Baryonic excitations in in heavy-quark QCD

Based on arXiv:2508.09927

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Collab. in

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The thermodynamic quantities which relate fluctuations of quark number.

nth quark number susceptibilities $\chi_n^{(Q)}$ is defined as

$$\chi_n^{(Q)} \equiv \frac{\partial^n}{\partial(\mu_q/T)^n} \frac{\Omega(T, \mu_q)}{VT} = 3^n \chi_n^{(B)}$$

T: Temperature
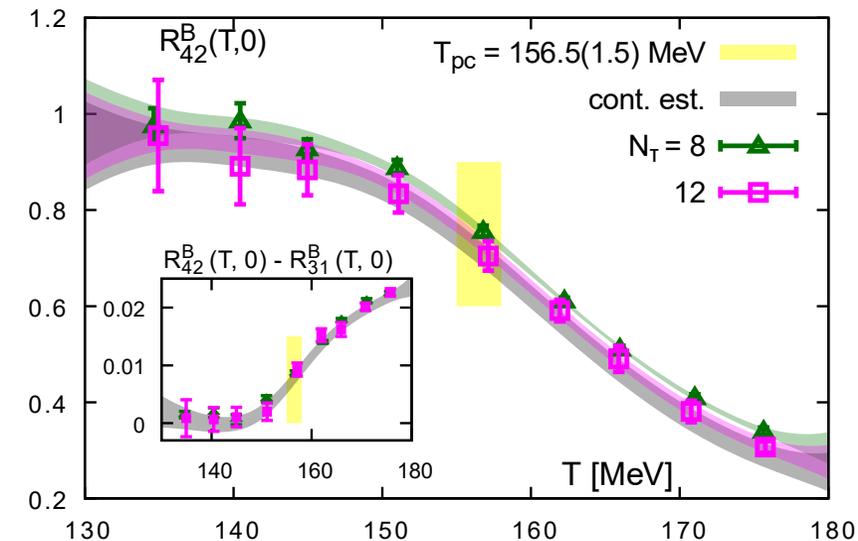μ_q: quark chemical potential
V: Spatial volume
$\chi_n^{(B)}$: nth baryon number susceptibility

This ratio can detect the phase of the system:

$$\frac{\chi_4^{(Q)}}{\chi_2^{(Q)}} = \begin{cases} 1 & (T \gg T_c), \\ 9 & (T \ll T_c). \end{cases} \Leftrightarrow \frac{\chi_4^{(B)}}{\chi_2^{(B)}} = \begin{cases} 1/9 & (T \gg T_c), \\ 1 & (T \ll T_c). \end{cases}$$

Ejiri, Karsch and Redlich (2006)

Purpose:
Analytical derivation of the above ratio in heavy-quark QCD.

SU(3) lattice QCD with $N_{\rm f}$ flavor Wilson fermion (P.B.C. for the temporal direction)

WF action: $\qquad S_{\rm Q} = \sum\limits_{f=1}^{N_{\rm f}} \sum\limits_{s,s'} \bar{q}_f(s)\left[1 - \kappa_f \underline{H_f(s,s')}\right] q_f(s') \qquad \kappa_f \simeq \dfrac{1}{m_f} \ll 1$

