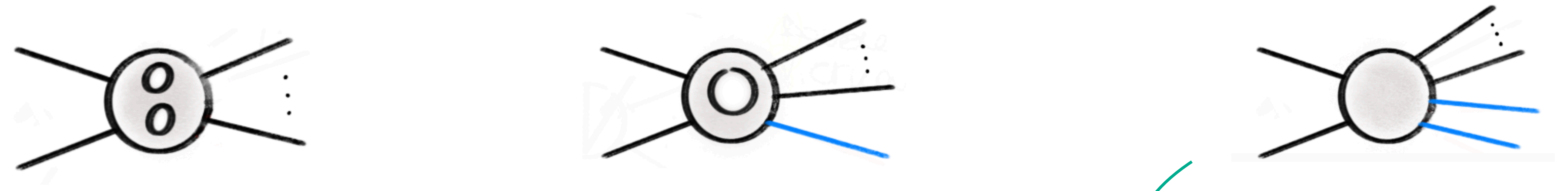
Details of the calculation: NNLO

Exploring the framework

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\mathrm{LO}}}{dX} + \frac{d\sigma_{\mathrm{NLO}}}{dX} + \boxed{\frac{d\sigma_{\mathrm{NNLO}}}{dX}} + \dots$$

Arbitrary number of massless QCD final-state emissions

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n} + \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}} + \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$



Numerous overlapping singularities!

Exploring the framework

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \frac{d\sigma_{\rm NLO}}{dX} + \boxed{\frac{d\sigma_{\rm NNLO}}{dX}} + \dots$$

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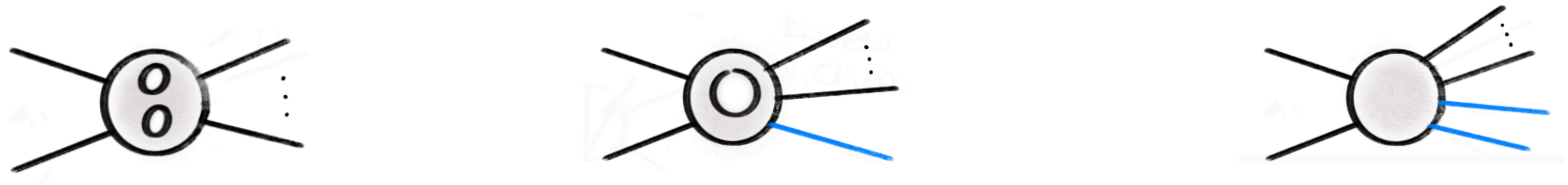
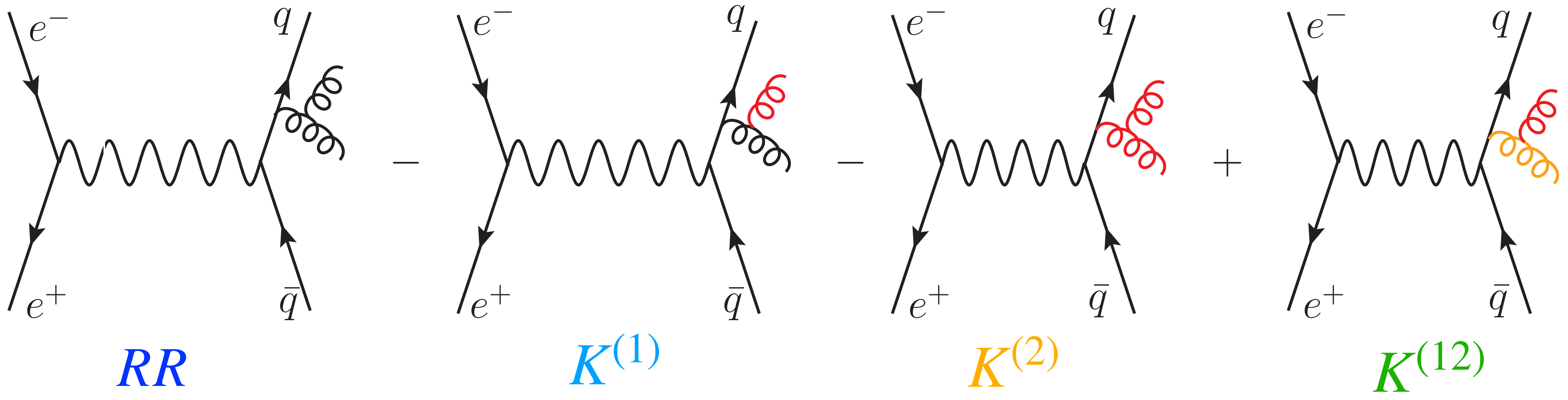


First step: divide the singular configurations into single-unresolved, double unresolved, and strongly-ordered



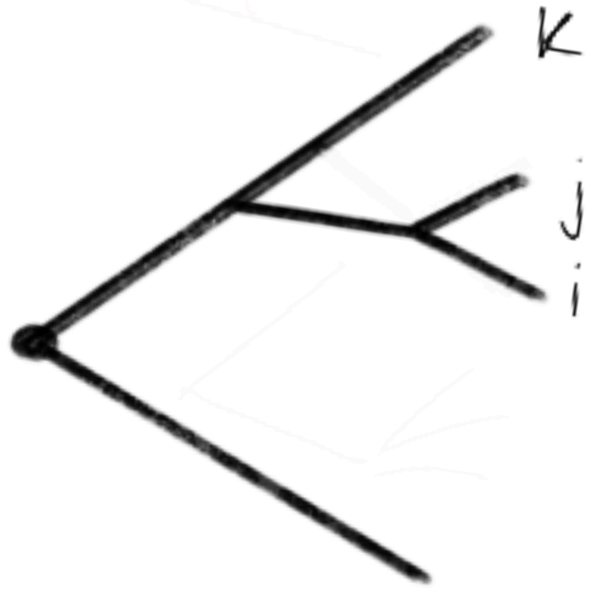
$$\boxed{\int d\Phi_{n+2}\Big[\, RR\,\delta_{n+2} - K^{(1)}\,\delta_{n+1} - \big(K^{(2)} - K^{(12)}\big)\delta_n \,\Big]}$$

Sector functions at NNLO

Second step: unitary partition of double-unresolved phase space Φ_{n+2} into sectors $\mathcal{W}_{ijkl}$

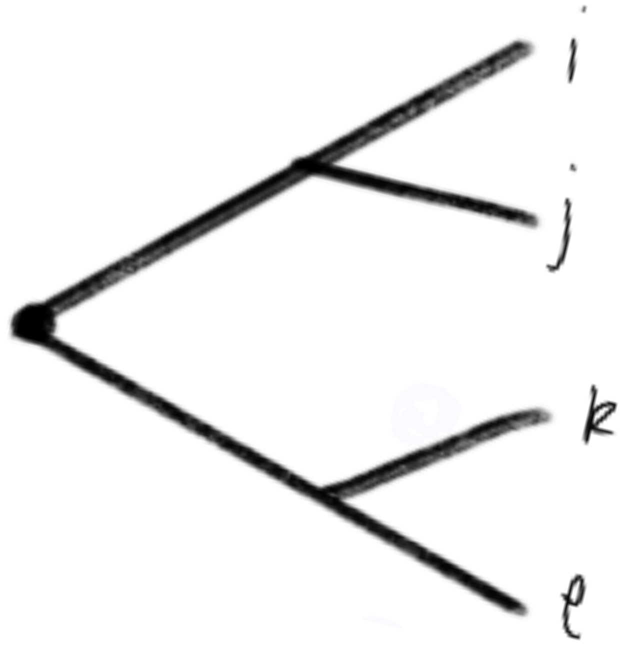
$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}\,, \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

