Connecting dilaton thermal fluctuation with the Polyakov loop at finite temperature

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The extreme environments and QCD phase diagram

Extreme environments:

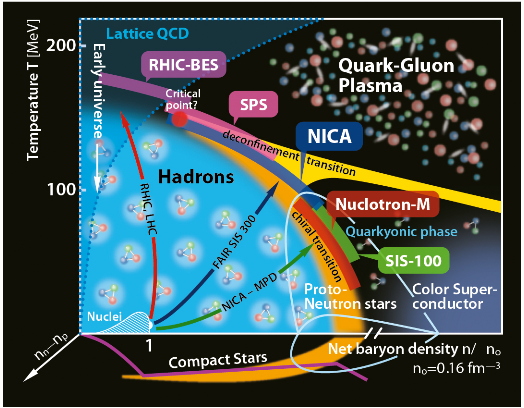
- Extremely High Temperature:

Matter in early universe from Big Bang at $t \approx 10^{-4}$s or in noncentral heavy ion collisions at RHIC and LHC has the quite high temperature: $T \approx 2.32 \times 10^{12}$K. $(T \sim 200\text{MeV})$

- Extremely High Density:

Matter in neutron stars is quite dense: baryon density $\rho \approx 5 - 10\rho_0$. $\rho_0 \approx 3 \times 10^{17}\text{kg/m}^3$ is the density of matter in atomic nuclei. $(\rho_0 \sim (190\text{MeV})^4)$

Universe 4 (2018) 3, 52, NA61/SHINE Collaboration



"Extreme" means that the energy scale of T and ρ is near $\Lambda_{\text{QCD}} \approx 200\text{MeV}$.

The trace anomaly and hadron masses

The scale symmetry of the QCD with massless quarks at the classical level:

$$x'^{\mu} = \lambda x^{\mu},\ \lambda \equiv e^{a},\ a \in \mathbb{R} \qquad S'_{\rm QCD} = S_{\rm QCD} \quad \Rightarrow \quad \theta^{\mu}{}_{\mu} = 0$$

The scale symmetry of the QCD with massive quarks is broken at the quantum level (**ITEP Lectures on PARTICLE PHYSICS AND FIELD THEORY, Volume one, M. A. Shifman**):

$$\theta^{\mu}{}_{\mu} = m_Q \overline{Q} Q + \frac{\beta(\alpha_{\rm s})}{4\alpha_{\rm s}} G^{a}_{\mu\nu} G^{a,\mu\nu},$$

Q: heavy quark field $\quad G^{a}_{\mu\nu}$: gluon field strength tensor

The mass formula of a hadron H_Q containing heavy quarks:

$$M_{H_Q} = \frac{1}{2M_{H_Q}} \langle H_Q | m_Q \overline{Q} Q | H_Q \rangle + \frac{1}{2M_{H_Q}} \langle H_Q | \frac{\beta(\alpha_{\rm s})}{4\alpha_{\rm s}} G^2 | H_Q \rangle$$

The origin of hadron masses can be understood from the QCD trace anomaly.

Are there some possible relations between QCD phase transitions and the trace anomaly? chiral transition deconfinement transition(this work)

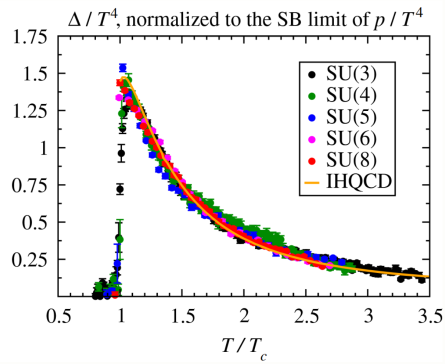
The scale restoration and deconfinement transition at high temperatures

- The trace anomaly in pure $\mathrm{SU(N_c)}$ gauge theory: $\theta^{\mu}{}_{\mu} = \dfrac{\beta(g_s)}{2g_s} G^a_{\mu\nu} G^{a,\mu\nu}$
- Polyakov loop: $\Phi = \frac{1}{N_c} \mathrm{Tr_c} \mathcal{P} \exp\left[-ig_s \int_0^\beta \mathrm{d}x_4 A_4\right]$, $\langle\langle\hat{\Phi}\rangle\rangle \propto \exp(-\Delta F/T)$

  ΔF : the least work to excite a quark in gluonic medium

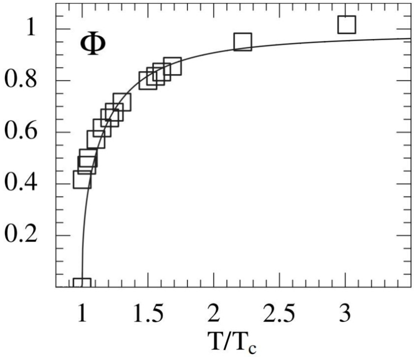
  $\langle\langle\hat{\Phi}\rangle\rangle = 0$: confinement phase $\quad\langle\langle\hat{\Phi}\rangle\rangle = 1$: deconfinement phase

PRL 103, 232001 (2009) Marco Panero



$\theta^{\mu}{}_{\mu} = \epsilon - 3p \to 0$ **at** $T \to +\infty$

PRD 73, 014019 (2006), Wolfram Weise et al.



