Spin physics in heavy-ion collisions

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The 2025 International Conference on the Structure of Baryons
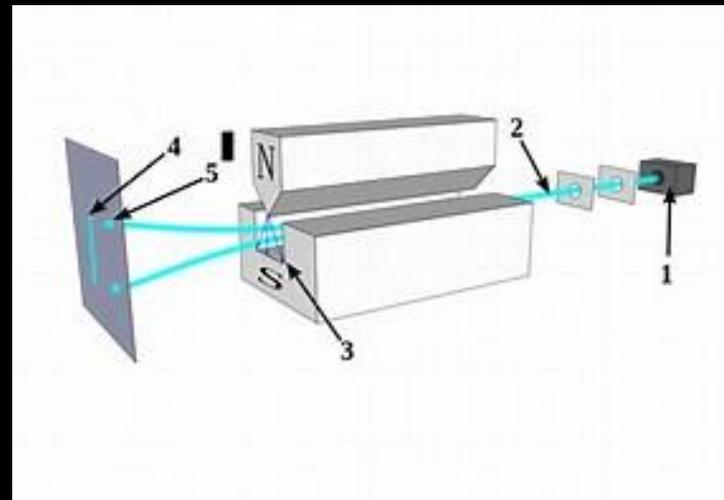November 10-14, 2025, Jeju, Republic of Korea

Outline

- **A microscopic model for the emergence of spin-vorticity coupling from spin-orbit coupling in parton-parton scatterings**

- **Spin Boltzmann (Kinetic) Equations for massive fermions from Wigner functions**

100 years of Spin
and Quantum Mechanics

Discovery of electron spin

- **Stern-Gerlach experiment (1922):** first observation of two discrete quantum states (spin) because sliver atoms have μ_B and split into two belt in inhomogeneous B field

Walter Gerlach & Otto Stern

Otto Stern, Nobel prize in Physics 1943

Discovery of electron spin

- **Fourth quantum number by Wolfgang Pauli (1924):** to explain anomalous Zeeman effect, which takes only two values.

- **Concept of electron spin by Ralph Kronig (1925):** can explain even splitting of alkali spectra (over-estimated by factor 2), but opposed by Pauli and Bohr, not published
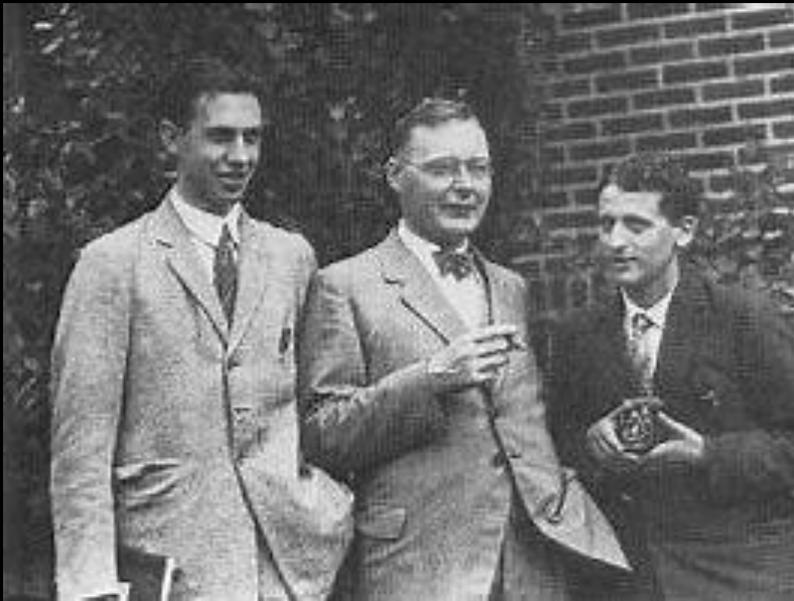
**Wolfgang Pauli,
Nobel prize in Physics 1945**

Ralph Kronig

Discovery of electron spin

- **Electron spin by Uhlenbeck and Goudsmit (1925):**

G.E. Uhlenbeck and S. Goudsmit, Naturwissenschaften 13 (1925) 953.

A subsequent publication by the same authors, Nature 117 (1926) 264.

George Uhlenbeck, Samuel Goudsmit

Rotation and Spin in HIC

Qun Wang (USTC/AUST), Spin physics in heavy ion collisions (Baryons 2025)

Rotation effects in HIC

- **Huge global orbital angular momenta are produced in non-central collisions with respect to reaction plane**