Hopping matrix

$$H_f(s,s') = \sum_{k=1}^{3}\left[(1-\gamma_k)U_k(s)\delta_{s-\hat{k},s'} + (1+\gamma_k)U_k^\dagger(s-\hat{k})\delta_{s+\hat{k},s'}\right]$$
$$+\left[(1-\gamma_4)e^{\mu_f a}U_4(s)\delta_{s-\hat{4},s'} + (1+\gamma_4)e^{-\mu_f a}U_4^\dagger(s-\hat{4})\delta_{s+\hat{4},s'}\right]$$
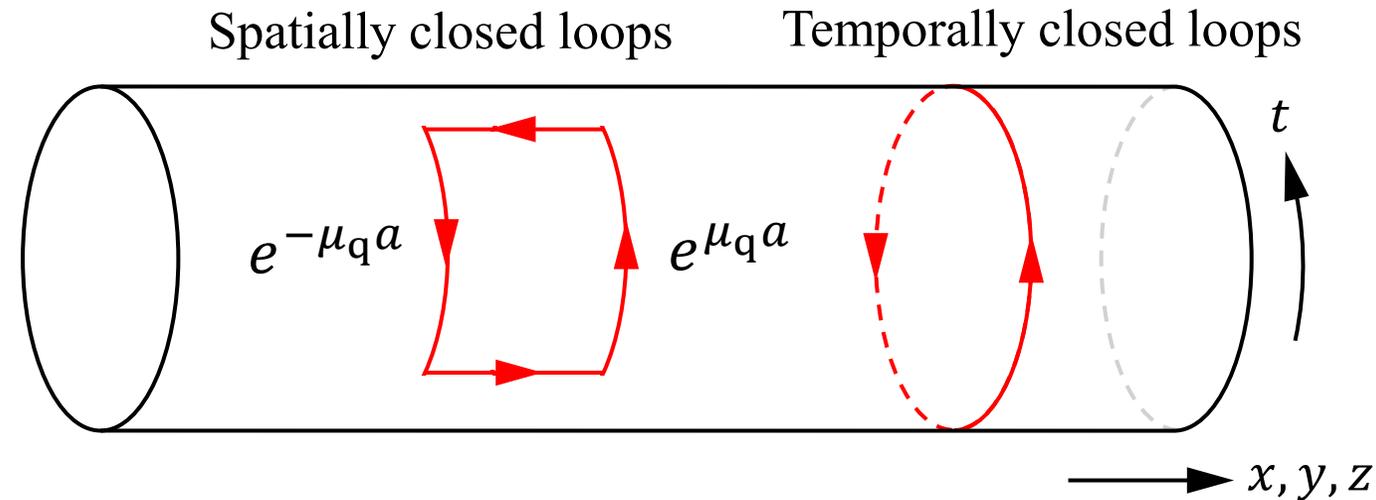
In this talk, we suppose degenerated quark masses and common chemical potential,

$$m_f = m \Longleftrightarrow \kappa_f = \kappa, \qquad \mu_f = \mu_{\rm q}.$$

Partition function

$$Z = \int \mathcal{D}U\mathcal{D}\bar{q}\mathcal{D}q \, \exp[-S_{\mathrm{g}} - S_{\mathrm{Q}}] = \int \mathcal{D}U \, e^{-S_{\mathrm{g}}} \exp\left[-N_{\mathrm{f}} \sum_{l=1}^{\infty} \frac{\kappa^l}{l} \boxed{\mathrm{Tr}\, H^l}\right]$$

Closed trajectories of length l.

Spatially closed loops   Temporally closed loops

$e^{-\mu_{\mathrm{q}}a}$ $e^{\mu_{\mathrm{q}}a}$

t

x, y, z

$\boldsymbol{\mu_q}$-dependences

- Spatially closed loops: μ_{q}-independent

- Temporally closed loops: μ_{q}-dependent
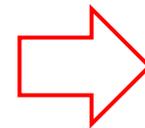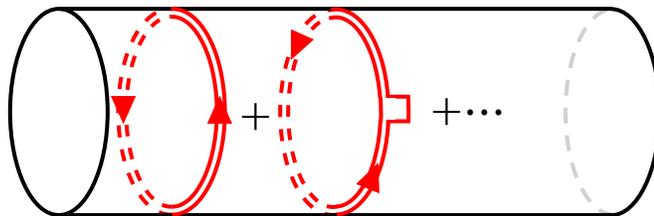
μ_q-dependence:

Specified by winding numbers of loop operators in the temporal direction.

Single winding $\Rightarrow$ $e^{\hat{\mu}_q}$ $\quad (\hat{\mu}_q \equiv \mu_q/T)$

Double winding $\Rightarrow$ $e^{2\hat{\mu}_q}$

Triple winding $\Rightarrow$ $e^{3\hat{\mu}_q}$

Fermion action

$$\ln \mathrm{Det}(1 - \kappa H) = -N_{\mathrm{f}} \sum_{l=1}^{\infty} \frac{\kappa^l}{l} \mathrm{Tr}\, H^l = \sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k$$

Spatially closed loops $(k = 0)$

$$\Lambda_0 = \quad \tau$$

$$\Lambda_0 \equiv (2\kappa)^4 N_{\mathrm{f}} \sum_{s} \sum_{\mu < \nu} \mathrm{Retr}_{\mathrm{c}}\, P_{\mu\nu}(s) + \mathcal{O}(\kappa^6)$$