- ***3 topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}\,, \qquad i\neq j\neq k$

$\mathcal{W}_{ijkj}\,, \qquad i\neq j\neq k$



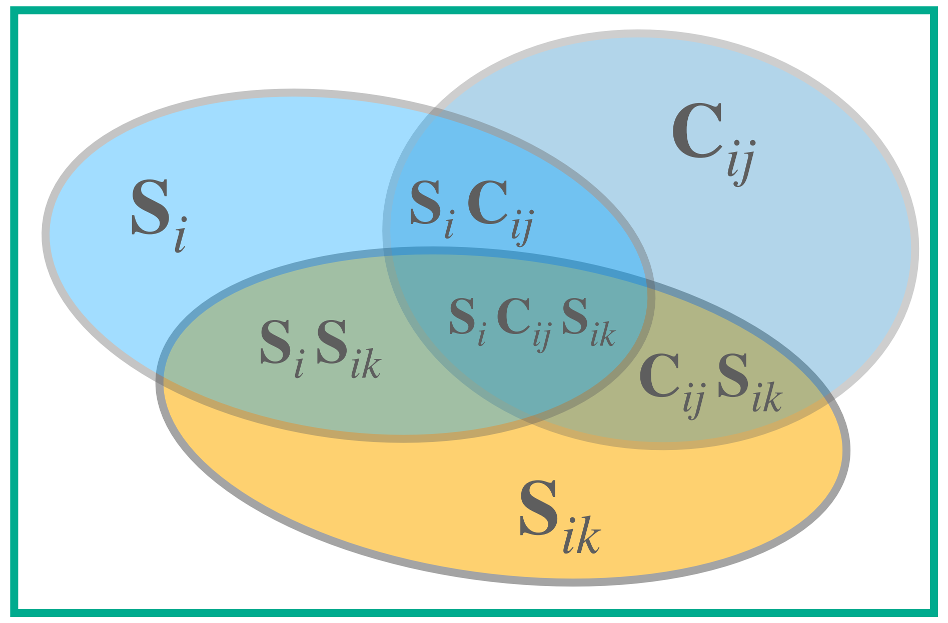
$\mathcal{W}_{ijkl}\,, \qquad i\neq j\neq k\neq l$

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

Single unresolved

Double unresolved

$\mathbf{S}_{ij}$	**double-soft** partons i and j
$\mathbf{C}_{ijk}$	**triple-collinear** partons (i, j, k)
$\mathbf{C}_{ijkl}$	**double-collinear** partons (i, j) and (k, l)
$\mathbf{SC}_{ijk}$	**soft** partons i and **collinear** partons (j, k)



Singular structure of the RR

- ***Limits on matrix elements:*** *RR factorises into universal kernel × lower multiplicity matrix elements [Catani, Grazzini 9810389, 9908523]*

Example: *double-soft*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\cancel{i}\cancel{j}}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\cancel{i}\cancel{j}}\big) \right]$$

triple collinear

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}}, k_{ijk}\big)$$

$I^{(i)}_{cd}$ = single eikonal
$I^{(ij)}_{cd}$ = double eikonal
$P^{\mu\nu}_{ijk}$ = triple splitting

} Functions of **Lorentz invariants**

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triple collinear

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}}, k_{ijk}\big)$$

Born-level kinematics does **not satisfy the mass-shell condition and momentum conservation**

Momentum mapping needed!

$I^{(i)}_{cd}$ = single eikonal
$I^{(ij)}_{cd}$ = double eikonal
$P^{\mu\nu}_{ijk}$ = triple splitting

Functions of **Lorentz invariants**

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- ***Limits on matrix elements:*** *RR factorises into universal kernel × lower multiplicity matrix elements [Catani, Grazzini 9810389, 9908523]*

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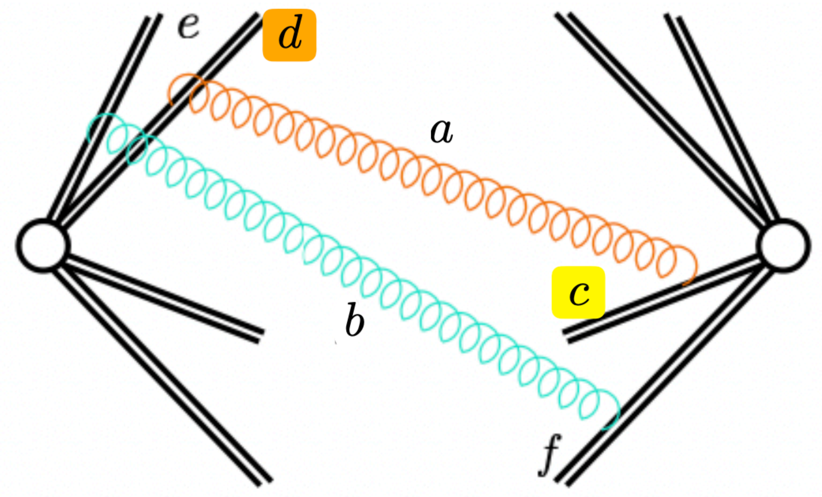
- ***Freedom in choosing the mapping:*** *adaptive parametrisation tuned to the specific kernel*

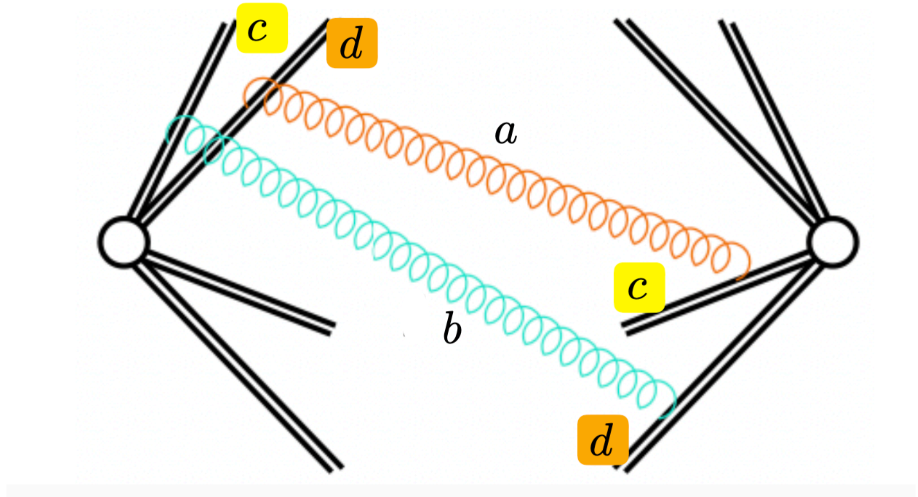
$$\bar{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c\neq i,j \\ d\neq i,j,c}}\left[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ef}\, B_{cdef}\left(\left\{\bar{k}^{(icd,jef)}\right\}\right) + 4\sum_{e\neq i,j,c,d} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ed}\, B_{cded}\left(\left\{\bar{k}^{(icd,jed)}\right\}\right)\right.$$

$$\left. + 2\, I^{(i)}_{cd}\, I^{(j)}_{cd}\, B_{cdcd}\left(\left\{\bar{k}^{(ijcd)}\right\}\right) + \left(I^{(ij)}_{cd} - \frac{1}{2}\, I^{(ij)}_{cc} - \frac{1}{2}\, I^{(ij)}_{dd}\right) B_{cd}\left(\left\{\bar{k}^{(ijcd)}\right\}\right)\right]$$



$$\{k\} \to \{\bar{k}\}^{(acd,bef)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\text{rad}}^{(acd)} \cdot d\Phi_{\text{rad}}^{(bef)}$$



$$\{k\} \to \{\bar{k}\}^{(abcd)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\text{rad},2}^{(abcd)}$$