$\langle\langle\hat{\Phi}\rangle\rangle \to 1$ **at** $T \to +\infty$

The scale restoration and deconfinement transition at high temperatures

The anomaly matching: $\theta^{\mu}{}_{\mu} = -\frac{m_\sigma^2}{4f_\sigma^2}\chi^4$ **PRD 21, 3393 (1980) Schechter, J.**

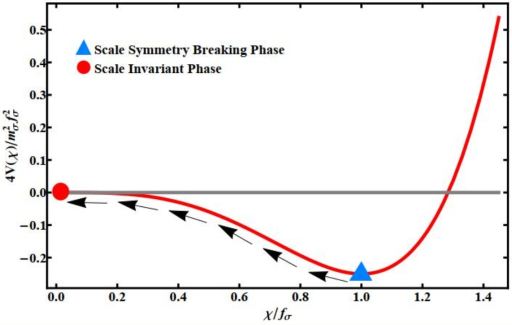
The low-energy effective potential of the trace anomaly:

Coleman–Weinberg-type potential

$$\mathcal{V}(\chi) = \frac{m_\sigma^2 f_\sigma^2}{4}\left(\frac{\chi}{f_\sigma}\right)^4\left[\ln\left(\frac{\chi}{f_\sigma}\right) - \frac{1}{4}\right]$$

PRD 95, 114011 (2017) Y.L. Li, Y.L. Ma et al.

The scale restoration at high temperatures: $\langle\langle\hat{\chi}\rangle\rangle \to 0$



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The scale restoration and deconfinement transition at high temperatures

¶ The scale symmetry will be restored and the deconfinement phase transition will happen at high temperatures

$$\lim_{T\to+\infty} \theta^{\mu}{}_{\mu} = 0 \qquad \lim_{T\to+\infty} \langle\langle \hat{\Phi} \rangle\rangle = 1$$

¶ Both the dilaton field χ and the Polyakov loop Φ are related to the gluonic configuration

$$\theta^{\mu}{}_{\mu} = \frac{\beta(g_s)}{2g_s} G^{a}_{\mu\nu} G^{a,\mu\nu} = -\frac{m_\sigma^2}{4f_\sigma^2}\chi^4 \qquad \Phi = \frac{1}{N_\mathrm{c}} \mathrm{Tr}_\mathrm{c} \mathcal{P} \exp\left[-ig_s \int_0^\beta \mathrm{d}x_4 A_4\right]$$

¶ The effective potential of the trace anomaly and that of the Polyakov loop have a similar logarithmic form

$$\mathcal{V}(\chi) = \frac{m_\sigma^2 f_\sigma^2}{4}\left(\frac{\chi}{f_\sigma}\right)^4\left[\ln\left(\frac{\chi}{f_\sigma}\right) - \frac{1}{4}\right] \quad \text{VS} \quad \mathcal{U}(\Phi) = a_3 T_0^4 \ln(1 - 6\Phi^2 + 8\Phi^3 - 3\Phi^4) + a_0 T^4 \Phi^2$$

PRC 81, 045201 (2010), V.A. Dexheimer et al.

A relation between $\langle\langle\hat{\chi}\rangle\rangle$ and $\langle\langle\hat{\Phi}\rangle\rangle$ for pure $\mathrm{SU}(3)$ systems

$$\mathcal{V}(\chi) = \frac{m_\sigma^2 f_\sigma^2}{4}\left(\frac{\chi}{f_\sigma}\right)^4\left[\ln\left(\frac{\chi}{f_\sigma}\right) - \frac{1}{4}\right] \quad \text{VS} \quad \mathcal{U}(\Phi) = a_3 T_0^4 \ln(1 - 6\Phi^2 + 8\Phi^3 - 3\Phi^4) + a_0 T^4 \Phi^2$$

suppose: $\dfrac{\chi}{f_\sigma} = 1 - a\Phi^*\Phi - \dfrac{b}{2}(\Phi^{*3} + \Phi^3) - c(\Phi^*\Phi)^2$ ($\mathbb{Z}_3$ symmetric)

$\chi = \langle\langle\hat{\chi}\rangle\rangle, \Phi = \langle\langle\hat{\Phi}\rangle\rangle$(mean-field approximation) $\qquad \chi(\Phi = 1) = 0 \Longrightarrow a + b + c = 1$

From the trace anomaly potential to the Polyakov potenital:

$$\begin{aligned}\mathcal{V}(\Phi) &= \frac{m_\sigma^2 f_\sigma^2}{4}\left(1 - a\Phi^2 - b\Phi^3 - c\Phi^4\right)^4\left[\ln\left(1 - a\Phi^2 - b\Phi^3 - c\Phi^4\right) - \frac{1}{4}\right] \\ &= \frac{m_\sigma^2 f_\sigma^2}{4}\ln\left(1 - a\Phi^2 - b\Phi^3 - c\Phi^4\right) + \cdots\end{aligned}$$

pure-gauge system: $\boldsymbol{\Phi^* = \Phi}$

$$m_\sigma = 640\text{MeV},\ f_\sigma = 150\text{MeV},\ a_3 = -0.4 \Longrightarrow T_0 = \left(\frac{m_\sigma^2 f_\sigma^2}{4|a_3|}\right)^{1/4} \approx 275\text{MeV}$$