$$L \sim 10^5 \hbar$$

- **Very strong magnetic fields are produced**

$$B \sim m_\pi^2 \sim 10^{18} \text{ Gauss}$$

- **How do orbital angular momenta be transferred to the matter in HIC?**

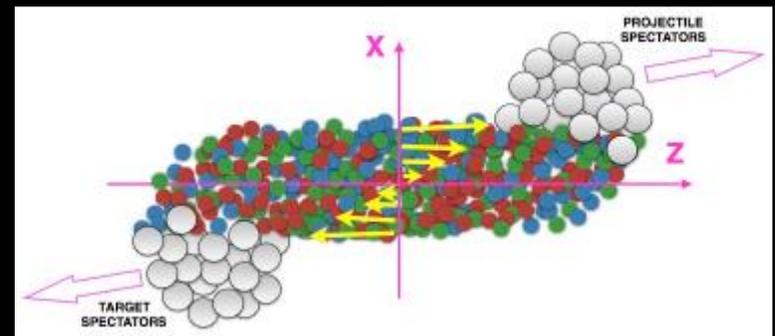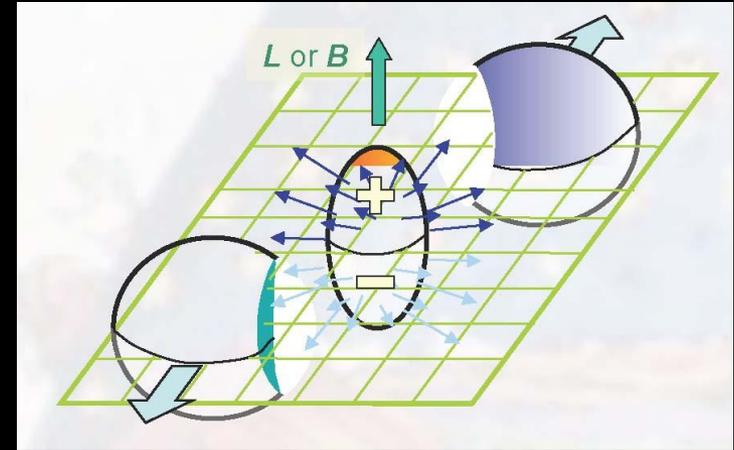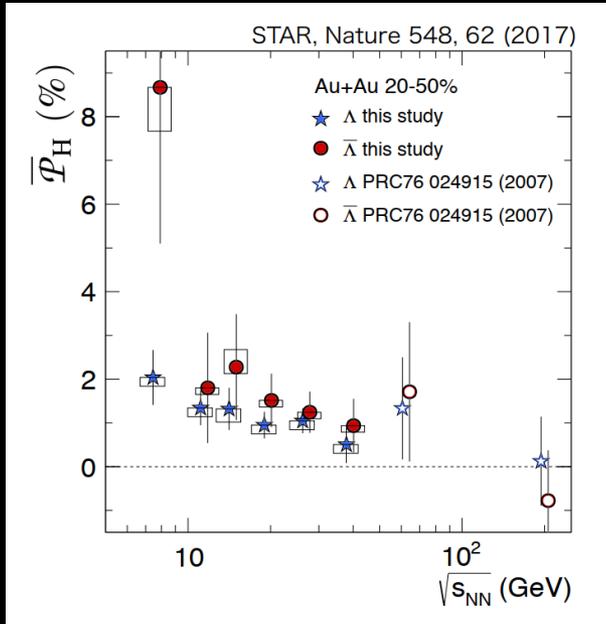- **How is spin coupled to local vorticity in the fluid?**

Figure taken from
Becattini et al, 1610.02506

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STAR: global polarization of Λ hyperon

STAR, Nature 548, 62 (2017)

Au+Au 20-50%
- ★ Λ this study
- ● $\overline{\Lambda}$ this study
- ☆ Λ PRC76 024915 (2007)
- ○ $\overline{\Lambda}$ PRC76 024915 (2007)

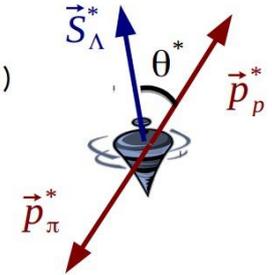$\overline{\mathcal{P}}_H$ (%) vs $\sqrt{s_{NN}}$ (GeV)

parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P_\Lambda} \cdot \mathbf{p_p^*})$$

α: Λ decay parameter (=0.642±0.013)
P_Λ: Λ polarization
p_p^*: proton momentum in Λ rest frame

$$\Lambda \rightarrow p + \pi^+$$
(BR: 63.9%, c τ ~7.9 cm)

Updated by BES III, PRL129, 131801 (2022)

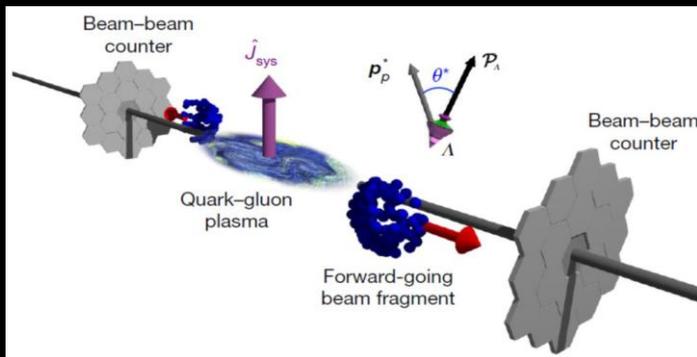ω = (9 ± 1)x10²¹/s, the largest angular velocity that has ever been observed in any system