Integration of the double-real counterterms

<u>Third step:</u> conterterms **integration.** Advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n}$$

$$+ \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}}$$

$$+ \int d\Phi_{n+2} \big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \big(K^{(2)} - K^{(12)} \big)\, \delta_{X_n}$$

→ **Finite by construction and integrable in $d = 4$**

- ***3 different integrated counterterms:*** *different phase-space and complexity*

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)}\,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}\,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}}\, K^{(12)}\,,$$

Integration of the double-real counterterms

Third step: conterterms **integration.** Advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(\qquad + I^{(12)} \right) \delta_{X_n} \right.$$

$$+ \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \longrightarrow$$

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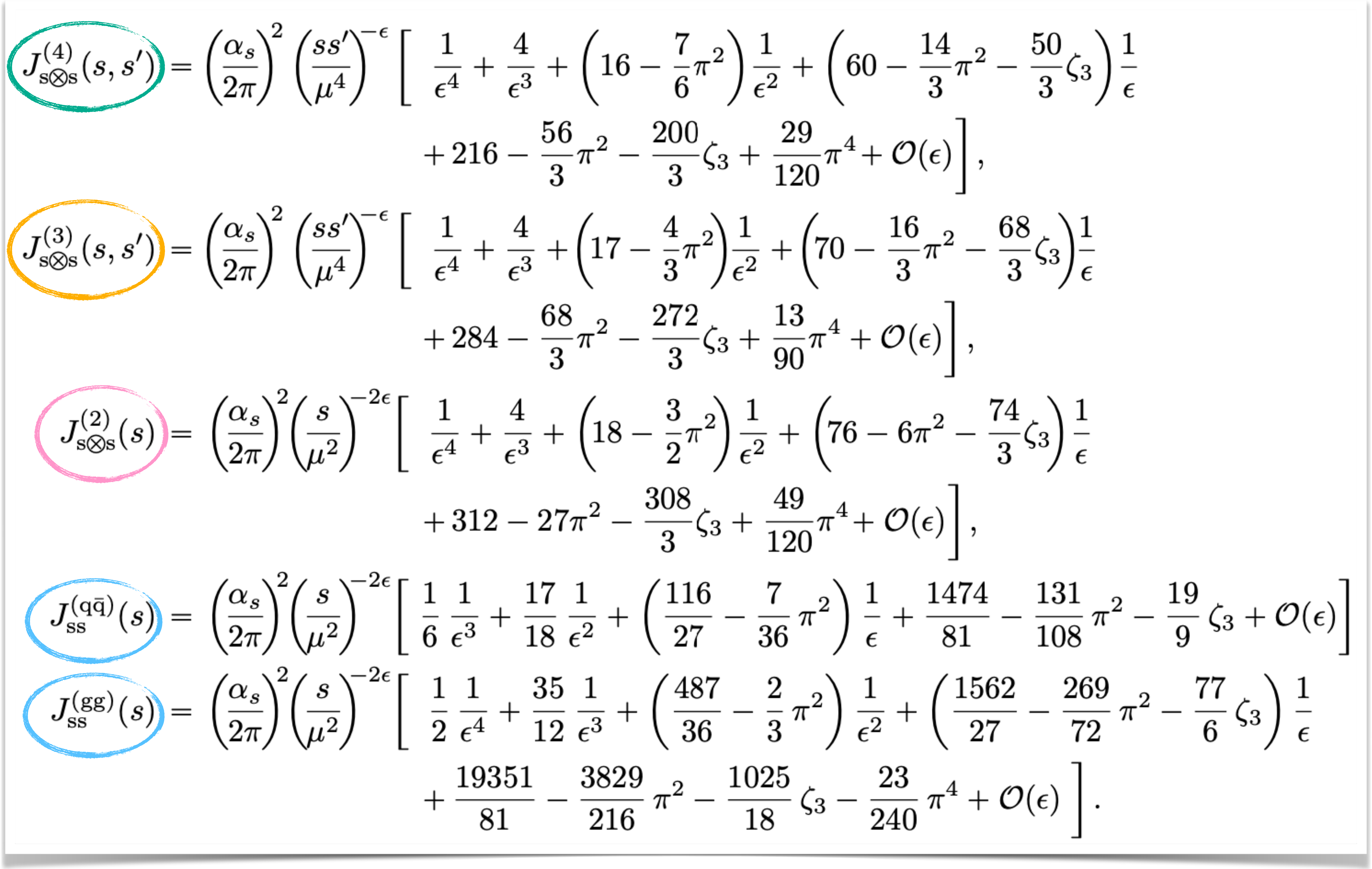
NNLO complexity: highly non trivial!

- ***Analytic integration*** *via* ***standard techniques*** *→ sectors sum rules + mapping adaptation [Magnea, C-SS et al. 2010.14493]*
- ***No approximations*** *→* ***simple and compact results*** *(at most simple* ***logarithmic dependence*** *on Mandelstam invariants)*

Integration of the double-real counterterms: example

- ***How the result looks like:***

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\,\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J^{ijcdef}_{\mathrm{s}\otimes\mathrm{s}}\,\bar{B}^{(icd,jef)}_{cdef} + 4\int d\Phi_n^{(icd,jed)}\,J^{ijcde}_{\mathrm{s}\otimes\mathrm{s}}\,\bar{B}^{(icd,jed)}_{cded}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J^{ijcd}_{\mathrm{s}\otimes\mathrm{s}}\,\bar{B}^{(ijcd)}_{cdcd} + J^{ijcd}_{\mathrm{ss}}\,\bar{B}^{(ijcd)}_{cd}\Big]\Bigg\},$$



$$J^{(4)}_{\mathrm{s}\otimes\mathrm{s}}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2}+\left(60-\frac{14}{3}\pi^2-\frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} + 216-\frac{56}{3}\pi^2-\frac{200}{3}\zeta_3+\frac{29}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(3)}_{\mathrm{s}\otimes\mathrm{s}}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(17-\frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(70-\frac{16}{3}\pi^2-\frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} + 284-\frac{68}{3}\pi^2-\frac{272}{3}\zeta_3+\frac{13}{90}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(2)}_{\mathrm{s}\otimes\mathrm{s}}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(18-\frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2}+\left(76-6\pi^2-\frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} + 312-27\pi^2-\frac{308}{3}\zeta_3+\frac{49}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(\mathrm{q\bar{q}})}_{\mathrm{ss}}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\left[\frac{1}{6}\,\frac{1}{\epsilon^3}+\frac{17}{18}\,\frac{1}{\epsilon^2}+\left(\frac{116}{27}-\frac{7}{36}\,\pi^2\right)\frac{1}{\epsilon}+\frac{1474}{81}-\frac{131}{108}\,\pi^2-\frac{19}{9}\,\zeta_3+\mathcal{O}(\epsilon)\right]$$

$$J^{(\mathrm{gg})}_{\mathrm{ss}}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{2}\,\frac{1}{\epsilon^4}+\frac{35}{12}\,\frac{1}{\epsilon^3}+\left(\frac{487}{36}-\frac{2}{3}\,\pi^2\right)\frac{1}{\epsilon^2}+\left(\frac{1562}{27}-\frac{269}{72}\,\pi^2-\frac{77}{6}\,\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81}-\frac{3829}{216}\,\pi^2-\frac{1025}{18}\,\zeta_3-\frac{23}{240}\,\pi^4+\mathcal{O}(\epsilon)\Bigg].$$