LQCD: $T_0 \approx 270\text{MeV}$

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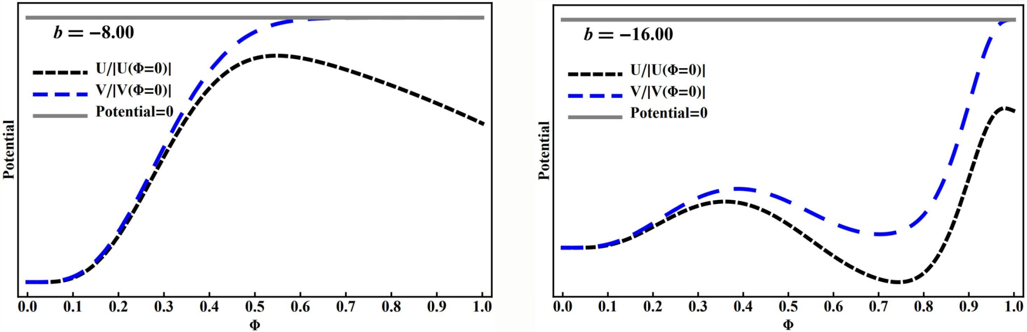
A relation between $\langle\langle\hat{\chi}\rangle\rangle$ and $\langle\langle\hat{\Phi}\rangle\rangle$ for pure $\mathrm{SU}(3)$ systems

$$\mathcal{U}(\Phi) = \mathcal{V}(\Phi) + \frac{m_\sigma^2 f_\sigma^2}{4} d\Phi^2 \qquad a = 6,\ d = -0.1$$



The minimum point jumps from $\langle\langle\hat{\Phi}\rangle\rangle = 0$ to $\langle\langle\hat{\Phi}\rangle\rangle \approx 0.75$ with the decrease of b and the first-order phase transition is manifested.

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A relation between $\langle\langle\hat{\chi}\rangle\rangle$ and $\langle\langle\hat{\Phi}\rangle\rangle$ for pure $\mathrm{SU}(3)$ systems

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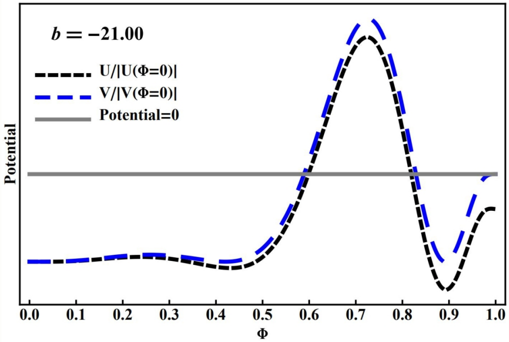

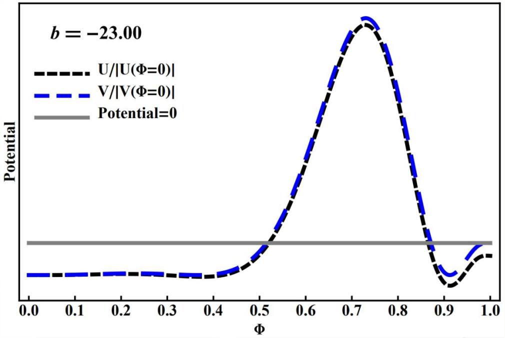

The value of potential $\mathcal{V}(\Phi = \langle\langle\hat{\Phi}\rangle\rangle) \to 0$ with the decrease of b indicating the restoration of scale symmetry.

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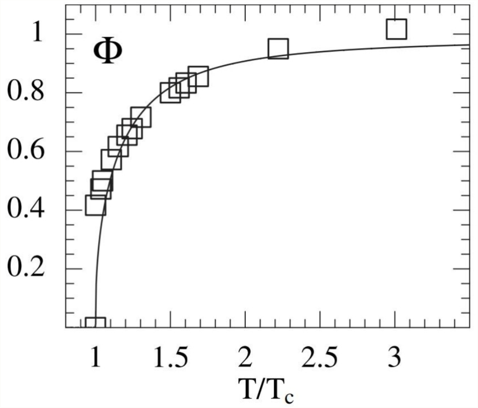
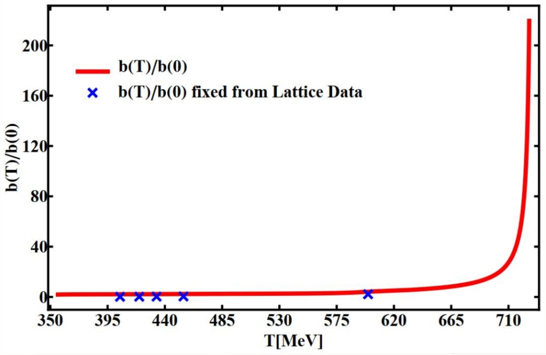
A relation between $\langle\langle\hat{\chi}\rangle\rangle$ and $\langle\langle\hat{\Phi}\rangle\rangle$ for pure $\mathrm{SU}(3)$ systems

PRD 73, 014019 (2006), Wolfram Weise et al.





The parameter $b(T)/b(0)$ is fixed by inputting the LQCD data of $\langle\langle\hat{\Phi}\rangle\rangle$ in $395\text{MeV} \lesssim T \lesssim 620\text{MeV}$ and extrapolated up to $T \approx 720\text{MeV}$ with the condition $\langle\langle\hat{\Phi}\rangle\rangle \to 1$. And the value begins increasing sharply at $T \approx 665\text{MeV}$.