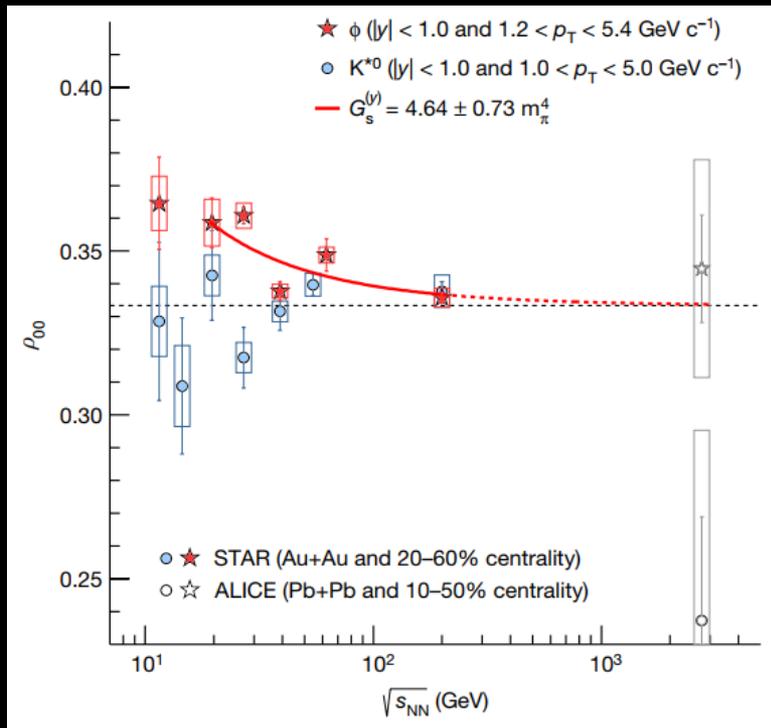*Liang, Wang, PRL (2005)*
Betz, Gyulassy, Torrieri, PRC (2007)
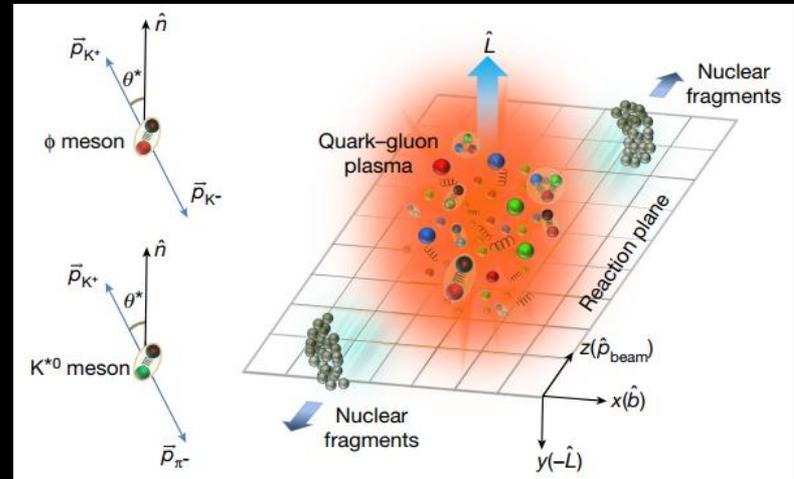*Becattini, Piccinini, Rizzo, PRC (2008)*
Gao et al., PRC (2008)

STAR: global spin alignments of vector mesons

STAR, Nature 614, 244 (2023);

★ ϕ ($|y| < 1.0$ and $1.2 < p_T < 5.4$ GeV c^{-1})
○ K^{*0} ($|y| < 1.0$ and $1.0 < p_T < 5.0$ GeV c^{-1})
— $G_s^{(y)} = 4.64 \pm 0.73$ m$_\pi^4$

○ ★ STAR (Au+Au and 20–60% centrality)
○ ☆ ALICE (Pb+Pb and 10–50% centrality)

Implication of correlation or fluctuation in polarization s and s-bar

Theory prediction:
Liang, Wang, PLB(2005);
Sheng, Oliva, QW, PRD(2020);
Sheng, Oliva, Liang, QW, Wang, PRL(2022).

$$P_\Lambda \sim \langle P_s \rangle, \qquad P_{\overline{\Lambda}} \sim \langle P_{\bar{s}} \rangle$$

$$\rho_{00}^\phi - \frac{1}{3} \sim \langle P_s P_{\bar{s}} \rangle \neq \langle P_s \rangle \langle P_{\bar{s}} \rangle \sim P_\Lambda P_{\overline{\Lambda}}$$

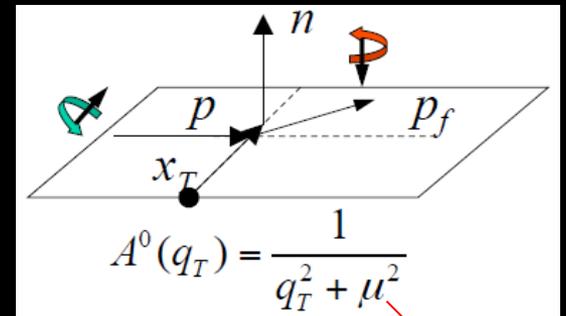# Emergence of spin-vorticity coupling from spin-orbit coupling

Quark polarization in potential scatterings

- **Quark scatterings at small angle in static potential at impact parameter x_T**
- **Unpolarized and polarized cross sections**

$$\frac{d\sigma}{d^2\vec{x}_T} = \frac{d\sigma_+}{d^2\vec{x}_T} + \frac{d\sigma_-}{d^2\vec{x}_T} = 4C_T\alpha_s^2 K_0(\mu x_T)$$

$$\frac{d\Delta\sigma}{d^2\vec{x}_T} = \frac{d\sigma_+}{d^2\vec{x}_T} - \frac{d\sigma_-}{d^2\vec{x}_T} \propto \vec{n}\cdot(\vec{x}_T\times\vec{p})$$

$$A^0(q_T) = \frac{1}{q_T^2 + \mu^2}$$

screening mass

$$\mu \sim T\sqrt{\alpha_S}$$

Spin quantization direction   **OAM**   **Spin-orbit coupling**

- **Polarization for small angle scattering and** $m_q \gg p, \mu$

$$P_q \approx -\pi\frac{\mu p}{4m_q^2} \sim -\frac{\Delta E_{LS}}{E_0}$$
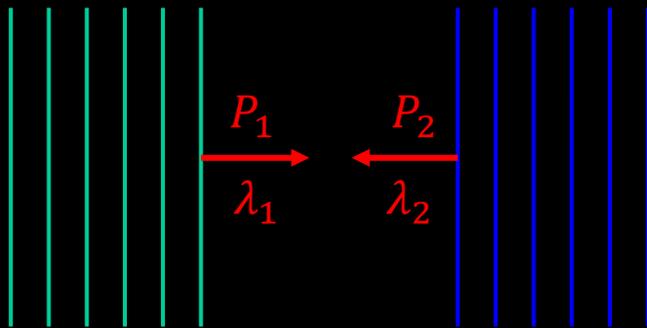
Liang, Wang, PRL 94, 102301(2005)