Temporally closed loops $(k \neq 0)$

$$\Lambda_k = \quad t$$

$$\Lambda_k \equiv (2\kappa)^{kN_t} 2N_{\mathrm{f}} N_{\mathrm{c}} \frac{(-1)^{k+1}}{k} L_k + \mathcal{O}\left(\kappa^{kN_t+2}\right)$$

$$L_k \equiv \frac{1}{N_{\mathrm{c}}} \sum_{s} \mathrm{tr}_{\mathrm{c}} \left[\prod_{j=0}^{kN_t-1} U_t(\boldsymbol{s} + j\hat{4}) \right].$$

k-winding Polyakov loop

The partition function

$$Z = Z_{\mathrm{g}} \left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k \right] \right\rangle$$

$$Z_{\mathrm{g}} = \int \mathcal{D}U \exp[-S_{\mathrm{g}}],$$

$$\langle \mathcal{O} \rangle = \lim_{\alpha \to 0} \frac{1}{Z_{\mathrm{g}}} \int \mathcal{D}U \exp[-S_{\mathrm{g}} - \alpha S_{\mathrm{Q}}] \mathcal{O}.$$

The grand potential

$$\Omega(T, \mu_{\mathrm{q}}) = -T \ln Z_{\mathrm{g}} - T \ln \left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k \right] \right\rangle$$

The partition function

$$Z = Z_g \left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_q} \Lambda_k \right] \right\rangle$$

$$Z_g = \int \mathcal{D}U \exp[-S_g] \, ,$$

$$\langle \mathcal{O} \rangle = \lim_{\alpha \to 0} \frac{1}{Z_g} \int \mathcal{D}U \exp[-S_g - \alpha S_Q] \mathcal{O} \, .$$

The grand potential

$$\Omega(T, \mu_q) = -T \ln Z_g - T \ln \left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_q} \Lambda_k \right] \right\rangle$$

$\rightarrow$ **cumulant expansion**

Cumulant expanded grand potential

$$\Omega(T, \mu_{\mathrm{q}}) = -T \ln Z_{\mathrm{g}} - T \underbrace{\sum_{\{m_k\}} \frac{e^{\sum_k k m_k \hat{\mu}_{\mathrm{q}}}}{\prod_k m_k!} \left\langle \prod_{k=-\infty}^{\infty} \Lambda_k^{m_k} \right\rangle_{\mathrm{c}}}_{\textcolor{red}{\Omega_{\mathrm{q}} \text{ : quark contributions}}}.$$

Cumulant expanded grand potential

$$\Omega\big(T, \mu_{\mathrm{q}}\big) = -T \ln Z_{\mathrm{g}} - T \sum_{\{m_k\}} \frac{e^{\sum_k k m_k \hat{\mu}_{\mathrm{q}}}}{\prod_k m_k!} \left\langle \prod_{k=-\infty}^{\infty} \Lambda_k^{m_k} \right\rangle_{\mathrm{c}}.$$

Ω_{q} **: quark contributions**

$$\frac{\Omega_{\mathrm{q}}\big(T, \mu_{\mathrm{q}}\big)}{-T} = X_0 + \sum_{w=1}^{\infty} \big(e^{w \hat{\mu}_{\mathrm{q}}} + e^{-w \hat{\mu}_{\mathrm{q}}}\big) X_w$$

X_w: an expectation value consists of several Λ_k operators with a total winding number given by the sum of those of the Λ_k's, $w = \sum_k k m_k$.

