Exploring Dense QCD through Hamiltonian Lattice Simulation in (1+1) Dimensions

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Hayata, KN and Hidaka, JHEP 07 (2024) 106

Sign problem

- Monte Carlo can not be directly applied to QCD at finite baryon density.

  - QCD at finite baryon density with Nc=2 and Nf=(even number)

    Kogut, Stephanov and Toublan (1999); Kogut, Sinclair, Hands and Morrison (2001); Muroya, Nakamura, Nonaka and Takaishi (2003); Hands, Kogut, Lombardo and Morrison (1999); Hands, Sitch and Skullerud (2008); Cotter, Giudice, Hands and Skullerud (2013); Braguta, Ilgenfritz, Kotov, Molochkov and Nikolaev (2016); Wilhelm, Holicki, Smith, Wellegehausen and von Smekal (2019); Iida, Itou and Lee (2020); Boz, Giudice, Hands and Skullerud (2020); Braguta (2023)……

  - Three-color QCD with isospin chemical potential

    Son and Stephanov (2001); Kogut and Sinclair (2002, 2004); Forcrand, Stephanov, Wenger (2007); Detmold, Orginos and Shi (2012); Endrődi (2014); Brandt, Endrődi, and Schmalzbauer (2018); Bornyakov et al. (2021); Brandt, Cuteri, Endrődi (2023)……

- Investigation of numerical calculation without sign problem is important!

  - Neutron stars and heavy ion collision experiments = Three-color QCD at μ_B

Hamiltonian formalism

- Hamiltonian formalism may be good for sign problem free calculation.
  - Lattice Hamiltonian is defined on a spatial lattice, and time is continuous.

- Minimization of the free energy without Monte Carlo→No sign problem
  - We use good variational ansatz and efficient algorithm for searching the ground state.

- There are many application of Hamiltonian formalism to gauge theories.
  - Schwinger model, SU(2) or SU(3) lattice gauge theory in (1+1) dim:

Silvi et.al., (2017); Rigobello et.al., (2023); Buyens et al. (2016); Bañuls et al. (2017); Farrell et al. (2024); Belyansky et al. (2023); Chen, Yan and Shi (2024)……
Kuhn et.al., (2015); Silvi et.al., (2017); Banuls et.al., (2017); Sala et.al., (2018); Cataldi et.al. (2024); Calajo et.al. (2024), Ciavarella and Bauer (2024)……

- Numerical calculation of thermodynamic quantities in (1+1)-d QCD at μ_B.

Hayata, KN and Hidaka (2024); Pai, Akiyama and Todo (2025); Cataldi et.al. (2024)

Hamiltonian formalism for lattice gauge theory in (1+1) dimension

- **Kogut-Susskind Hamiltonian:** $H = H_{\rm E} + H_{\rm hop} + H_{\rm m}$ Kogut and Susskind, PRD (1975)

$$H_{\rm E} = \frac{ag_0^2}{2}\sum_{n=1}^{N-1} E_i^2(n),$$

$$H_{\rm hop} = \frac{1}{2a}\sum_{n=1}^{N-1}\left(\chi^\dagger(n+1)U(n)\chi(n) + \chi^\dagger(n)U^\dagger(n)\chi(n+1)\right),$$

$$H_{\rm m} = m_0\sum_{n=1}^{N}(-1)^n\chi^\dagger(n)\chi(n)$$

- **Quark (baryon) number:** $N_{\rm q} = \sum_{n=1}^{N}\chi^\dagger(n)\chi(n)$

- **Full Hamiltonian:** $H_{\rm tot} \equiv H_{\rm E} + H_{\rm hop} + H_{\rm m} - \mu_{0\rm B}N_{\rm B}$

Optimization of the free energy

Variational ansatz = Matrix Product state

$$|\psi\rangle = \sum_{\alpha_1,\cdots\alpha_N} \sum_{i_1,i_2,\cdots i_N}^{2} A^{i_1}_{\alpha_1} A^{i_2}_{\alpha_1\alpha_2} \cdots A^{i_N}_{\alpha_N} |i_1, i_2, \cdots i_N\rangle$$

Products of matrices — Spin basis

Perez-Garcia, et.al., (2007) ; McCulloch, et.al., (2007)

- Optimization of the free energy by Density Matrix Renormalization Group
  - Variational minimization and discarding tiny singular values

White, PRL (1992)

- We numerically determine Ground state: $|\mathrm{GS}\rangle$
- We can calculate expectation values: $\langle \mathrm{GS}|\hat{\mathcal{O}}|\mathrm{GS}\rangle$

- This method has been applied to one-dimensional quantum spin systems and has achieved great success.