- **With initial polarization P_i, the final polarization P_f after one scattering is**
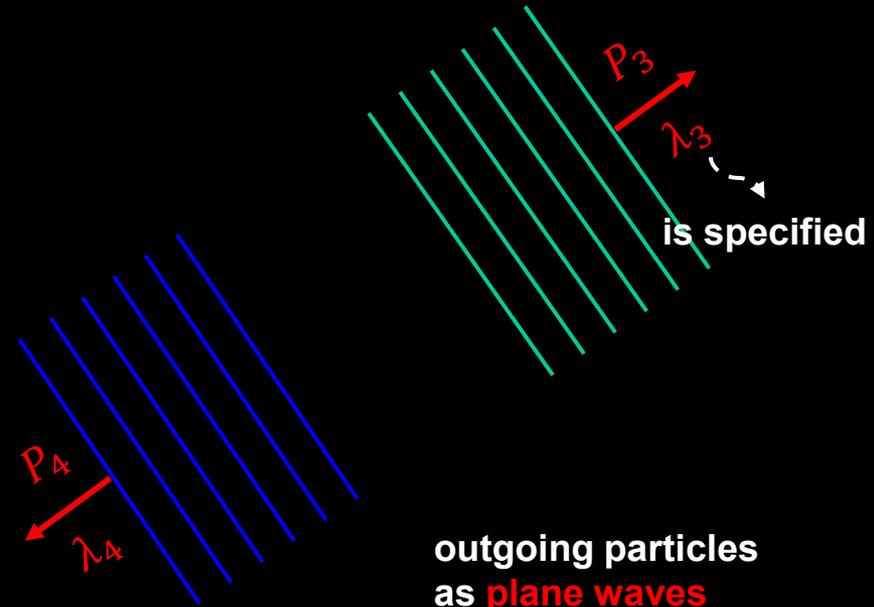
$$P_f = P_i - \frac{(1-P_i^2)\pi\mu p}{2E(E+m) - P_i\pi\mu p}.$$

Huang, Huovinen, Wang, PRC84, 054910(2011)

Collisions of particles as plane waves

P_1 λ_1 P_2 λ_2

**incident particles
as plane waves**

P_3 λ_3

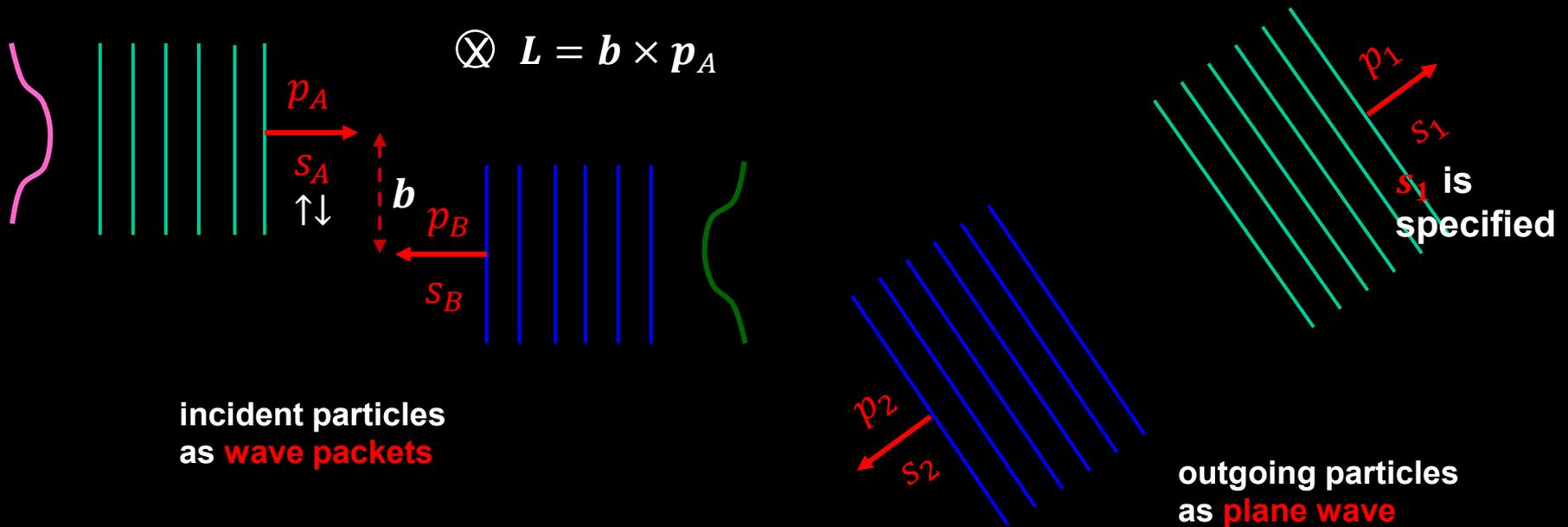**is specified**

P_4 λ_4

**outgoing particles
as plane waves**

**Particle collisions as plane waves:
since there is no favored position for particles, so the OAM vanishing**

$$\langle \widehat{x} \times \widehat{p} \rangle = 0 \quad \Longrightarrow \quad \left(\frac{d\sigma}{d\Omega} \right)_{\lambda_3 = \uparrow} = \left(\frac{d\sigma}{d\Omega} \right)_{\lambda_3 = \downarrow}$$

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Collisions of particles as wave packets