$$\chi_n^{(\mathrm{Q})}(T,\mu_{\mathrm{q}}) \equiv -\frac{\partial^n}{\partial\hat{\mu}_{\mathrm{q}}{}^n}\frac{\Omega(T,\mu_{\mathrm{q}})}{TV} = -\frac{\partial^n}{\partial\hat{\mu}_{\mathrm{q}}{}^n}\frac{\Omega_{\mathrm{q}}(T,\mu_{\mathrm{q}})}{TV} \ .$$

$$\textcolor{red}{\Omega_{\mathrm{q}}(T,\mu_{\mathrm{q}}) = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\mathrm{q}}} + e^{-w\hat{\mu}_{\mathrm{q}}}\right)X_w}$$

$$= \frac{1}{V}\sum_{w=1}^{\infty} w^n\left[e^{w\hat{\mu}_{\mathrm{q}}} + (-1)^n e^{-w\hat{\mu}_{\mathrm{q}}}\right]X_w \ .$$

$$\chi_n^{(\mathrm{Q})}(T,\mu_{\mathrm q}) \equiv -\frac{\partial^n}{\partial \hat{\mu}_{\mathrm q}{}^n}\frac{\Omega(T,\mu_{\mathrm q})}{TV} = -\frac{\partial^n}{\partial \hat{\mu}_{\mathrm q}{}^n}\frac{\Omega_{\mathrm q}(T,\mu_{\mathrm q})}{TV}\ .$$

$$\textcolor{red}{\Omega_{\mathrm q}(T,\mu_{\mathrm q}) = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\mathrm q}} + e^{-w\hat{\mu}_{\mathrm q}}\right)X_w}$$

$$= \frac{1}{V}\sum_{w=1}^{\infty} w^n\left[e^{w\hat{\mu}_{\mathrm q}} + (-1)^n e^{-w\hat{\mu}_{\mathrm q}}\right]X_w\ .$$

The ratio of 4$^{\text{th}}$ to 2$^{\text{nd}}$:

$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_{\mathrm q})}{\chi_2^{(Q)}(T,\mu_{\mathrm q})} = \frac{\sum_{w=1}^{\infty} w^4 C_w X_w}{\sum_{w'=1}^{\infty} w'^2 C_{w'} X_{w'}}\ . \quad \left(C_w = e^{w\hat{\mu}_{\mathrm q}} + e^{-w\hat{\mu}_{\mathrm q}}\right)$$

@ deconfined phase ($T > T_c$)

$\mathbb{Z}_3$-**symmetry is broken** and all X_w can have nonzero value.

→ The leading contribution: X_1.

$$\frac{\chi_4^{(Q)}(T,\mu_q)}{\chi_2^{(Q)}(T,\mu_q)} = \frac{C_1 X_1 + 2^4 C_2 X_2 + \cdots}{C_1 X_1 + 2^2 C_2 X_2 + \cdots}$$

$$= \mathbf{\color{red}{1}} + 12 \frac{C_2}{C_1} \underbrace{\frac{X_2}{X_1}}_{\mathcal{O}(\kappa^{N_t})}.$$

@ confined phase ($T < T_c$)

@ deconfined phase ($T > T_c$)

$\mathbb{Z}_3$**-symmetry is broken** and all X_w can have nonzero value.

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$$\frac{\chi_4^{(Q)}(T,\mu_q)}{\chi_2^{(Q)}(T,\mu_q)} = \frac{C_1 X_1 + 2^4 C_2 X_2 + \cdots}{C_1 X_1 + 2^2 C_2 X_2 + \cdots}$$

$$= \mathbf{\color{red}{1}} + 12 \frac{C_2}{C_1} \underline{\frac{X_2}{X_1}}.$$

$$\underbrace{\qquad\qquad}_{\mathcal{O}(\kappa^{N_t})}$$

@ confined phase ($T < T_c$)

$\mathbb{Z}_3$**-symmetry is preserved** and X_{3n} ($n \in \mathbb{Z}$) can have nonzero value.

$\to$ The leading contribution: X_3.

$$\frac{\chi_4^{(Q)}(T,\mu_q)}{\chi_2^{(Q)}(T,\mu_q)} = \frac{3^4 C_3 X_3 + 6^4 C_6 X_6 + \cdots}{3^2 C_3 X_3 + 6^2 C_6 X_6 + \cdots}$$

$$= \mathbf{\color{red}{9}} + 140 \frac{C_6}{C_3} \underline{\frac{X_6}{X_3}}.$$

$$\underbrace{\qquad\qquad}_{\mathcal{O}(\kappa^{3N_t})}$$

@ deconfined phase ($T > T_c$)

$\mathbb{Z}_3$**-symmetry is broken** and all X_w can have nonzero value.