Dimensionless Hamiltonian

- **We perform DMRG optimization with MPS using iTensor.**
  https://itensor.github.io/ITensors.jl/stable/ITensorType.html; Also see reference by Fishman, White and Stoudenmire (2022)

- **Full Hamiltonian w/ n_B normalized g_0** $J = ag_0/2\,,\ \epsilon = 1/(2ag_0)\,,\ m = m_0/g_0\,,\ \mu = \mu_0/g_0$

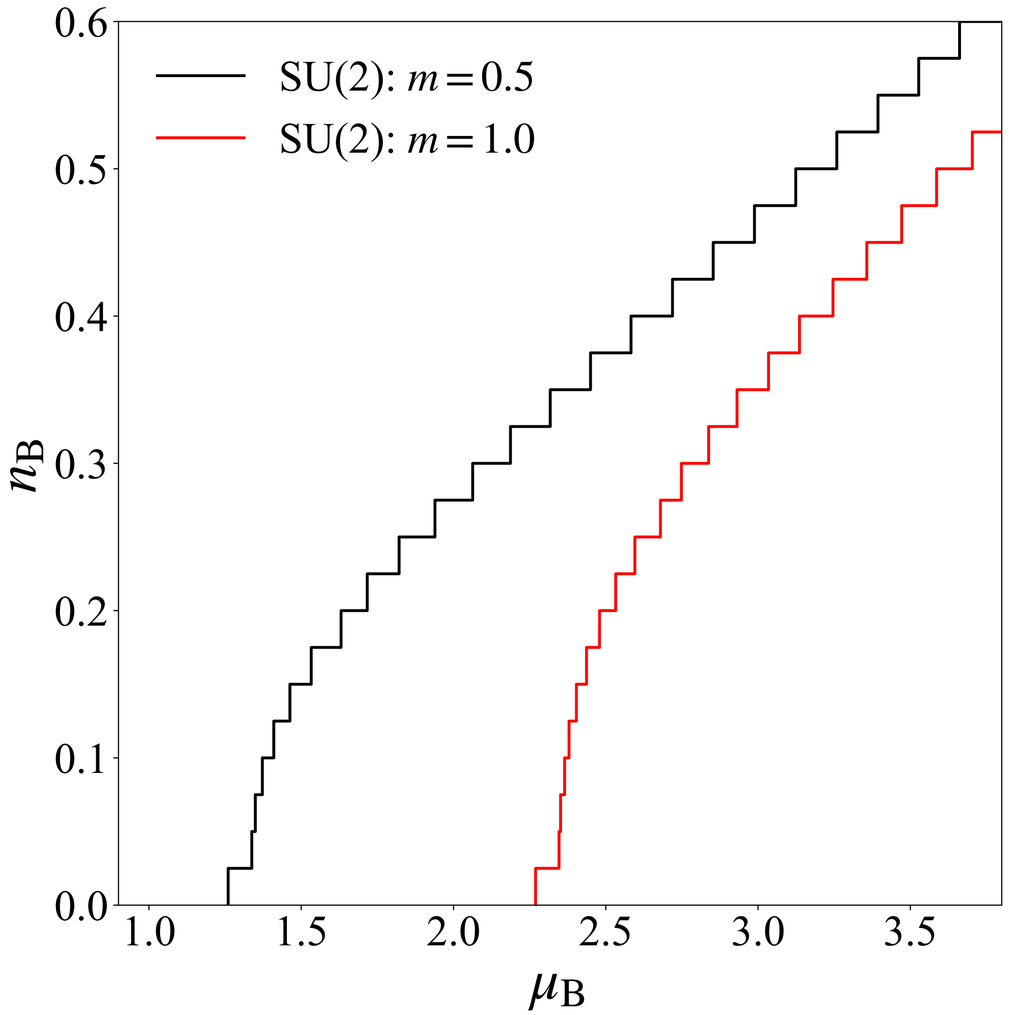
$$H_{\rm E}/g_0 = J\sum_{n=1}^{N-1} E_i^2(n)\,, \quad H_{\rm hop}/g_0 = \epsilon\sum_{n=1}^{N-1}\left(\chi^\dagger(n+1)U(n)\chi(n) + {\rm h.c.}\right)\,,$$

$$H_{\rm m}/g_0 = m\sum_{n=1}^{N}(-1)^n\chi^\dagger(n)\chi(n)\,, \quad \mu_{0\rm B}N_{\rm B}/g_0 = \mu_{\rm B}\frac{1}{N_{\rm c}}\sum_{n=1}^{N}\chi^\dagger(n)\chi(n)\,.$$

Groundwork for numerical calculation

- Since I can solve Gauss law constraint in open boundary condition, SU(2) link variables can be eliminated and represented by Staggared fermions.
- After Wigner-Jordan transformation, our Hamiltonian is represented by Pauli matrices.