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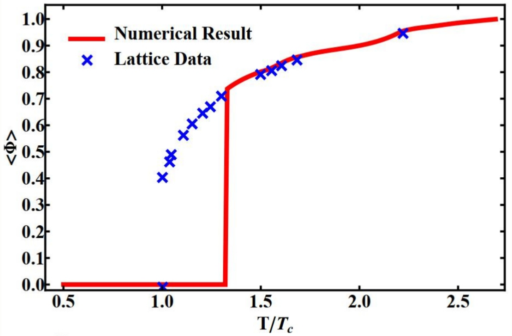
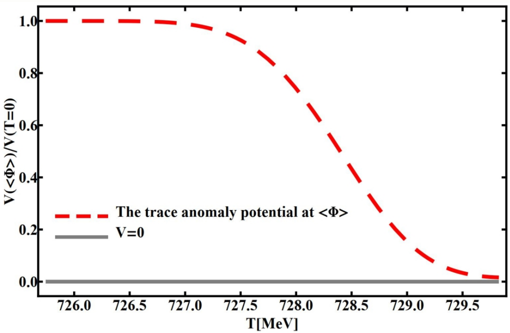
A relation between $\langle\langle\hat{\chi}\rangle\rangle$ and $\langle\langle\hat{\Phi}\rangle\rangle$ for pure $\mathrm{SU}(3)$ systems

Minimize $\mathcal{U}(\Phi) = \mathcal{V}(\Phi) + \frac{m_\sigma^2 f_\sigma^2}{4} d\Phi^2 \quad m_\sigma = 640\text{MeV},\ f_\sigma = 150\text{MeV},\ a = 6,\ d = -0.1,$

$b(T)$ fitted using $\langle\langle\hat{\Phi}\rangle\rangle_{\text{LQCD}}$





The critical temperature of the first-order deconfinement phase transition is much high than the Lattice QCD (LQCD) result in pure $\mathrm{SU}(3)$ gauge system: $\widetilde{T}_c = T(\langle\langle\hat{\Phi}\rangle\rangle \approx 0.728) \approx 356\text{MeV} > 270\text{MeV}$

Symmetry **16**, 718 (2024), **Bing-Kai Sheng** and Yong-Liang Ma

Incorporate the dilaton thermal fluctuation

The gluon field is regarded as static and homogeneous background field(PRD **73**, 014019 (2006), C. Ratti, et al.; PLB **591**, 277 (2004), K. Fukushima). Only the dilaton thermal fluctuation is considered.

$$\delta_{\chi,\mathrm{th}} = \chi - \langle\!\langle \hat{\chi} \rangle\!\rangle = \delta^{(1)}_{\chi,\mathrm{th}}(T) + \delta^{(2)}_{\chi,\mathrm{th}}(\Phi, \Phi^*; T)$$

$\delta^{(1)}_{\chi,\mathrm{th}}$: the fluctuation of dilaton particles near T_c

$\delta^{(2)}_{\chi,\mathrm{th}}$: the fluctuation at much higher temperature, which is related to the Polyakov loop.

The thermodynamic potential density:

$$\Omega_\chi = -\frac{T \ln\left[\mathrm{Tr} e^{-\beta V \widetilde{\mathcal{V}}(\hat{\chi})}\right]}{V} = \widetilde{\mathcal{V}}(\overline{\chi}),\ \overline{\chi} = \langle\!\langle \hat{\chi} \rangle\!\rangle + \delta^{(1)}_{\chi,\mathrm{th}}(T) + \delta^{(2)}_{\chi,\mathrm{th}}(\langle\!\langle \hat{\Phi} \rangle\!\rangle, \langle\!\langle \hat{\Phi} \rangle\!\rangle^*; T)$$

The temperature-modified potential $\widetilde{\mathcal{V}}(\chi)$ with $\tilde{m}_\sigma(T)$ and $\tilde{f}_\sigma(T) = \langle\!\langle \hat{\chi} \rangle\!\rangle$ is considered and the stationary phase approximation yields the expectation value of Φ and Φ^*, i.e., $\partial\Omega_\chi/\partial\Phi = \partial\Omega_\chi/\partial\Phi^* = 0$ at $\Phi = \langle\!\langle \hat{\Phi} \rangle\!\rangle$. ($\tilde{f}_\sigma(T=0) = f_\sigma = 150\mathrm{MeV}$)

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

Incorporate the dilaton thermal fluctuation

Construct the explicit expressions for $\delta^{(1)}_{\chi,\mathrm{th}}$ and $\delta^{(2)}_{\chi,\mathrm{th}}$ guided by the scale covariance of

$$\delta_{\chi,\mathrm{th}} = \delta^{(1)}_{\chi,\mathrm{th}}(T) + \delta^{(2)}_{\chi,\mathrm{th}}(\Phi, \Phi^*; T)$$

The scale transformation of dilaton under $x'^{\mu} = \lambda x^{\mu}, \lambda \equiv e^{a}, a \in \mathbb{R}$

$$\chi'(\boldsymbol{x}', t') = \lambda^{-1}\chi(\boldsymbol{x}, t)$$

and the thermal fluctuation has the same transformation: $\delta'_{\chi,\mathrm{th}} = \lambda^{-1}\delta_{\chi,\mathrm{th}}$.

The scale covariance requires that $\boldsymbol{\delta^{(1)}_{\chi,\mathrm{th}}}$ and $\boldsymbol{\delta^{(2)}_{\chi,\mathrm{th}}}$ have the same transformation behavior as $\boldsymbol{\delta_{\chi,\mathrm{th}}}$.