→ The leading contribution: X_1.

$$\frac{\chi_4^{(Q)}(T, \mu_q)}{\chi_2^{(Q)}(T, \mu_q)} = \frac{C_1 X_1 + 2^4 C_2 X_2 + \cdots}{C_1 X_1 + 2^2 C_2 X_2 + \cdots}$$

$$= \mathbf{\color{red}{1}} + 12 \underbrace{\frac{C_2}{C_1} \frac{X_2}{X_1}}_{\mathcal{O}(\kappa^{N_t})}.$$

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$$= \mathbf{\color{red}{9}} + 140 \underbrace{\frac{C_6}{C_3} \frac{X_6}{X_3}}_{\mathcal{O}(\kappa^{3N_t})}.$$

$$\boxed{\frac{\chi_4^{(Q)}}{\chi_2^{(Q)}} = \begin{cases} \mathbf{1} & (T > T_c) \\ \mathbf{9} & (T < T_c) \end{cases} \textbf{ is analytically reproduced!}}$$

Theory:

SU(3) lattice QCD with N_f flavor Wilson fermion.

Assumption:
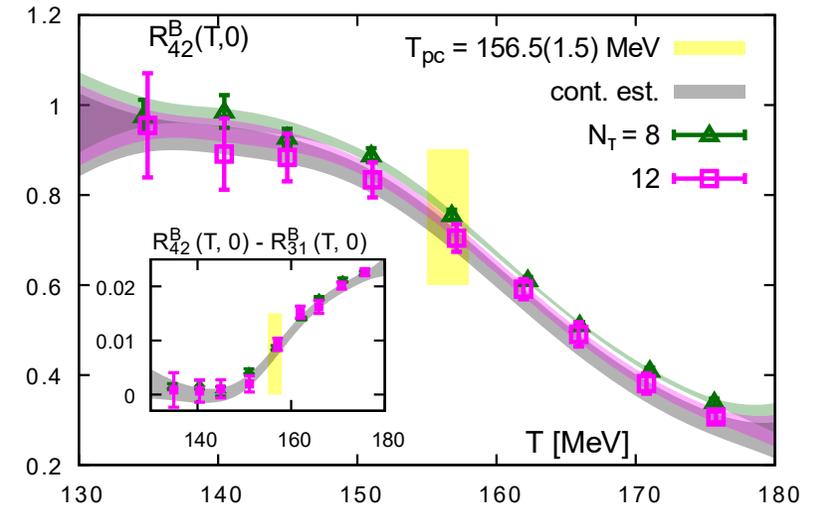
- Heavy quark
  ($\mathbb{Z}_3$ symmetry)

Method:

1. Hopping parameter expansion (HPE)
2. Cumulant expansion

Result:

$\chi_4^{(Q)}/\chi_2^{(Q)}$ is analytically calculated as

$$\frac{\chi_4^{(Q)}}{\chi_2^{(Q)}} = \begin{cases} 1 & (T > T_c) \\ 9 & (T < T_c) \end{cases} \quad \text{at the LO of HPE.}$$
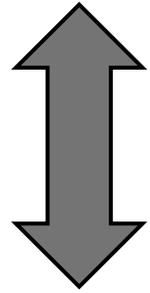

A. Bazavov, D. Bollweg, H.-T. Ding, P. Enns, J. Goswami, P. Hegde, O. Kaczmarek, F. Karsch, R. Larsen et al. (HotQCD Collaboration), Phys. Rev. D **101**, 074502 (2020)

Thank you for listening!

Strategy

$$\Omega_{\mathrm{SB}}(T, \mu_{\mathrm{q}}) = -TgN_{\mathrm{site}}\left(e^{\hat{\mu}_{\mathrm{q}}} + e^{-\hat{\mu}_{\mathrm{q}}}\right)\underline{e^{-\beta M}}$$

Same at the LO

$$\Omega_{\mathrm{q}}(T, \mu_{\mathrm{q}}) = -TX_0 - T\sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\mathrm{q}}} + e^{-w\hat{\mu}_{\mathrm{q}}}\right)\underline{X_w}$$

The Mass of the excitation can be extracted from LO behavior of X_w.