Subtracting RV singularities

Forth step: regularisation of the second line → delicate interplay between different counterterms *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \Big)\, \delta_{X_n}$$

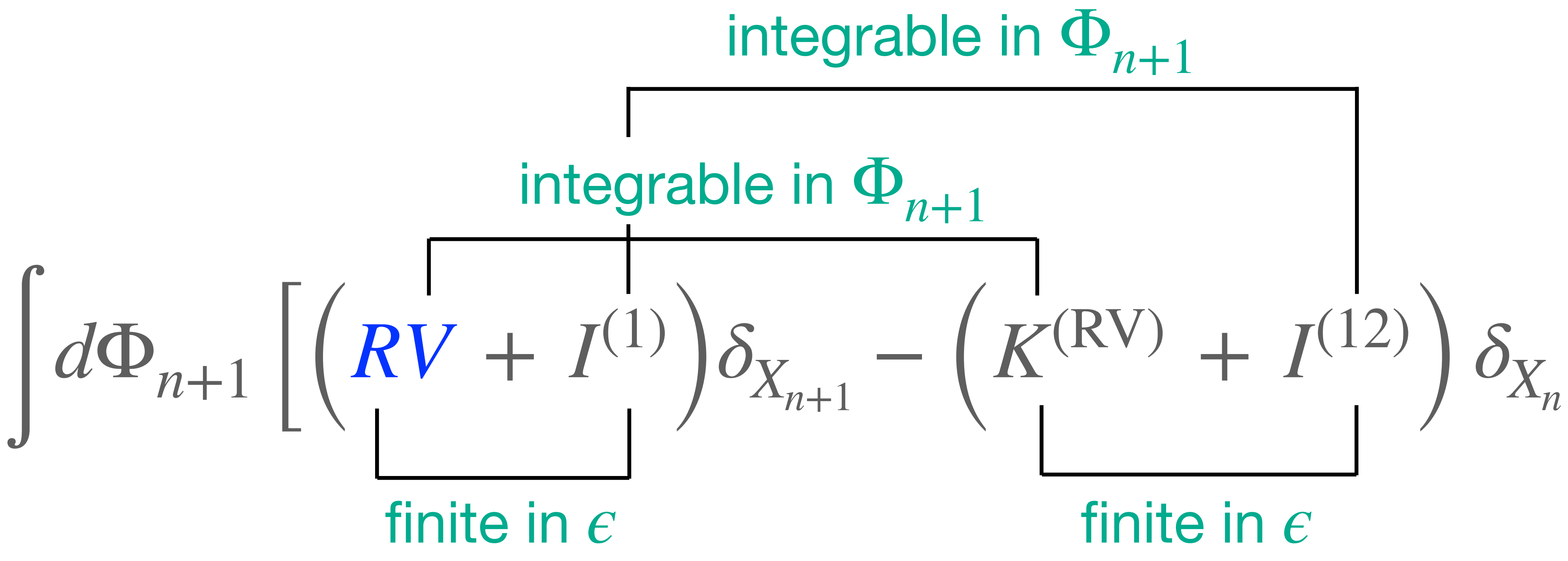
$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\rm RV)} + I^{(12)} \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big)\, \delta_{X_n}$$

$RV + I^{(1)}$ → finite in ϵ

$I^{(1)} - I^{(12)}$ → integrable

- ***Intricate cancellation pattern** involving both **poles and phase-space singularities***



integrable in Φ_{n+1}

integrable in Φ_{n+1}

$$\int d\Phi_{n+1} \Big[\Big(\underbrace{RV + I^{(1)}}_{\text{finite in } \epsilon} \Big) \delta_{X_{n+1}} - \Big(\underbrace{K^{(\rm RV)} + I^{(12)}}_{\text{finite in } \epsilon} \Big)\, \delta_{X_n}$$

Combination with double virtual

Fifth step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right] \right]$$

$$I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathrm{RV})}$$

Combination with double virtual

Fifth step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\rm RV)} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Bigg[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\rm hc} \gamma_l^{\rm hc} \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B} \\
& + \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2\left(1-\zeta_2\right) \sum_{j,c\neq j,r} \gamma_j^{\rm hc} \left(2 - \mathbf{L}_{cr}\right) \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \Bigg[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 + \left(4 - \mathbf{L}_{cd}\right) \sum_j \gamma_j^{\rm hc} \mathbf{L}_{jr} \Bigg] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \Bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2\left(1-\zeta_3\right) \mathbf{L}_{cd} \Bigg] \mathbf{B}_{cdcd} \\
& + \left(1-\zeta_2\right) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd} \mathbf{L}_{ef} \left[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \right] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \left[\ln \frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,{\rm Li}_3\left(-\frac{s_{ce}}{s_{de}} \right) \right] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \Bigg[\Sigma_\phi - \sum_j \gamma_j^{\rm hc} \mathbf{L}_{jr} \Bigg] \mathbf{V}^{\rm fin} + \sum_{c,d\neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}_{cd}^{\rm fin} \Bigg\} + \mathbf{VV}^{\rm fin}
\end{aligned}$$

Combination with double virtual

Fifth step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$VV + I^{(2)} + I^{(\rm RV)} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \Bigg[\boxed{I^{(0)}} + \sum_j \boxed{I_j^{(1)}} \mathbf{L}_{jr} + \sum_j \boxed{I_j^{(2)}} \mathbf{L}_{jr}^2$$

$$+ \sum_j \Big[\boxed{I_{jr}^{(0)}} + \boxed{I_{jr}^{(1)}} \mathbf{L}_{jr} \Big] \mathbf{B}_{jr} - 2\,(1-$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd} \Big[\boxed{I_{cd}^{(0)}} + \boxed{I_{cd}^{(1)}} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2$$

$$+ \sum_{c,d\neq c} \Bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2\Big($$

$$+ (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd}\, \mathbf{L}_{ed}\, \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}}$$

$$+ \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \Bigg[\ln\frac{s_{ce}}{s_{de}}\, \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2$$

$$+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ \Bigg[\Sigma_\phi - \sum_j \gamma_j^{\rm hc}\, \mathbf{L}_{jr} \Bigg] \mathbf{V}^{\rm fin} + \sum_{c,d\neq c} \mathbf{L}_{cd} \Bigg(\quad 2 \quad \Bigg) \quad \Bigg\}$$

$$\begin{aligned}
I^{(0)} = \; & N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F \left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right] \\
& + N_g^2 \left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right] \\
& + N_q C_F \Bigg[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right) \\
& \qquad + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\Bigg] \\
& + N_g \Bigg[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right) \\
& \qquad + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\Bigg] \\
I_j^{(1)} = \; & \delta_{f_a\{q,\bar q\}} C_F \Bigg[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3}N_g\beta_0 \\
& \qquad + C_F\left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\Bigg] \\
& + \delta_{f_a g}\Bigg[N_q C_F C_A(10 - 7\zeta_2) - N_q C_F\beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A\beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right) \\
& \qquad - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4}C_F C_A - \frac{3}{4}C_F\beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\Bigg] \\
I_j^{(2)} = \; & \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15)C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\,C_{f_j}^2 \\
I_{jr}^{(0)} = \; & (-1 + 3\zeta_2 - 2\zeta_3)C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3)C_{f_j} + (5 + 2\zeta_3)\gamma_j \\
I_{jr}^{(1)} = \; & (1 - \zeta_2)C_A + \frac{1}{2}(4 + 7\zeta_2)C_{f_j} - (2 + \zeta_2)\gamma_j \\
I_{cd}^{(0)} = \; & \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right)C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\,C_{f_d} \\
I_{cd}^{(1)} = \; & -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right)C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi
\end{aligned}$$

Take home message

Local Analytic Sector Subtraction provides a fully local infrared subtraction scheme at NNLO for generic coloured massless final states.