The scale transformation of the Polyakov loop:

$$\Phi'(\boldsymbol{x}') = \frac{1}{N_{\mathrm{c}}}\mathrm{tr}_{\mathrm{c}}\left\{\mathcal{P}\exp\left[-ig_{\mathrm{s}}\int_0^{\beta'}\mathrm{d}x'_4 A'^{a}_4(\boldsymbol{x}', x'_4)T^a\right]\right\} =? \quad A'^{a}_4(\boldsymbol{x}', x'_4) = \lambda^{-1}A^{a}_4(\boldsymbol{x}, x_4)$$

key problem: $\boldsymbol{\beta' = 1/T' =?}$

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

The scale transformation of temperature

The Liouville equation: $\dfrac{\partial\hat{\rho}(t,\beta_1,\cdots,\beta_L)}{\partial t}+\dfrac{1}{i}\left[\hat{\rho}(t,\beta_1,\cdots,\beta_L),\hat{H}(t)\right]=0$

should be covariant under the scale transformation for density operator

$$\hat{\rho}'(t',\beta_1',\cdots,\beta_L')=\hat{D}(t',a)\hat{\rho}(t',\beta_1',\cdots,\beta_L')[\hat{D}(t',a)]^\dagger\equiv\mathscr{D}_\rho\big(\hat{\rho}(t,\beta_1,\cdots,\beta_L)\big)$$

$\hat{D}(t,a)=\exp[-ia\hat{Q}_{\rm D}(t)]$, $\hat{Q}_{\rm D}(t)=t\hat{H}+\int \mathrm{d}^3x\sum_r\left[\boldsymbol{x}\cdot(\hat{\pi}_r\boldsymbol{\nabla}\hat{\phi}^r)+\hat{\pi}_r\hat{\phi}^r\Delta_{\phi^r}+\text{h.c.}\right]$, scale dimension:$\Delta_{\phi^r}$

The density operator is scale invariant due to the covariance. $\mathscr{D}_{\boldsymbol{\rho}}$: **identity mapping**

grand canonical ensemble: $\hat{\rho}(T,V,\mu_1,\cdots,\mu_K)=\dfrac{1}{Z}\exp\left[-\beta\left(\hat{H}-\sum_{i=1}^K\mu_i\hat{Q}_i\right)\right]$

The scale transformations of Hamiltonian and conserved charges of global unitary symmetry: $\hat{H}'=\lambda^{-1}\hat{H},\hat{Q}_i'=\hat{Q}_i,i=1,\cdots,K$

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

The scale transformation of temperature

The Liouville equation: $\dfrac{\partial\hat{\rho}(t,\beta_1,\cdots,\beta_L)}{\partial t}+\dfrac{1}{i}\Big[\hat{\rho}(t,\beta_1,\cdots,\beta_L),\hat{H}(t)\Big]=0$

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$$\hat{\rho}'(t',\beta_1',\cdots,\beta_L')=\hat{D}(t',a)\hat{\rho}(t',\beta_1',\cdots,\beta_L')[\hat{D}(t',a)]^\dagger\equiv\mathscr{D}_\rho\big(\hat{\rho}(t,\beta_1,\cdots,\beta_L)\big)$$

$\hat{D}(t,a)=\exp[-ia\hat{Q}_{\rm D}(t)]$, $\hat{Q}_{\rm D}(t)=t\hat{H}+\int \mathrm{d}^3x\sum_r\big[\boldsymbol{x}\cdot(\hat{\pi}_r\boldsymbol{\nabla}\hat{\phi}^r)+\hat{\pi}_r\hat{\phi}^r\Delta_{\phi^r}+\mathrm{h.c.}\big]$, scale dimension: Δ_{ϕ^r}

The density operator is scale invariant due to the covariance. $\mathscr{D}_{\boldsymbol{\rho}}$: **identity mapping**

The scale invariance of $\hat{\rho}(T,V,\mu_1,\cdots,\mu_K)$ yields the scale transformation rules for temperature, chemical potentials and partition function:

$$T'=\lambda^{-1}T,\quad \mu_i'=\lambda^{-1}\mu_i,\ i=1,\cdots,K,\quad Z'(T',V',\mu_1',\cdots,\mu_K')=Z(T,V,\mu_1,\cdots,\mu_K)$$

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Self-consistency check

$T' = \lambda^{-1}T, \mu'_i = \lambda^{-1}\mu_i,\ i = 1, \cdots, K$ and $Z'(T', V', \mu'_1, \cdots, \mu'_K) = Z(T, V, \mu_1, \cdots, \mu_K)$ give the scale transformations of pressure, particle numbers, entropy, internal energy, enthalpy, Helmholtz-free energy, Gibbs-free energy and thermodynamic potential, respectively, i.e., $P' = \lambda^{-4}P,\ N'_i = N_i(i = 1, \cdots, K),\ S' = S,\ U' = \lambda^{-1}U,\ H' = \lambda^{-1}H,\ F' = \lambda^{-1}F,\ G' = \lambda^{-1}G,\ \Psi' = \lambda^{-1}\Psi$

The four fundamental laws of thermodynamics must be scale-covariant:

- ¶ The zeroth law: Three thermodynamic systems $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the coordinates $\{x^\mu\}$, $\mathcal{A}$ equilibrates with $\mathcal{C}$ ($T_\mathcal{A} = T_\mathcal{C}$) and also equilibrates with $\mathcal{B}$ ($T_\mathcal{A} = T_\mathcal{B}$), then $\mathcal{B}$ and $\mathcal{C}$ must equilibrate with each other ($T_\mathcal{B} = T_\mathcal{A} = T_\mathcal{C}$). We perform the scale transformation $T' = \lambda^{-1}T$, and the coordinates $\{x'^\mu = \lambda x^\mu\}$. $T'_\mathcal{A} = T'_\mathcal{C}$ means that $\mathcal{A}$ equilibrates with $\mathcal{C}$ and $T'_\mathcal{A} = T'_\mathcal{B}$ indicates that $\mathcal{A}$ equilibrates with $\mathcal{B}$, then equation $T'_\mathcal{B} = T'_\mathcal{A} = T'_\mathcal{C}$ tells us that $\mathcal{B}$ must equilibrate with $\mathcal{C}$. For this reason, the zeroth law is scale-covariant.