$\otimes$ $\boldsymbol{L} = \boldsymbol{b} \times \boldsymbol{p}_A$

p_A

s_A

$\uparrow\downarrow$

$\boldsymbol{b}$

p_B

s_B

incident particles as wave packets

p_1

s_1

s_1 **is specified**

p_2

s_2

outgoing particles as plane wave

Particle collisions as wave packets: there is a transverse distance between two wave packets (impact parameter) giving non-vanishing OAM and then the polarization of one final particle

$$\boldsymbol{L} = \boldsymbol{b} \times \boldsymbol{p}_A \qquad\longrightarrow\qquad \left(\frac{d\sigma}{d\Omega}\right)_{s_1=\uparrow} \neq \left(\frac{d\sigma}{d\Omega}\right)_{s_1=\downarrow}$$

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Quark-quark scattering at fixed impact parameter

For the quark-quark scattering of spin-momentum states

$$q_1(P_1, \lambda_1) + q_2(P_2, \lambda_2) \to q_1(P_3, \lambda_3) + q_2(P_4, \lambda_4)$$

where $P_i = (E_i, \vec{p}_i)$ and λ_i denote spin states, the difference cross section (λ_3 is specified)

$c_{qq} = 2/9$ (color factor)

$$d\sigma_{\lambda_3} = \frac{c_{qq}}{4F} \sum_{\lambda_1 \lambda_2 \lambda_4} \mathcal{M}(Q)\mathcal{M}^*(Q)(2\pi)^4\delta^{(4)}(P_1 + P_2 - P_3 - P_4)\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}$$

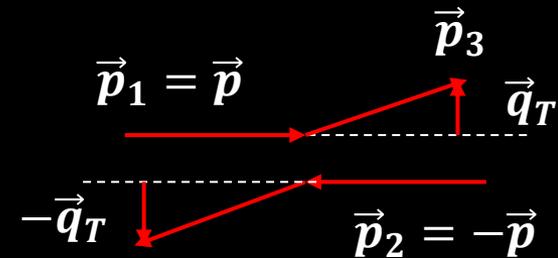fixed   sum over ↑↓ $Q = P_3 - P_1 = P_2 - P_4$

(momentum transfer)

$F = 4\sqrt{(P_1 \cdot P_2)^2 - m_1^2 m_2^2}$ (flux factor)

Integrate $\vec{p}_4$ and $p_{3z}^{\pm} = \pm\sqrt{p^2 - q_T^2}$

to remove $\delta^{(4)}(P_1 + P_2 - P_3 - P_4)$

$\vec{p}_3$

$\vec{p}_1 = \vec{p}$ $\vec{q}_T$

$-\vec{q}_T$ $\vec{p}_2 = -\vec{p}$

Quark-quark scattering at fixed impact parameter

We obtain $d\sigma_{\lambda_3}$ for scattered quark with spin state λ_3

$$d\sigma_{\lambda_3} = \frac{c_{qq}}{16F} \sum_{\lambda_1 \lambda_2 \lambda_4} \sum_{i=+,-} \frac{1}{(E_1 + E_2)|p^i_{3z}|} \mathcal{M}(Q_i)\mathcal{M}^*(Q_i) \frac{d^2\vec{q}_T}{(2\pi)^2}$$

for small angle scattering, only $i = +$ is relevant

Jacobian

momentum transfer in small angle scattering

Then we can introduce impact parameter $\vec{x}_T = (x_T, \phi)$

$$d\sigma_{\lambda_3} = \frac{c_{qq}}{16F} \sum_{\lambda_1,\lambda_2,\lambda_3} \int d^2\vec{x}_T \int \frac{d^2\vec{q}_T}{(2\pi)^2} \int \frac{d^2\vec{k}_T}{(2\pi)^2} \exp\left[i\left(\vec{k}_T - \vec{q}_T\right)\cdot\vec{x}_T\right] \frac{\mathcal{M}(\vec{q}_T)\mathcal{M}^*(\vec{k}_T)}{\Lambda(\vec{q}_T)\Lambda^*(\vec{k}_T)}$$
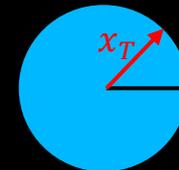
$\Rightarrow d^2\sigma_{\lambda_3}/d^2\vec{x}_T$

$$\boxed{1}$$

$$=$$
$$\sqrt{(E_1 + E_2)|p^+_{3z}(q_T)|}$$

If we integrate over $\vec{x}_T$ in whole space we obtain

$$\sigma_{\lambda_3} = \int_0^\infty dx_T \, x_T \int_0^{2\pi} d\phi \, \frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} \quad \longrightarrow \quad \sigma_\uparrow = \sigma_\downarrow$$

x_T

If we integrate over $\vec{x}_T$ in half-space we obtain

$$\sigma_{\lambda_3} = \int_0^\infty dx_T \, x_T \int_0^\pi d\phi \frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} \quad \longrightarrow \quad \sigma_\uparrow \neq \sigma_\downarrow$$