Take home message

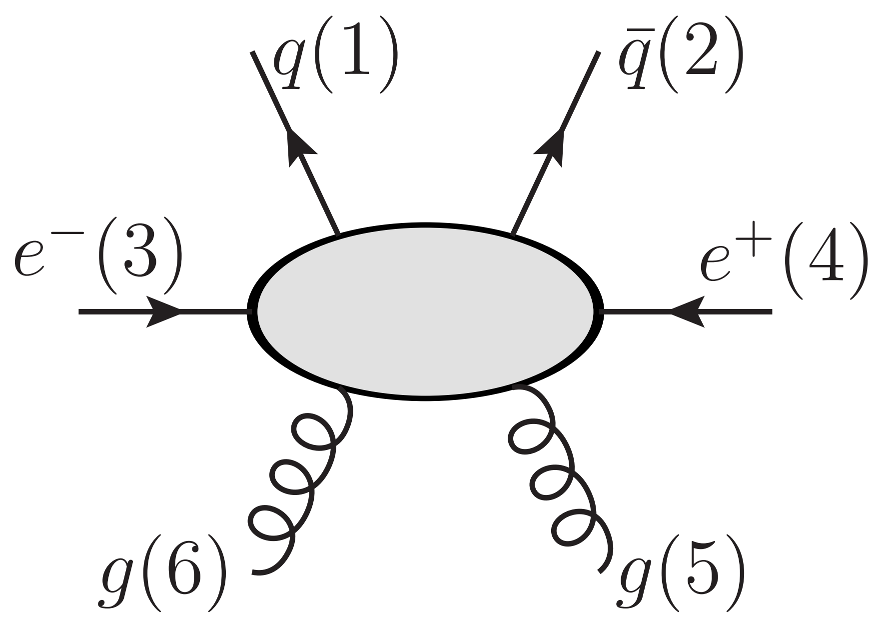
Local Analytic Sector Subtraction provides a fully local infrared subtraction scheme at NNLO for generic coloured massless final states.

Thank you for your attention!

Backup

Phase space partitions

Examples: Local Analytic Sector Subtraction $e^+e^- \to \gamma^* \to q\bar{q} + g\,g$ *[Magnea, C.S-S. et al. 1806.09570]*



$$1 = \mathscr{W}_{1225} + \mathscr{W}_{1226} + \mathscr{W}_{1252} + \mathscr{W}_{1256} + \ldots + \mathscr{W}_{6152}$$

$$\mathscr{W}_{abcd} = \frac{\sigma_{abcd}}{\sum_{m,n,p,q} \sigma_{mnpq}} \qquad \sigma_{abcd} = \frac{1}{(e_a\,w_{ab})^\alpha}\frac{1}{(e_c + \delta_{bc}\,e_a)\,w_{cd}}\,, \quad \alpha > 1$$

Angular+energy partition

Advantages:

1. Compact definition
2. Triple-collinear sectors do not require further partition
3. Structure of collinear singularities fully defined
4. Valid for arbitrary number of FS partons
5. **Defined in terms of Lorentz invariants**

Disadvantages:

1. Numerous sectors → consequence of being fully general
   → non minimal structure
2. Non-trivial recombination before integration

Sector functions at NLO in the analytic sector subtraction

Sector functions $\mathcal{W}_{ij}$:

1) Select the minimum number of singularities

$$\mathbf{S}_i \mathcal{W}_{ab} = 0 \,, \qquad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = 0 \,, \qquad \forall a, b \notin \{i, j\} \,.$$

2) Sum properties

$$\sum_{i, j \neq i} \mathcal{W}_{ij} = 1 \qquad \mathbf{S}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1 \,, \qquad \mathbf{C}_{ij} \sum_{a, b \in \{ij\}} \mathcal{W}_{ab} = 1 \,.$$

3) Explicit form

$$CM : q^{\mu} = (\sqrt{s}, \overrightarrow{0}) \,, \qquad e_i = \frac{s_{qi}}{s} \,, \qquad \omega_{ij} = \frac{s \, s_{ij}}{s_{qi} \, s_{qj}} \,,$$

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k, l \neq k} \sigma_{kl}} \,, \qquad \sigma_{ij} = \frac{1}{e_i \, \omega_{ij}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c \neq a} 1/\omega_{ac}} \,, \qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = \left(\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja} \right) \frac{e_b}{e_a + e_b}$$

The idea of mappings

$$\int d\Phi_{n+1}\left(R_{n+1}-K_{n+1}\right) \xrightarrow{\{k\}_{n+1}\,\to\,\{\bar{k}_n\}^{(abc)}} \int d\Phi_{n+1}\left(R_{n+1}-\overline{K}_{n+1}\right)$$

$$S_i\,R_{n+1}\big(\{k\}\big) \propto \sum_{a,c\neq i}\frac{s_{cd}}{s_{ci}\,s_{di}}\,B_n\big(\{k\}_{\not{i}}\big)$$

$$\overline{S}_i\,R_{n+1}\big(\{k\}\big) \propto \sum_{a,c\neq i}\frac{s_{cd}}{s_{ci}\,s_{di}}\,B_n\big(\{\bar{k}\}^{(icd)}\big)$$

$$C_{ij}\,R_{n+1}\big(\{k\}\big) \propto \frac{1}{s_{ij}}\,P^{\mu\nu}_{ij}(s_{ir},s_{jr})\,B^{\mu\nu}_n\big(\{k\}_{\not{i}\not{j}},k_{ij}\big)$$

$$\overline{C}_{ij}\,R_{n+1}\big(\{k\}\big) \propto \frac{1}{s_{ij}}\,P^{\mu\nu}_{ij}(s_{ir},s_{jr})\,B^{\mu\nu}_n\big(\{\bar{k}\}^{(ijr)}\big)$$

$$S_i\,C_{ij}\,R_{n+1}\big(\{k\}\big) \propto \frac{s_{jr}}{s_{ij}\,s_{ir}}\,B_n\big(\{k\}_{\not{i}}\big)$$

$$\overline{S}_i\,\overline{C}_{ij}\,R_{n+1}\big(\{k\}\big) \propto \frac{s_{jr}}{s_{ij}\,s_{ir}}\,B_n\big(\{\bar{k}\}^{(ijr)}\big)$$

Why a mapping?