Quark excitation

$$\Omega_{\text{SB}}(T, \mu_{\text{q}}) = -T \cdot 2N_{\text{f}}N_{\text{c}} \cdot N_{\text{site}}\left(e^{\hat{\mu}_{\text{q}}} + e^{-\hat{\mu}_{\text{q}}}\right)\underline{e^{-\beta M_{\text{Q}}}}$$

$$\Omega_{\text{q}}(T, \mu_{\text{q}}) = -TX_0 - T \cdot 2N_{\text{f}}N_{\text{c}} \cdot N_{\text{site}}\left(e^{\hat{\mu}_{\text{q}}} + e^{-\hat{\mu}_{\text{q}}}\right)(2\kappa)^{N_t}\underline{\frac{L_1}{N_{\text{site}}}}$$

$$2\kappa = e^{-m_{\text{q}}a}$$

Mass of quark excitation

$$M_{\text{Q}} = -\frac{1}{\beta}\ln\left[(2\kappa)^{N_t}\frac{\langle L_1\rangle}{N_{\text{site}}}\right] = m_{\text{q}} + \underline{F_{\text{Q}}}$$
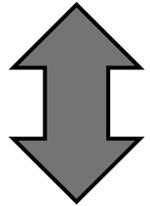
Free energy of a quark

Baryon excitation

b: baryons
g_b: degeneracy
M_b: mass of baryon b

$$\Omega_{\mathrm{SB}}(T, \mu_{\mathrm{q}}) = -TN_{\mathrm{site}}\left(e^{3\widehat{\mu}_{\mathrm{q}}} + e^{-3\widehat{\mu}_{\mathrm{q}}}\right) \sum_b g_b e^{-M_b/T}$$

$$\Omega_{\mathrm{q}}(T, \mu_{\mathrm{q}}) = -TX_0 - TN_{\mathrm{site}}\left(e^{3\widehat{\mu}_{\mathrm{q}}} + e^{-3\widehat{\mu}_{\mathrm{q}}}\right)(2\kappa)^{3N_t}\left(\frac{2}{3}N_{\mathrm{f}}\frac{\langle L_3 \rangle}{N_{\mathrm{site}}} - 2N_{\mathrm{f}}^2\frac{\langle L_2 L_1 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}} + \frac{8N_{\mathrm{f}}^3}{3!}\frac{\langle L_1^3 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}}\right)$$

How can we distinguish each sectors of baryonic excitations?

Symmetric ψ_{space}

Spin 3/2:

$$g_{3/2}^{(\mathrm{S})} = 4 \times \frac{(N_{\mathrm{f}} + 2)(N_{\mathrm{f}} + 1)N_{\mathrm{f}}}{6}$$

Spin 1/2:

$$g_{1/2}^{(\mathrm{M})} = 2 \times 2 \frac{(N_{\mathrm{f}} + 1)N_{\mathrm{f}}(N_{\mathrm{f}} - 1)}{6}$$

Anti-symmetric ψ_{space}

Spin 3/2:

$$g_{3/2}^{(\mathrm{A})} = 4 \times \frac{N_{\mathrm{f}}(N_{\mathrm{f}} - 1)(N_{\mathrm{f}} - 2)}{6}$$

Spin 1/2:

$$g_{1/2}^{(\mathrm{M})} = 2 \times \frac{(N_{\mathrm{f}} + 1)N_{\mathrm{f}}(N_{\mathrm{f}} - 1)}{6}$$

Contributions from each baryons can be distinguished by the N_{f}-dependences.

Baryon contribution:

$$\Omega_{\mathrm{q}}\left(T, \mu_{\mathrm{q}}\right) = \cdots + \#(2\kappa)^{3N_t} \underbrace{\left(\frac{2}{3} N_{\mathrm{f}} \frac{\langle L_3 \rangle}{N_{\mathrm{site}}} - 2N_{\mathrm{f}}^2 \frac{\langle L_2 L_1 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}} + \frac{4N_{\mathrm{f}}^3}{3} \frac{\langle L_1^3 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}}\right)}_{\color{red}{}}$$

$$= g_{3/2}^{(S)} \times \frac{2}{3} \times \frac{\langle L_3 \rangle - 3\langle L_2 L_1 \rangle_{\mathrm{c}} + 2\langle L_1^3 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}} \qquad \textbf{Spin 3/2}$$
Flavor sym.

$$+ g_{1/2}^{(M)} \times \frac{2}{3} \times \frac{\langle L_3 \rangle + 3\langle L_2 L_1 \rangle_{\mathrm{c}} + 2\langle L_1^3 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}} \qquad \textbf{Spin 1/2}$$
Flavor mixed

$$+ g_{3/2}^{(A)} \times \frac{1}{3} \times \frac{-\langle L_3 \rangle + 4\langle L_1^3 \rangle_{\mathrm{c}}}{N_{\mathrm{site}}} \qquad \textbf{Spin 3/2}$$
Flavor anti-sym.