Self-consistency check

$T' = \lambda^{-1}T, \mu'_i = \lambda^{-1}\mu_i,\ i = 1, \cdots, K$ and $Z'(T', V', \mu'_1, \cdots, \mu'_K) = Z(T, V, \mu_1, \cdots, \mu_K)$ give the scale transformations of pressure, particle numbers, entropy, internal energy, enthalpy, Helmholtz-free energy, Gibbs-free energy and thermodynamic potential, respectively, i.e., $P' = \lambda^{-4}P,\ N'_i = N_i(i = 1, \cdots, K),\ S' = S,\ U' = \lambda^{-1}U,\ H' = \lambda^{-1}H,\ F' = \lambda^{-1}F,\ G' = \lambda^{-1}G,\ \Psi' = \lambda^{-1}\Psi$

The four fundamental laws of thermodynamics must be scale-covariant:

- The first law:

$$\begin{aligned} &\mathrm{d}U = \delta Q - P\mathrm{d}V + \mu_i \mathrm{d}N_i \quad \Rightarrow \quad \lambda^{-1}\mathrm{d}U = \lambda^{-1}\delta Q - \lambda^{-1}P\mathrm{d}V + \lambda^{-1}\mu_i \mathrm{d}N_i \\ &\Rightarrow \quad \mathrm{d}U' = \delta Q' - P'\mathrm{d}V' + \mu'_i \mathrm{d}N'_i \end{aligned}$$

- The second law:

$$\mathrm{d}S' \geq \frac{\delta Q'}{T'} \quad \Rightarrow \quad \mathrm{d}S \geq \frac{\lambda^{-1}\delta Q}{\lambda^{-1}T} \quad \Rightarrow \quad \mathrm{d}S \geq \frac{\delta Q}{T}$$

$\delta Q' = \lambda^{-1}\delta Q$ can be derived by a reversible process.

Self-consistency check

$T' = \lambda^{-1}T, \mu_i' = \lambda^{-1}\mu_i,\ i = 1, \cdots, K$ and $Z'(T', V', \mu_1', \cdots, \mu_K') = Z(T, V, \mu_1, \cdots, \mu_K)$ give the scale transformations of pressure, particle numbers, entropy, internal energy, enthalpy, Helmholtz-free energy, Gibbs-free energy and thermodynamic potential, respectively, i.e., $P' = \lambda^{-4}P,\ N_i' = N_i(i = 1, \cdots, K),\ S' = S,\ U' = \lambda^{-1}U,\ H' = \lambda^{-1}H,\ F' = \lambda^{-1}F,\ G' = \lambda^{-1}G,\ \Psi' = \lambda^{-1}\Psi$

The four fundamental laws of thermodynamics must be scale-covariant:

- ¶ The third law: For any thermodynamic system, it is impossible to refrigerate it to arrive at zero temperature with finite procedures. In $\{x^\mu\}$, the third law is valid. In $\{x'^\mu = \lambda x^\mu\}$, $T' = \lambda^{-1}T,\ \lambda > 0$ tells us that with finite procedures, we cannot arrive at $T' = 0$. Hence, the third law is scale covariant as well. Moreover, $T' = \lambda^{-1}T,\ \lambda > 0$ also indicates that we cannot find a well-defined coordinate $\{x'^\mu\}$ in which we could arrive at $T' = 0$. In other words, if we expect to obtaining $T' = 0$ using scale transformation then λ is of divergence and $x'^\mu = \lambda x^\mu$ are divergent and the coordinate $\{x'^\mu\}$ is not well-defined in this case.

The scale transformation of $\Phi(\boldsymbol{x})$

The Polyakov loop is a scale-invariant object:

$$
\begin{aligned}
\Phi'(\boldsymbol{x}') &= \frac{1}{N_\mathrm{c}}\mathrm{tr}_\mathrm{c}\left\{\mathcal{P}\exp\left[-ig_\mathrm{s}\int_0^{\beta'}\mathrm{d}x_4' A_4'^a(\boldsymbol{x}', x_4')T^a\right]\right\} \\
&= \frac{1}{N_\mathrm{c}}\mathrm{tr}_\mathrm{c}\left\{\mathcal{P}\exp\left[-ig_\mathrm{s}\int_0^{\lambda^{-1}\beta'}\lambda\mathrm{d}x_4 \lambda^{-1} A_4^a(\boldsymbol{x}, x_4)T^a\right]\right\} \\
&= \frac{1}{N_\mathrm{c}}\mathrm{tr}_\mathrm{c}\left\{\mathcal{P}\exp\left[-ig_\mathrm{s}\int_0^{\lambda^{-1}\lambda\beta}\mathrm{d}x_4 A_4^a(\boldsymbol{x}, x_4)T^a\right]\right\} \\
&= \frac{1}{N_\mathrm{c}}\mathrm{tr}_\mathrm{c}\left\{\mathcal{P}\exp\left[-ig_\mathrm{s}\int_0^{\beta}\mathrm{d}x_4 A_4^a(\boldsymbol{x}, x_4)T^a\right]\right\} \\
&= \Phi(\boldsymbol{x})
\end{aligned}
$$

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

Expressions of the dilaton thermal fluctuation

$$\delta_{\chi,\mathrm{th}} = \delta^{(1)}_{\chi,\mathrm{th}}(T) + \delta^{(2)}_{\chi,\mathrm{th}}(\Phi, \Phi^*; T)$$