1. $\{k\}_{\not{i}}$ is a set of n momenta that do not satisfy n-body momentum conservation away from the exact S_i limit
2. $\{k\}_{\not{i}\not{j}}, k_{ij}$ is a set of n momenta where $k_{ij} = k_i + k_j$ is off-shell away from the exact C_{ij} limit
3. Factorise the $n+1$-body PS intro a n-body and radiation phase space is necessary to integrate K only in the latter

Collinear limit: single mapping > *dipole = (ijr)*

Soft limit: different mapping for each contribution > *dipole = (icd)*

Lesson from NLO

- **Unitary partition** of radiative phase-space with **sector functions** $\mathscr{W}_{ij}$
- Collection of relevant IRC limits for a given sector
- **Catani-Seymour** final-state **dipole mapping**
- Promotion to counterterms: **improved limits**
- **Locality of the cancellation** ensured by **consistency relations**
- $\mathscr{W}_{ij}$ **sum rules+ mapping adaptation = simple analytic counterterm integration**

$$K = \sum_{i,j} K_{ij} \propto \overline{\mathbf{S}}_i R \left[\overbrace{\sum_j \overline{\mathbf{S}}_i \mathscr{W}_{ij}}^{=1} \right] + \overline{\mathbf{C}}_{ij} R \left[\overbrace{\overline{\mathbf{C}}_{ij} \left(\mathscr{W}_{ij} + \mathscr{W}_{ji} \right)}^{=1} \right] - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R \left[\overbrace{\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \mathscr{W}_{ij}}^{=1} \right]$$

$$\Longrightarrow K = \sum_i \overline{\mathbf{S}}_i R + \sum_{i,j\neq i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) R$$

Remarks:

1. The integrated counterterm has to **match the poles of V**, which is **not split** into sectors
2. The sector functions would have made the **integration** much **more involved**

Integration of the double-real counterterms: example

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi^{(ijcd)}_{n+2}\, I^{(ij)}_{cd}\, B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - \boxed{s_{ij}\,s_{cd}}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\int d\Phi^{(ijcd)}_{n+2}\,\frac{s_{ij}\,s_{cd}^2}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}} \quad \propto \int_0^1 \frac{dx'\,dy'\,dz'\,dx\,dy\,dz\,(z-1)^2(1-y)^{1-2\epsilon}y^{-2\epsilon-1}(1-y')^{1-2\epsilon}y'^{-\epsilon}[(1-z)z]^{-\epsilon}[(1-z')z']^{-\epsilon-1}}{[x(1-x)x'(1-x')]^{\epsilon+1/2}(y'(z-1)-z)\left(y'z'(1-z)+(1-z')z+2(2x'-1)\sqrt{y'(z-1)z(z'-1)z'}\right)}$$

- Integrate over x → simple Beta functions
- Integrate over y → simple Beta function
- Integrate over x' → Master Integral $I_{x'}$ → Hypergeometric and Theta functions
- Integrate over z' → partial fractioning $\frac{I_{x'}}{[z'(1-z')]^{1+\epsilon}} = \frac{I_{x'}}{[z'(1-z')]^{\epsilon}}\left[\frac{1}{z} + \frac{1}{1-z}\right]$
  → Master Integral $I_{x'z'} + J_{x'z'}$ → Hypergeometric functions
- Integrate over z → Integral representation of Hyp. → auxiliary t variable
- Integrate over y' → poles extraction

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

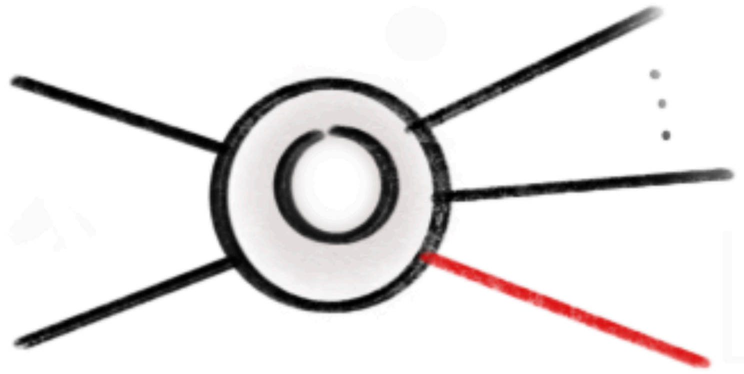
$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right.$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

🟨 ***1loop single unresolved***



$$K_{ij}^{(\mathbf{RV})} \equiv K_{ij,\,\mathrm{expected}}^{(\mathbf{RV})} + \Delta_{ij} = \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) \right] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

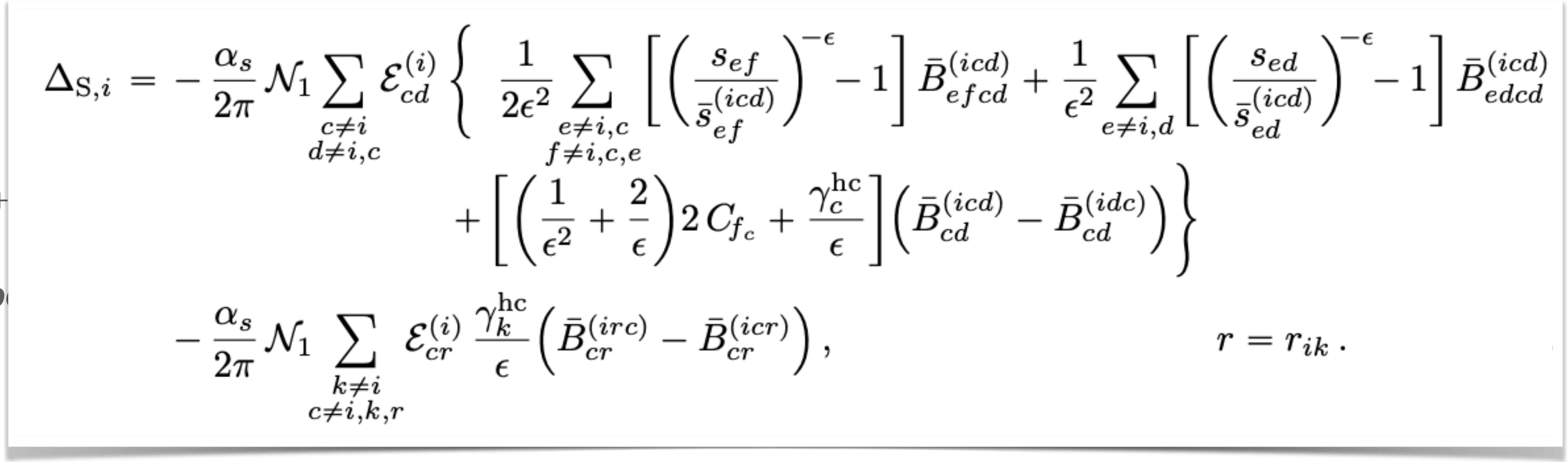
Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_X \right.$$

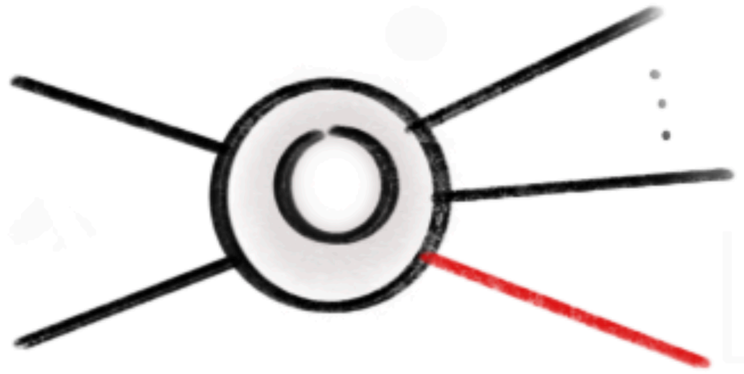
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+} } \right.$$



$$\Delta_{\mathrm{S},i} = -\frac{\alpha_s}{2\pi}\mathcal{N}_1 \sum_{\substack{c\neq i \\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \left\{ \frac{1}{2\epsilon^2} \sum_{\substack{e\neq i,c \\ f\neq i,c,e}} \left[\left(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}} \right)^{-\epsilon} - 1 \right] \bar{B}^{(icd)}_{efcd} + \frac{1}{\epsilon^2} \sum_{e\neq i,d} \left[\left(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}} \right)^{-\epsilon} - 1 \right] \bar{B}^{(icd)}_{edcd} \right.$$
$$\left. + \left[\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \right) 2\, C_{f_c} + \frac{\gamma^{\mathrm{hc}}_c}{\epsilon} \right] \left(\bar{B}^{(icd)}_{cd} - \bar{B}^{(idc)}_{cd} \right) \right\}$$
$$- \frac{\alpha_s}{2\pi}\mathcal{N}_1 \sum_{\substack{k\neq i \\ c\neq i,k,r}} \mathcal{E}^{(i)}_{cr} \frac{\gamma^{\mathrm{hc}}_k}{\epsilon} \left(\bar{B}^{(irc)}_{cr} - \bar{B}^{(icr)}_{cr} \right), \qquad r = r_{ik}\,.$$