The differential cross section for spin-independent and spin-dependent part

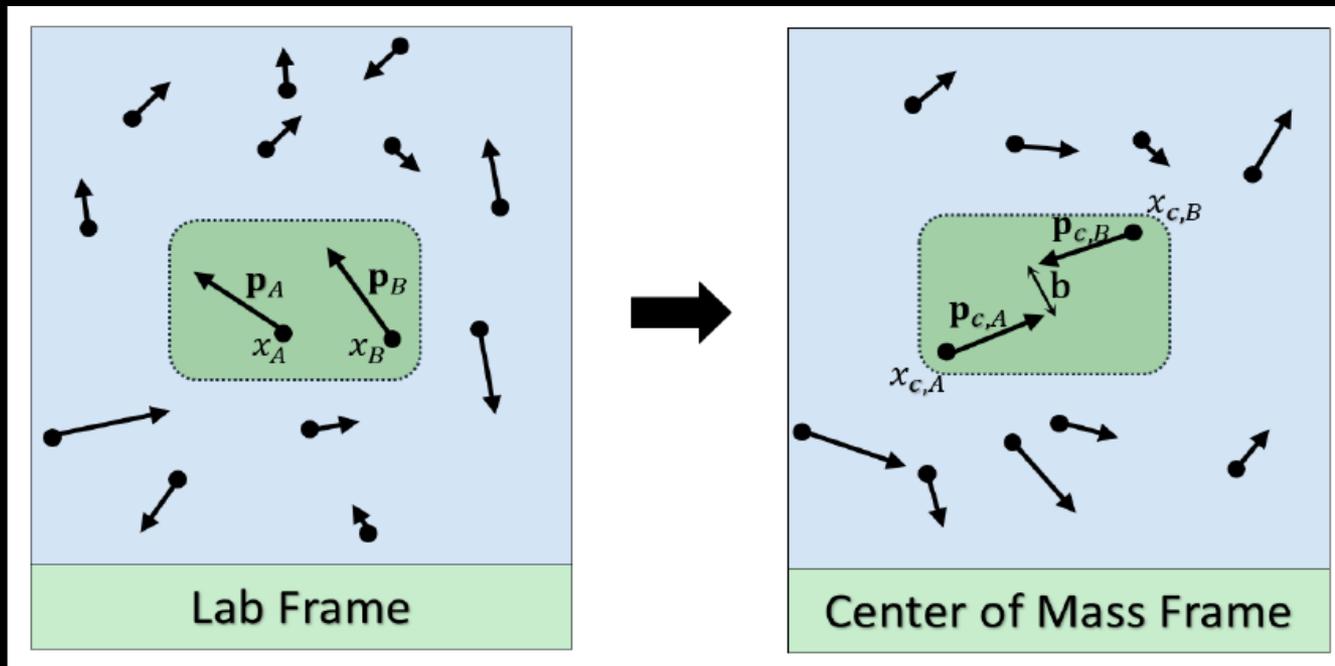$$\frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} = \frac{d^2\sigma}{d^2\vec{x}_T} + \lambda_3 \frac{d^2\Delta\sigma}{d^2\vec{x}_T}$$

$$\frac{d^2\sigma}{d^2\vec{x}_T} = \frac{1}{2}\left(\frac{d^2\sigma_\uparrow}{d^2\vec{x}_T} + \frac{d^2\sigma_\downarrow}{d^2\vec{x}_T}\right) = F(x_T)$$

$$\frac{d^2\Delta\sigma}{d^2\vec{x}_T} = \frac{1}{2}\left(\frac{d^2\sigma_\uparrow}{d^2\vec{x}_T} - \frac{d^2\sigma_\downarrow}{d^2\vec{x}_T}\right) = \underline{\vec{n}\cdot(\vec{x}_T \times \vec{p})}\Delta F(x_T)$$

$$\Delta\sigma = \int_0^\infty dx_T x_T \int_0^\pi d\phi \frac{d^2\Delta\sigma}{d^2\vec{x}_T}$$

$$\sigma = \int_0^\infty dx_T x_T \int_0^{2\pi} d\phi \frac{d^2\Delta\sigma}{d^2\vec{x}_T}$$

spin-orbit coupling

$$P_q = \frac{\Delta\sigma}{\sigma}$$

Gao, Chen, Deng, et al., PRC 77, 044902 (2008)

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Ensemble average in thermal QGP for global polarization through spin-orbit couplings in parton scatterings

Lab Frame

Center of Mass Frame

Zhang, Fang, QW, Wang, PRC 100, 064904 (2019)

From spin-orbit coupling to spin-vorticity coupling: ensemble average

- **Quark polarization rate per unit volume: 10D + 6D integration**

$$\frac{d^4 P_{AB \to 12}(X)}{dX^4} = \frac{\pi}{(2\pi)^4} \frac{\partial(\beta u_\rho)}{\partial X^\nu} \int \frac{d^3 p_A}{(2\pi)^3 2E_A} \frac{d^3 p_B}{(2\pi)^3 2E_B}$$

6D integral

$$\times |v_{c,A} - v_{c,B}| [\Lambda^{-1}]^\nu_j e_{c,i} \epsilon_{ikh} \hat{p}^h_{c,A}$$

Lorentz boost

$$\times f_A(X, p_A) f_B(X, p_B) (p^\rho_A - p^\rho_B) \Theta_{jk}(\mathbf{p}_{c,A})$$

10D integral

$$\equiv \frac{\partial(\beta u_\rho)}{\partial X^\nu} W^{\rho\nu}$$

16D integral !!

3-vector form

$$\frac{d^4 P_{AB \to 12}(X)}{dX^4} = 2W \nabla_X \times (\beta \mathbf{u})$$

- **Numerical challenge !!!** We have developed ZMCintegral-3.0, a Monte Carlo integration package that runs on multi-GPUs [Wu, Zhang, Pang, QW, Comp. Phys. Comm. (2020) (1902.07916)]

- **Another challenge:** there are more than 5000 terms in polarized amplitude squared for 2-to-2 parton scatterings

$$I^{q_a q_b \to q_a q_b}_M(s_2) = \sum_{s_A, s_B, s_1} \sum_{i,j,k,l} \mathcal{M}(\{s_A, k_A; s_B, k_B\} \to \{s_1, p_1; s_2, p_2\}) \mathcal{M}^*(\{s_A, k'_A; s_B, k'_B\} \to \{s_1, p_1; s_2, p_2\})$$