According to $T' = \lambda^{-1}T$, $\Phi'(\boldsymbol{x}') = \Phi(\boldsymbol{x})$ and the scale covariance of the above relation at high temperature:

$$\delta^{(1)}_{\chi,\mathrm{th}}(T) = \bar{\delta}T, \quad \frac{\delta^{(2)}_{\chi,\mathrm{th}}}{\tilde{f}_\sigma} = \sum_{k=0}^{+\infty}\sum_{k'=0}^{+\infty} \frac{\bar{c}_{kk'}T}{2}\left[\varphi^k(\varphi^*)^{k'} + (\varphi^*)^k\varphi^{k'}\right], \varphi \equiv \Phi^{N_c}$$

with

$$\bar{c}_{kk'} = \begin{cases} \bar{c}_{k'k}, & k \neq k' \\ 0, & k = k' \neq 0 \end{cases},$$

Only $\mathbb{Z}_{N_c}$-symmetric terms are retained ~~U(1)-symmetric terms~~

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

Analytical expressions of $\langle\langle\hat{\Phi}\rangle\rangle$

$$\frac{\delta^{(2)}_{\chi,\mathrm{NLO}}}{\tilde{f}_\sigma} = \frac{T}{T_{0,N_c}}\left(1 - a_{N_c}\varphi\right)$$
$$\frac{\delta^{(2)}_{\chi,\mathrm{NNLO}}(\Phi,\Phi;T)}{\tilde{f}_\sigma} = \frac{T}{T_{0,N_c}}\left(1 - a_{N_c}\varphi - b_{N_c}\varphi^2\right)$$

Consider the next-to-leading order (NLO) and the next-to-next-to-leading order (NNLO) of $\delta^{(2)}_{\chi,\mathrm{th}}/\tilde{f}_\sigma$ with respect to φ and minimize Ω_χ, we have the following analytic expressions of $\langle\langle\hat{\Phi}\rangle\rangle$ in the deconfinement phase:

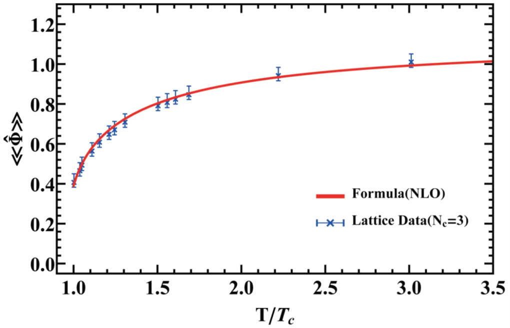
$$\langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NLO}} = \left[\frac{1}{a_{N_c}}\left(1 - \frac{T_{0,N_c}}{T}\right)\right]^{\frac{1}{N_c}} \qquad \langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NNLO}} = \left[\frac{-a_{N_c} + \sqrt{a^2_{N_c} - 4b_{N_c}\left(T_{0,N_c}/T - 1\right)}}{2b_{N_c}}\right]^{\frac{1}{N_c}}$$

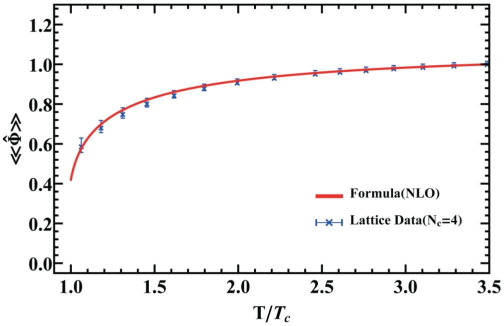
where $T_{0,N_c} \equiv 1/\bar{c}_{00},\ a_{N_c} \equiv -2\bar{c}_{01}/\bar{c}_{00}$ and $b_{N_c} \equiv -2\bar{c}_{02}/\bar{c}_{00}$.

arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

Fix the parameters in $\langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NLO}}$

Gauge group	SU(3)	SU(4)	SU(5)
a_{N_c}	0.696735	0.720968	0.557514
95% CI for a_{N_c}	$[0.670448, 0.723021]$	$[0.706921, 0.735015]$	$[0.500769, 0.614259]$
$T_{0,N_c}/T_{c,N_c}$	0.958802	0.977624	0.999476
95% CI for $T_{0,N_c}/T_{c,N_c}$	$[0.951533, 0.966071]$	$[0.968212, 0.987036]$	$[0.987320, 1.011632]$





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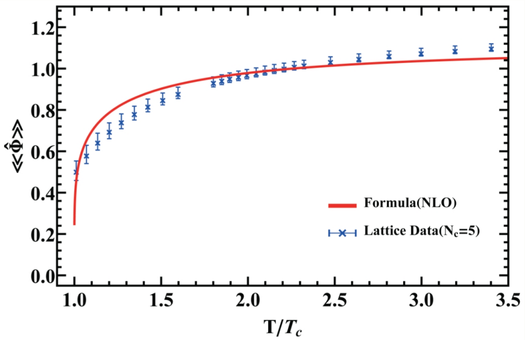
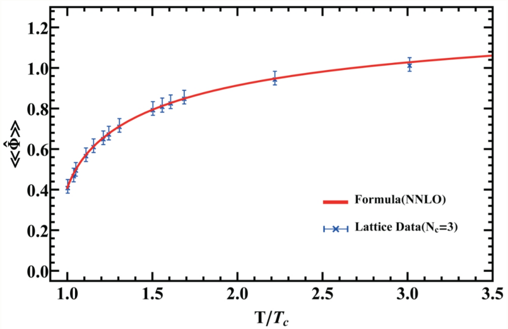