- ***Intricate cancellation pattern*** *involving both p*

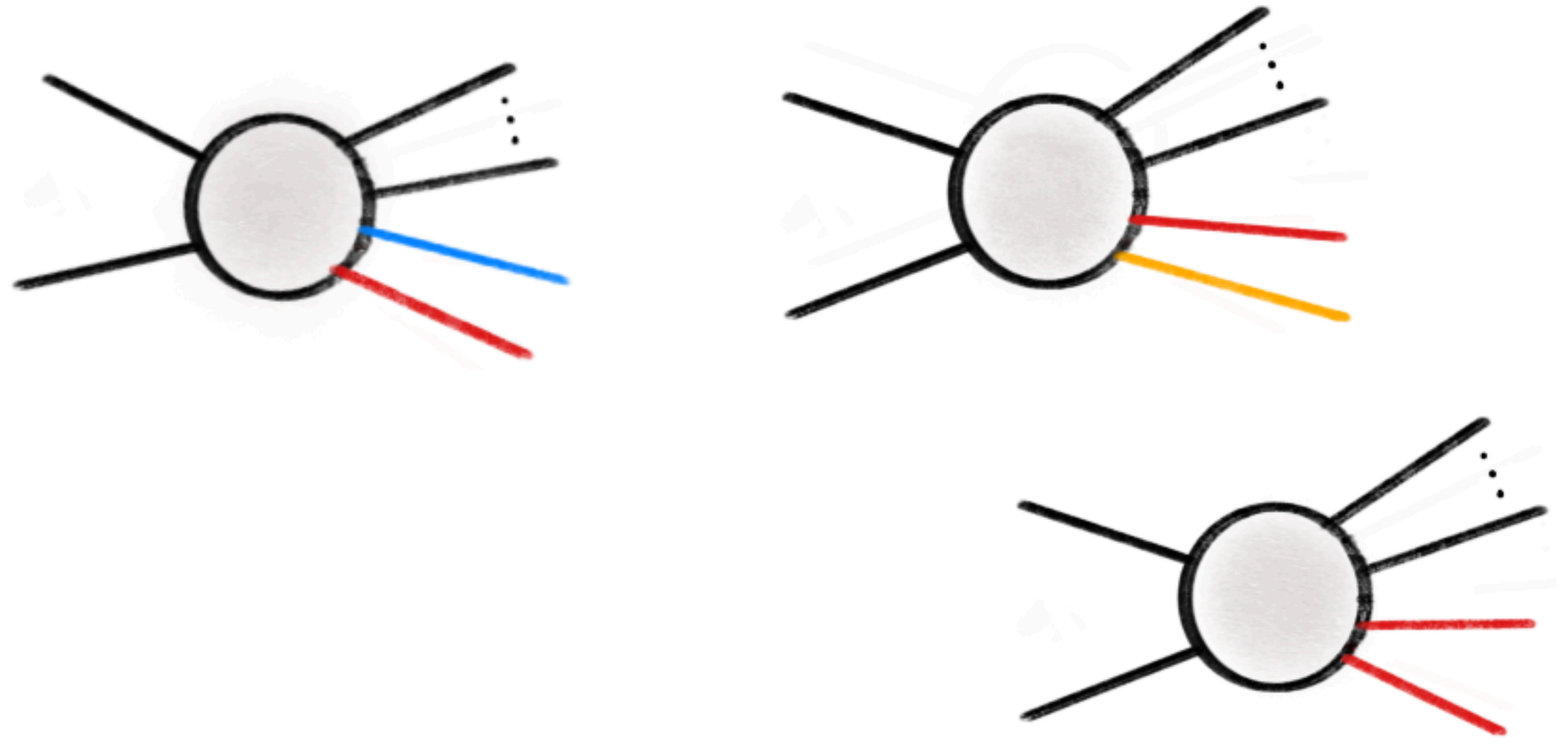
🟨 ***1loop single unresolved***

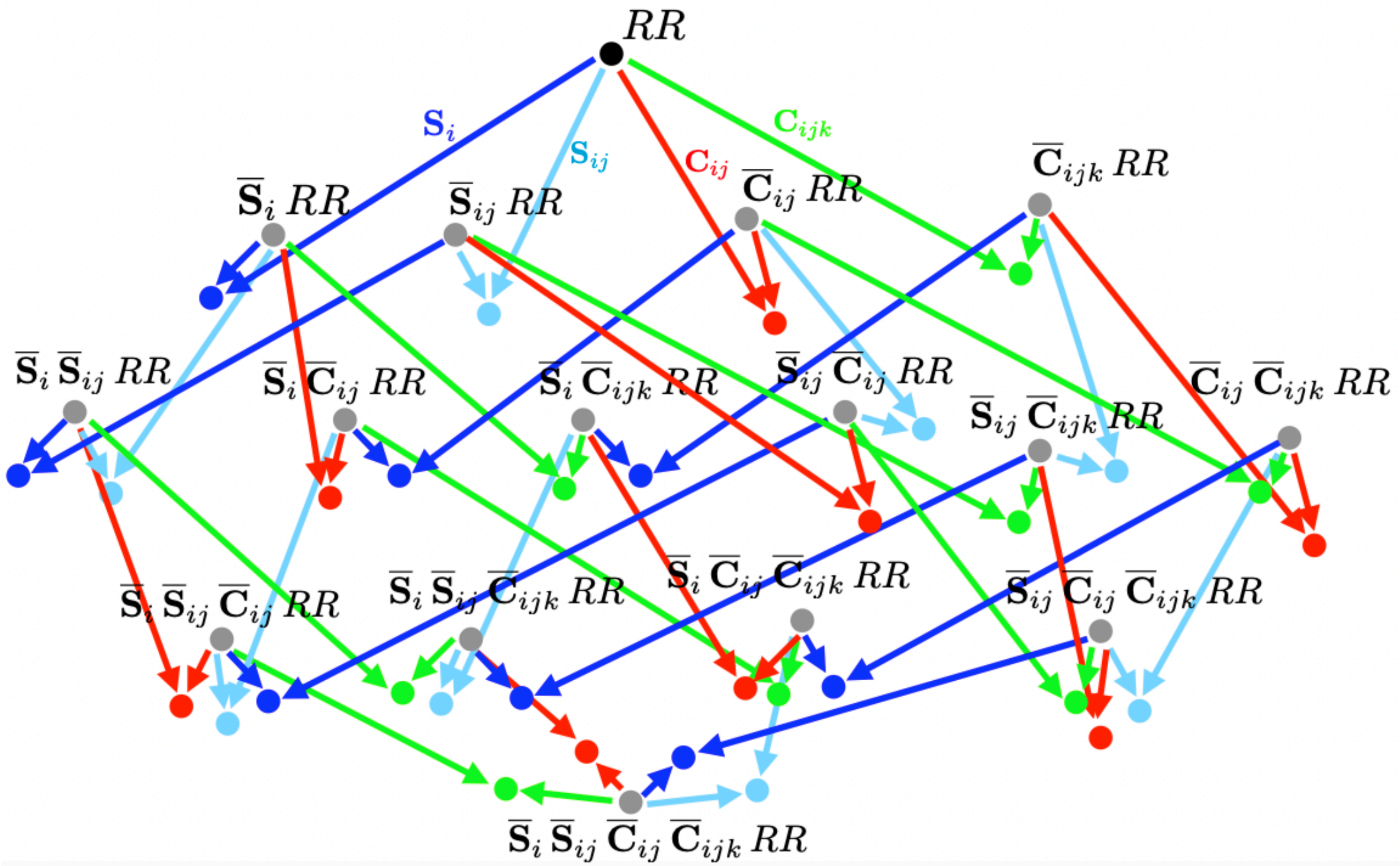


$$K^{(\mathbf{RV})}_{ij} \equiv K^{(\mathbf{RV})}_{ij,\,\mathrm{expected}} + \Delta_{ij} = \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) \right] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