Nonlocal collisions in Boltzmann equations

Extend phase space by introducing a classical spin variable s^μ

$$f(x,p) \Longrightarrow f(x,p,s)$$

$$\int dS(p) = \frac{1}{\kappa(p)} \int d^4 s \, \delta(s^2+3)\delta(s\cdot p)$$

Boltzmann equation with non-local collisions

$$p \cdot \partial f(x,p,s) = C[f]$$

$$C[f] = C_{p+s}[f] + C_s[f]$$

$$= \int d\Gamma_1 d\Gamma_2 d\Gamma' \mathcal{W}$$

$$\times \left[f(x+\underline{\Delta}_1, p_1, s_1) f(x+\underline{\Delta}_2, p_2, s_2) - f(x+\underline{\Delta}, p, s) f(x+\underline{\Delta}', p', s') \right]$$

$$+ \int d\Gamma_2 dS_1(p) \mathcal{M} f(x+\underline{\Delta}_1, p, s_1) f(x+\underline{\Delta}_2, p_2, s_2)$$

$$d\Gamma = \int d^4 p \, \delta(p^2 - m^2) \int dS(p)$$

spin-momentum integral

**Δ : space-shift
Side-jump for chiral fermions:
Chen, Son, Stephanov (2015)**

**Weickgenannt, Speranza, Sheng, QW, Rischke (2021);
Wagner, Weickgenannt, Rischke (2022)**

Qun Wang (USTC/AUST), Spin physics in heavy ion collisions (Baryons 2025)

AMPT transport model
-- Li, Pang, QW, Xia, PRC96, 054908(2017)
-- Wei, Deng, Huang, PRC99, 014905(2019)
UrQMD + vHLLE hydro
-- Karpenko, Becattini, EPJC 77, 213(2017)
PICR hydro
-- Xie, Wang, Csernai, PRC 95,031901(2017)
Chiral Kinetic Equation + Collisions
-- Sun, Ko, PRC96, 024906(2017)
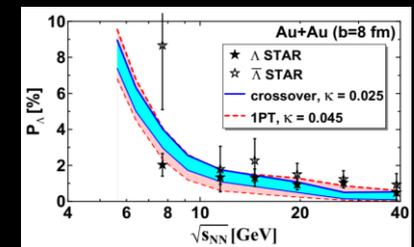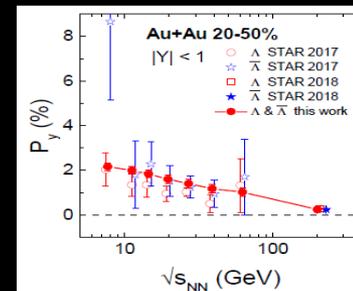-- Liu, Sun, Ko, PRL125, 062301(2020)
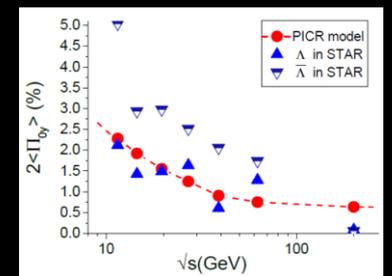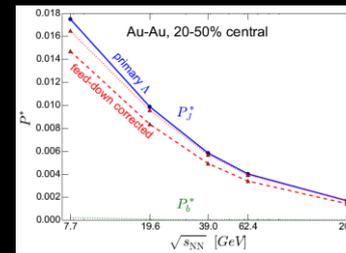**AVE+3FD**
-- Ivanov, 2006.14328

Reviews:
-- Huang, Liao, QW, Xia (2021)
-- Becattini, Buzzegoli, Niida, Pu, Tang, QW (2024)

vorticity $\omega \Rightarrow S$ spin   Becattini et al. (2013);
on freeze-out hypersurface   Fang et al. (2016)

Vortex formation in collisions of BEC as topological realization of global spin polarization

With field interaction
$v = 0.34c_s$

Vortices (spins)

Without field interaction
$v = 0.34c_s$

With field interaction
$v = 0$

Deng, QW, Zhang (2022)

Quantum kinetic equations with spin or Spin Boltzmann (kinetic) equations with Wigner functions

Wigner functions for massive fermions

Wigner function (4x4 matrix) for spin 1/2 massive fermions

$$W_{\alpha\beta}(x,p) = \int d^4 y \exp\left(\frac{i}{\hbar} p \cdot y\right) \left\langle \overline{\psi}_\beta\left(x - \frac{y}{2}\right) \psi_\alpha\left(x + \frac{y}{2}\right)\right\rangle$$

Heinz (1983); Vasak-Gyulassy-Elze (1987); Zhuang-Heinz (1996); Iancu-Blaizot (2001); QW-Redlich-Stoecker-Greiner (2002)

Wigner function decomposition in 16 generators of Clifford algebra

spin 4-vector

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5 \mathscr{P} + \gamma^\mu \mathscr{V}_\mu + \gamma^5 \gamma^\mu \mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

scalar   p-scalar   vector   axial-vector   tensor

$$j^\mu = \int d^4 p \mathscr{V}^\mu, \qquad j_5^\mu = \int d^4 p \mathscr{A}^\mu, \qquad T^{\mu\nu} = \int d^4 p \, p^\mu \mathscr{V}^\nu$$

Recent reviews:
Hidaka-Pu-QW-Yang, PPNP (2022)
Gao-Liang-QW, IJMPA (2021)

Vasak-Gyulassy-Elze, Ann. Phys. 173, 462 (1987);
Elze-Gyulassy-Vasak, Nucl. Phys. B 276, 706 (1986);

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Qun Wang (USTC/AUST), Spin physics in heavy ion collisions (Baryons 2025)