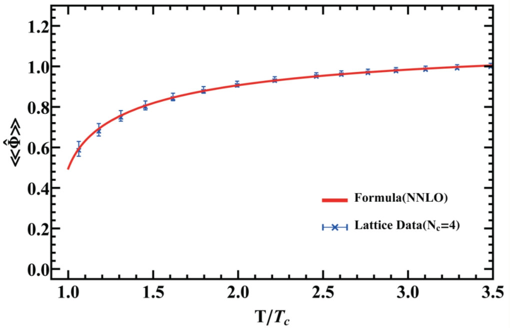
arXiv:2506.13549, **Bing-Kai Sheng** and Yong-Liang Ma

Fix the parameters in $\langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NNLO}}$

$b_{N_c} = -0.186231$

Gauge group	SU(3)	SU(4)	SU(5)
a_{N_c}	0.834206	0.905033	0.752401
95% CI for a_{N_c}	[0.826099, 0.842314]	[0.898431, 0.911634]	[0.745672, 0.759129]
$T_{0,N_c}/T_{c,N_c}$	0.943952	0.946053	0.993848
95% CI for $T_{0,N_c}/T_{c,N_c}$	[0.940958, 0.946947]	[0.939536, 0.952571]	[0.984730, 1.002966]





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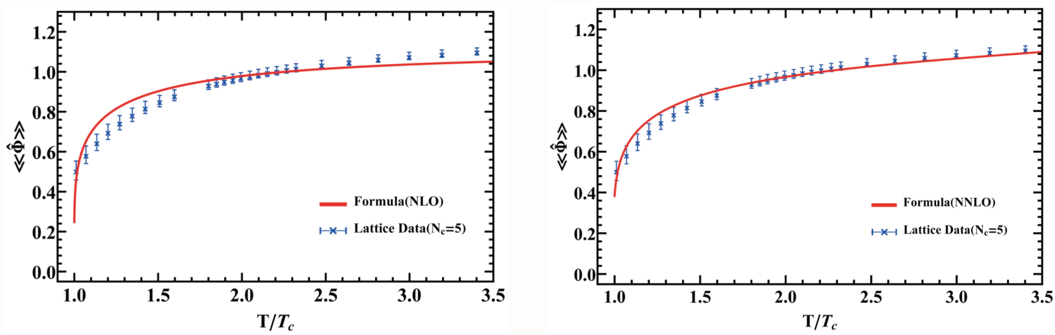

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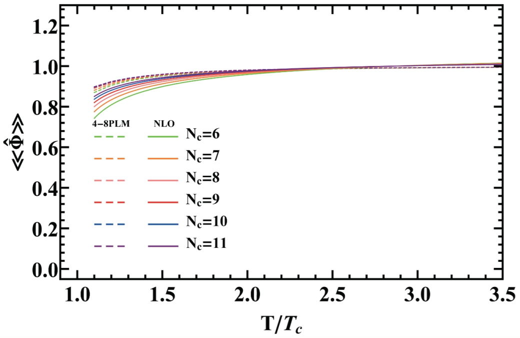
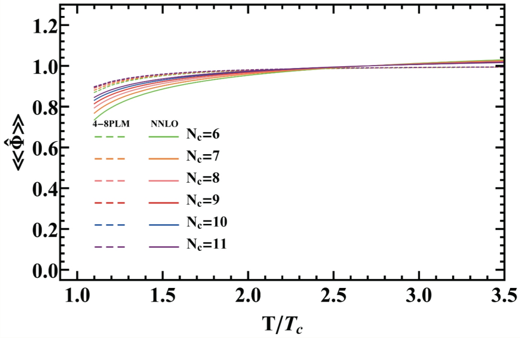
Large-N_c behavior

Compare $\langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NLO}}$ and $\langle\langle\hat{\Phi}\rangle\rangle_{\mathrm{NNLO}}$ with the results evaluated from $4-8$PLM model for the Polyakov loop(JHEP 09, 060 (2021), Z. Kang, J. Zhu, and S. Matsuzaki)





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Summary and Discussion

- Our recent work gives a possible hint that there is some relationship between the QCD trace anomaly and the confinement phenomenon and one can investigate the deconfinement phase transition starting from the trace anomaly.
- The scale transformation rules of all thermodynamic quantities are derived base on quantum statistical mechanics and these rules are model-independent.
- One interesting thing is that for a given scale-symmetric thermodynamic system, we find the basic building blocks for the equation of state (EoS) such as $T^4, \mu^4, T^2\mu^2, \cdots$ and can construct the EoS $P = \mathcal{C}_1 T^4 + \mathcal{C}_2 \mu^4 + \mathcal{C}_3 T^2 \mu^2 + \cdots$ directly instead of calculating the partition function. A typical example is the Stefan-Boltzmann law $P \propto T^4$ or $P \propto \mu^4$.

Summary and Discussion

- At present, the dynamic of dilatons has not been involved yet near the critical temperature. The interaction between the dilatonic excitation and the Polyakov loop need to be studied and the EoS of glueball matter should be calculated.
- The first-order deconfinement phase transition of $N_{\mathrm{c}} \geq 3$ has still not been derived starting from the trace anomaly and further investigations need to be implemented.
- In these two works, the quark degree of freedom is excluded. Once it is incorporated, it is of interest to study the possible relation among chiral phase transition, deconfinement phase transition and scale restoration at high temperature and/or even high baryon density.

Thank You!