Counterterm definition

- ***Locality of the cancellation*** *ensured by* ***consistency relations***
  - **Tower of nested limits** that have “horizontal” and “vertical” consistency relations.
  - **Consistency** relations have to **hold simultaneously** for **all the mapped limits**.
  - The **number of consistency relations grows rapidly** as the number of unresolved limits increases.
  - **Inconsistencies at the bottom** of the tower usually require a **redefinition** of the mapped limits **at the top** (and, as a consequence, of the entire cascade).
  - The definition of consistent mapped limits has to be set <u>**once for all**</u>, and is almost process-independent.





Selection of displayed limits

$\mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk}$

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\cancel{i}\cancel{j}}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\cancel{i}\cancel{j}}\big) \right]$$

$$I_{cd}^{(i)} = \frac{s_{cd}}{s_{ic}\, s_{id}} \qquad I_{cd}^{(ij)} = 2\, T_R\, I_{cd}^{(q\bar{q})(ij)} - 2\, C_A\, I_{cd}^{(gg)(ij)} \qquad s_{ab} = 2 p_a \cdot p_b$$

$$I_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})} \qquad I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2 s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})} \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}}, k_{ijk}\big) \qquad P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] \right.$$

$$z_a = \frac{s_{ar}}{s_{ir} + s_{jr} + s_{kr}}\,, \quad z_{ab} = z_a + z_b$$

$$\left. + \frac{s_{ijk}^2}{2 s_{ij}s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i(1 + 2z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivia**l and various **structures**, to be integrated over a **6-dim PS.**

Double real singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j}\left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big)\right]$$

$$I_{cd}^{(ij)} = 2\, T_R\, I_{cd}^{(q\bar{q})(ij)} - 2\, C_A\, I_{cd}^{(gg)(ij)}$$

$$I_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})}$$

$$I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}\,, s_{jr}\,, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)$$

$$P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2\left\{\frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_iz_j - 1}{z_{ij}} + \frac{z_iz_j - 2}{z_k} + \frac{(1 - z_kz_{ij})^2}{z_iz_kz_{jk}} + \frac{5}{2}\,z_k + \frac{3}{2}\right] + \frac{s_{ijk}^2}{2s_{ij}s_{ik}}\left[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik}z_{ij}} + \frac{1 - 2z_iz_{jk}}{z_jz_k} + 2z_jz_k + z_i(1 + 2z_i) - 4\right] + \frac{3(1-\epsilon)}{4}\right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\,\frac{s_{ijk}}{s_{ij}}\left\{\left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2}\right)\frac{1}{s_{ik}}\right]\tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_iz_j}{z_{ij}z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_k^2\, q_k^{\mu\nu}\right\} + perm\,.$$

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Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\,RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j}\Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\,I^{(j)}_{ef}\,B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I^{(ij)}_{cd}\,B_{cd}\big(\{k\}_{\not{i}\not{j}}\big)\Bigg] \qquad I^{(ij)}_{cd} = 2\,T_R\,I^{(q\bar{q})(ij)}_{cd} - 2\,C_A\,I^{(gg)(ij)}_{cd}$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} \qquad I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd}+s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\,RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\,P^{\mu\nu}_{ijk}\left(s_{ir}\,,s_{jr}\,,s_{kr}\right)B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}},k_{ijk}\big) \qquad P^{\mu\nu}_{ijk}B_{\mu\nu} = P_{ijk}\,B + Q^{\mu\nu}_{ijk}\,B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2\left\{\frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_iz_j-1}{z_{ij}}+\frac{z_iz_j-2}{z_k}+\frac{(1-z_kz_{ij})^2}{z_iz_kz_{jk}}+\frac{5}{2}\,z_k+\frac{3}{2}\right] + \frac{s_{ijk}^2}{2s_{ij}s_{ik}}\left[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}}+\frac{1+2z_i(1+z_i)}{z_{ik}z_{ij}}+\frac{1-2z_iz_{jk}}{z_jz_k}+2z_jz_k+z_i(1+2z_i)-4\right]+\frac{3(1-\epsilon)}{4}\right\}+perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\,\frac{s_{ijk}}{s_{ij}}\left\{\left[\frac{2z_j}{z_k}\frac{1}{s_{ij}}+\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}\right)\frac{1}{s_{ik}}\right]\tilde{k}_i^2\,q_i^{\mu\nu}+\left[\frac{2z_i}{z_k}\frac{1}{s_{ij}}-\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}-\frac{z_i}{z_k}+\frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_j^2\,q_j^{\mu\nu}-\left[\frac{2z_iz_j}{z_{ij}z_k}\frac{1}{s_{ij}}+\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}-\frac{z_i}{z_j}+\frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_k^2\,q_k^{\mu\nu}\right\}+perm\,.$$

Key problem: several **different invariants** combined into **non-trivial** and various **structures**, to be integrated over a **6-dim PS.**

↘

Key solution: split the **different structures** according to the contributing Lorentz invariants and **tune the mapping !**

Double real singular kernels:

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$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \Bigg]$$

$$I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar{q})(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd}$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s^2_{ij}(s_{ic} + s_{jc})(s_{id} + s_{jd})}$$

$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s^2_{ij}(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\boxed{\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}\left(s_{ir}\,, s_{jr}\,, s_{kr}\right) B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)}$$

$$P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \left\{ \frac{(1-\epsilon)s^2_{ijk}}{4s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] + \frac{s^2_{ijk}}{2s_{ij}s_{ik}} \left[\frac{2z_i z_j z_{ik}(1 - 2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik}z_{ij}} + \frac{1 - 2z_i z_{jk}}{z_j z_k} + 2z_j z_k + z_i(1 + 2z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

How the results look like:

$$\boxed{\int d\Phi_{n+2}\, \overline{\mathbf{C}}_{ijk}\, RR \;=\; \int d\Phi_n\big(\bar{k}^{(ijrk)}\big)\, J_{\rm cc}\big(\bar{s}^{ijkr}_{kr}\big)\, B\big(\bar{k}^{(ijrk)}\big)}$$

$$J^{(3g)}_{\rm cc}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[\frac{15}{\epsilon^4} + \frac{63}{\epsilon^3} + \left(\frac{853}{3} - 22\,\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{10900}{9} - \frac{275}{3}\,\pi^2 - 376\,\zeta_3\right)\frac{1}{\epsilon} + \frac{180739}{36} - \frac{3736}{9}\,\pi^2 - 1555\,\zeta_3 + \frac{41}{10}\,\pi^4 + \mathcal{O}(\epsilon) \right]$$