Axial vector component of WF (spin vector) has many contributions

$$j_5^\mu = j_{5,\text{thermal}}^\mu + j_{5,\text{shear}}^\mu + j_{5,\text{accel}}^\mu + j_{5,\text{chemical}}^\mu + j_{5,\text{EM}}^\mu$$

Thermal vorticity

Shear viscous tensor

Fluid acceleration

Gradient of chemical potential

Electromagnetic fields

$$j_{5,\text{thermal}}^\mu = a \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu \partial_\alpha \frac{u_\beta}{T}$$

$$\partial_{(\sigma} u_{\nu)} \equiv \frac{1}{2}\left(\partial_\mu u_\nu + \partial_\nu u_\mu\right)$$

$$j_{5,\text{shear}}^\mu = -a \frac{1}{T(u\cdot p)} \epsilon^{\mu\nu\alpha\beta} p_\alpha u_\beta p^\sigma \partial_{(\sigma} u_{\nu)}$$

$$j_{5,\text{accel}}^\mu = a \frac{1}{2T} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \left(Du_\beta - \frac{1}{T}\partial_\beta T\right)$$

$$j_{5,\text{chemical}}^\mu = a \frac{1}{(u\cdot p)} \epsilon^{\mu\nu\alpha\beta} p_\alpha u_\beta \partial_\nu \frac{\mu}{T}$$

$$j_{5,\text{EM}}^\mu = a \frac{1}{T(u\cdot p)} \epsilon^{\mu\nu\alpha\beta} p_\alpha u_\beta E_\nu + a \frac{B^\mu}{T}$$

Hidaka, Pu, Yang (2018); Yi, Pu, Yang (2021)

Becattini, et al, (2021)
Fu, Liu, et al., (2021)

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Spin DOF: Matrix Valued Spin Distributions (MVSD)

Relativistic MVSD for fermion in QFT

$$p^\mu \equiv \frac{1}{2}(p_1^\mu + p_2^\mu)$$

$$q^\mu \equiv p_1^\mu - p_2^\mu$$

$$f_{rs}(x,p) \equiv \int \frac{d^4 q}{2(2\pi)^3} \exp\left(-\frac{i}{\hbar} q \cdot x\right) \delta(p \cdot q) \langle a^\dagger(s, \mathbf{p}_2) a(r, \mathbf{p}_1) \rangle$$

on-shell condition

$$r, s = \uparrow, \downarrow \text{ or } 1, 2$$

Relativistic MVSD can be parameterized in un-polarized and polarized parts

$$j = 1, 2, 3$$

Pauli matrices in spin space (rs-space)

$$f_{rs}^{(+)}(x, \mathbf{p}) = \frac{1}{2} f_q(x, \mathbf{p}) \left[\delta_{rs} - P_\mu^q(x, \mathbf{p}) n_j^{(+)\mu}(\mathbf{p}) \tau_{rs}^j\right],$$

$$f_{rs}^{(-)}(x, -\mathbf{p}) = \frac{1}{2} f_{\bar{q}}(x, -\mathbf{p}) \left[\delta_{rs} - P_\mu^{\bar{q}}(x, -\mathbf{p}) n_j^{(-)\mu}(\mathbf{p}) \tau_{rs}^j\right],$$

un-polarized dist.

spin polarization dist.

Four-vectors of three basis directions in rest frame of q and $\bar{q}$ (one is the spin quantization direction)

MVSD:
Becattini et al. (2013)
Sheng, Weickgenannt, et al. (2021)
Sheng, QW, Rischke (2022)

Spin Boltzmann equation for massive fermions

At leading order in $O(\hbar^0)$ spin Boltzmann equation (SBE) with local collision terms

$$\frac{1}{E_p} p \cdot \partial_x \text{tr}\left[f^{(0)}(x,p)\right] = \mathscr{C}_{\text{scalar}}\left[f^{(0)}\right]$$

$$\frac{1}{E_p} p \cdot \partial_x \text{tr}\left[n_j^{(+)\mu} \tau_j f^{(0)}(x,p)\right] = \mathscr{C}_{\text{pol}}^\mu\left[f^{(0)}\right]$$

➡️ $f_{rs}^{(0)}(x,p)$

At next-to-leading order in $O(\hbar^1)$, SBE describes how $f^{(1)}(x,p)$ evolves for given $f^{(0)}(x,p)$ and $\partial_x f^{(0)}(x,p)$ [non-local terms]

$$\frac{1}{E_p} p \cdot \partial_x \text{tr}\left[f^{(1)}(x,p)\right] = \mathscr{C}_{\text{scalar}}\left[f^{(0)}, \partial_x f^{(0)}, f^{(1)}\right]$$

$$\frac{1}{E_p} p \cdot \partial_x \text{tr}\left[n_j^{(+)\mu} \tau_j f^{(1)}(x,p)\right] = \mathscr{C}_{\text{pol}}^\mu\left[f^{(0)}, \partial_x f^{(0)}, f^{(1)}\right]$$

determined by leading order SBE

$\partial_\mu u_\nu, \ \partial_\mu T, \ \partial_\mu \mu_B$

Convenient for simulation !

Sheng, Speranza, Rischke, QW, Weickgenannt (2021)

spin transport for massive fermions from WF or KB equation was also studied in: Yang, Hattori, Hidaka (2020); Gao, Liang (2021); Wang, Zhuang (2021)

Summary

- **Spin physics is an emergent field in heavy-ion collisions**

- **A microscopic model for the emergence of spin-vorticity coupling from spin-orbit coupling in parton-parton scatterings**

- **Spin Boltzmann (Kinetic) Equations for massive fermions from Wigner functions**