Classification of each baryons seems to be achieved in heavy-quark QCD.

However, difficult to calculate them in an analytical manner.

Strong coupling expansion:

$$\Omega_{\mathrm{q}}(T, \mu_{\mathrm{q}}) = \cdots + \#(2\kappa)^{3N_t}\left(\frac{2}{3}N_{\mathrm{f}}\frac{\langle L_3\rangle}{N_{\mathrm{site}}} - 2N_{\mathrm{f}}^2\frac{\langle L_2 L_1\rangle_{\mathrm{c}}}{N_{\mathrm{site}}} + \frac{4N_{\mathrm{f}}^3}{3}\frac{\langle L_1^3\rangle_{\mathrm{c}}}{N_{\mathrm{site}}}\right)$$

$$\simeq g_{3/2}^{(S)} \times 4 \times e^{-3m_{\mathrm{q}}/T}$$

Spin 3/2
Flavor sym.

$$+ g_{1/2}^{(M)} \times 2 \times e^{-3m_{\mathrm{q}}/T}$$

Spin 1/2
Flavor mixed

$$+ g_{3/2}^{(A)} \times 0 \times$$

Spin 3/2
Flavor anti-sym.

Very simple and reasonable results are derived.

For a single variable

Let $P(x)$ be a probability distribution function of x ,
a cumulant generating function is defined as

$$C(\theta) = \ln\langle e^{\theta x}\rangle_{P(x)} \quad .$$

$$\langle f(x)\rangle_{P(x)} \equiv \int dx\, f(x)P(x)$$

mth cumulant of x is calculated as a mth derivative of $C(\theta)$ with respect to θ,

$$\langle x^m\rangle_{\mathrm{c}} = \frac{\partial^m}{\partial\theta^m}\ln\langle e^{\theta x}\rangle_{P(x)}\bigg|_{\theta=0} \quad ,$$

and thus

$$C(\theta) = \sum_{m=0}^{\infty} \frac{\langle x^m\rangle_{\mathrm{c}}}{m!}\theta^m \quad .$$

For multiple variables

Let $P(\vec{x})$ be a probability distribution function of $\vec{x} = (x_1, \cdots, x_N)$,
a cumulant generating function is defined as

$$C(\vec{\theta}) = \ln \left\langle e^{\sum_{k=1}^{N} \theta_k x_k} \right\rangle_{P(\vec{x})}.$$

$$\langle f(\vec{x}) \rangle_{P(\vec{x})} \equiv \int d\vec{x}\, f(\vec{x}) P(\vec{x})$$
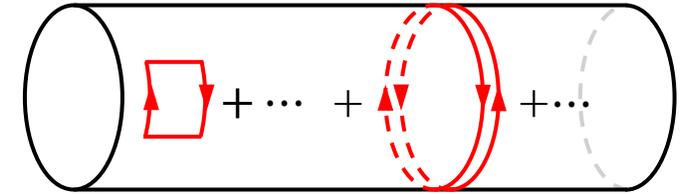
$$\vec{\theta} = (\theta_1, \cdots, \theta_N)$$

A cumulant of $\vec{x}$ is calculated as derivatives of $C(\vec{\theta})$ with respect to $\vec{\theta}$,

$$\langle x_1^{m_1} \cdots x_N^{m_N} \rangle_{\text{c}} = \frac{\partial^{m_1}}{\partial \theta_1^{m_1}} \cdots \frac{\partial^{m_N}}{\partial \theta_N^{m_N}} \ln \left\langle e^{\sum_{k=1}^{N} \theta_k x_k} \right\rangle_{P(\vec{x})} \Bigg|_{\vec{\theta}=0},$$
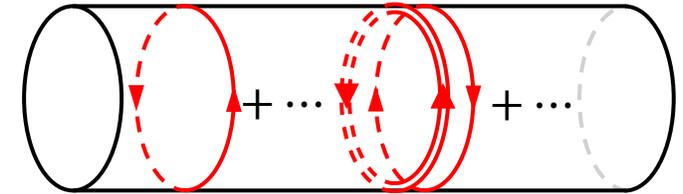
and thus

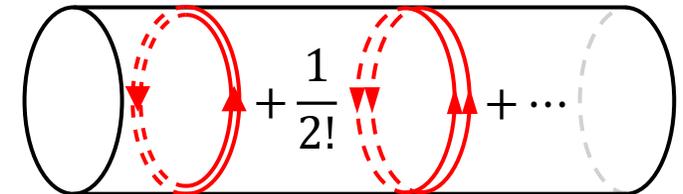$$C(\vec{\theta}) = \left\langle \prod_{k=1}^{n} \sum_{m_k=0}^{\infty} \frac{(\theta_k x_k)^{m_k}}{m_k!} \right\rangle_{\text{c}}.$$

$$X_0 = \underline{\langle\Lambda_0\rangle}_{\text{LO}} + \langle\Lambda_1\Lambda_{-1}\rangle_{\text{c}} + \cdots = \mathcal{O}(\kappa^4)$$
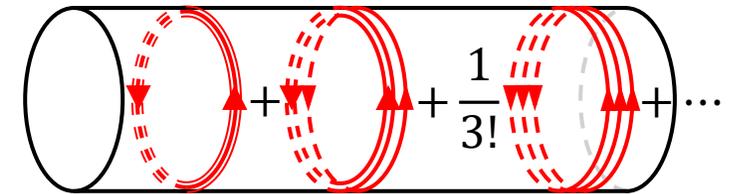
$$X_1 = \underline{\langle\Lambda_1\rangle}_{\text{LO}} + \langle\Lambda_2\Lambda_{-1}\rangle_{\text{c}} + \cdots = \mathcal{O}(\kappa^{N_t})$$

$$X_2 = \underline{\langle\Lambda_2\rangle + \frac{1}{2!}\langle\Lambda_1^2\rangle_{\text{c}}}_{\text{LO}} + \cdots = \mathcal{O}(\kappa^{2N_t})$$

$$X_3 = \underline{\langle\Lambda_3\rangle + \langle\Lambda_2\Lambda_1\rangle_{\text{c}} + \frac{1}{3!}\langle\Lambda_1^3\rangle_{\text{c}}}_{\text{LO}} + \cdots = \mathcal{O}(\kappa^{3N_t})$$

Note: X_w is a real (because $\Omega_{\text{q}} \in \mathbb{R}$ by its definition), i.e., $\boldsymbol{X_w = X_w^* = X_{-w}}$.

$$\frac{\Omega_{\text{q}}(T,\mu_{\text{q}})}{-T} = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\text{q}}}X_w + e^{-w\hat{\mu}_{\text{q}}}X_{-w}\right) = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\text{q}}} + e^{-w\hat{\mu}_{\text{q}}}\right)X_w$$

In our case, $C(\vec{\theta}) = \ln\left\langle e^{\sum_{k=1}^{N} \theta_k x_k}\right\rangle_{P(\vec{x})}$ is

$$\ln\left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}}\Lambda_k\right]\right\rangle = \ln\left\langle \exp\left[\sum_{k=-\infty}^{\infty} \theta_k\Lambda_k\right]\right\rangle\Bigg|_{\theta_k = e^{k\hat{\mu}_{\mathrm{q}}}} .$$

Applying the cumulant expansion method,

$$\ln\left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}}\Lambda_k\right]\right\rangle = \left\langle \prod_{k=-\infty}^{\infty} \sum_{m_k=0}^{\infty} \frac{e^{km_k\hat{\mu}_{\mathrm{q}}}\Lambda_k^{m_k}}{m_k!}\right\rangle_c$$

$$= \sum_{\{m_k\}} \frac{e^{\sum_k km_k\hat{\mu}_{\mathrm{q}}}}{\prod_k m_k!}\left\langle \prod_{k=-\infty}^{\infty} \Lambda_k^{m_k}\right\rangle_c .$$