Baryon number



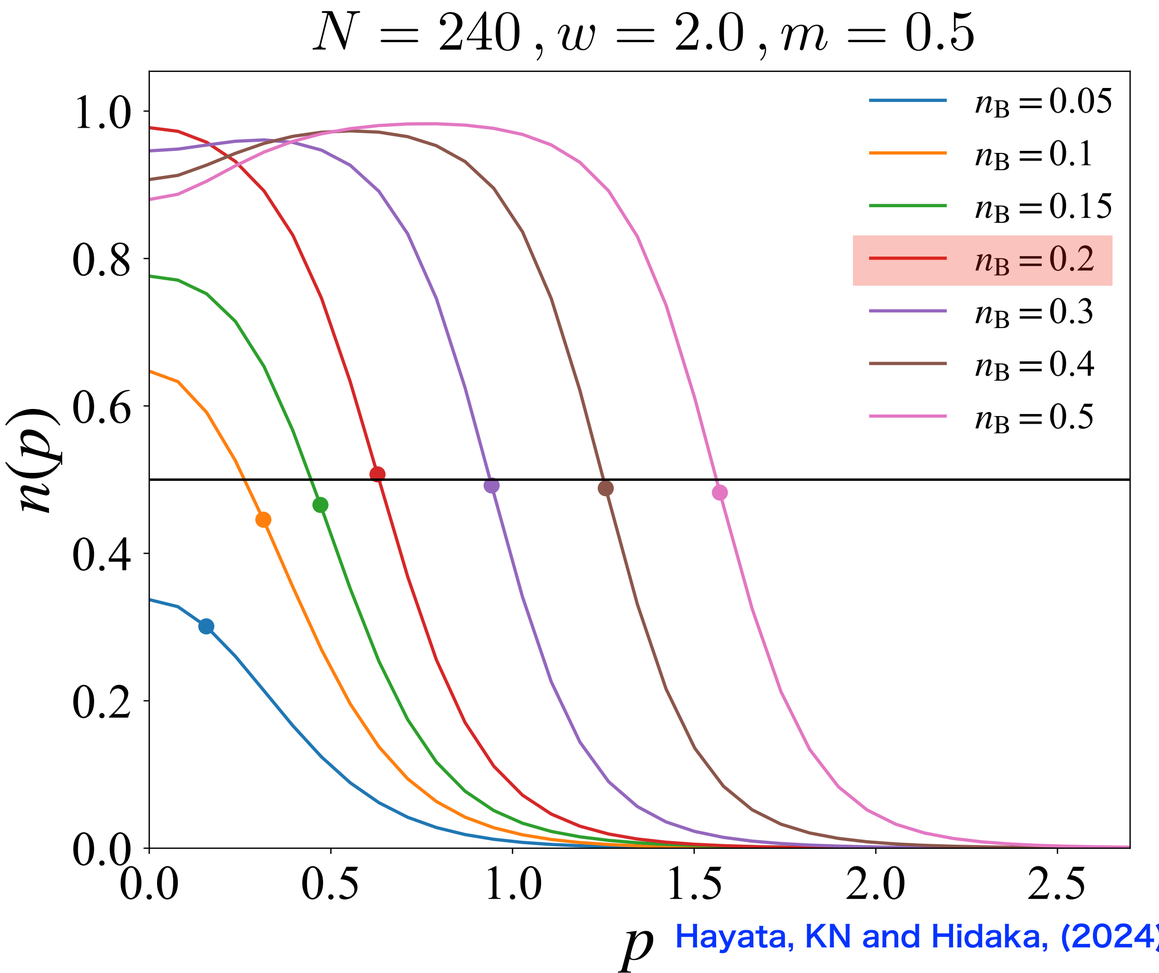
- N=160, N_c=2, and w=2.0
- Stepwise behavior
  - Due to the finite volume lattice
- Silver Blaze phanomena
- Threshold value

$$\mu_{\mathrm{c}} = \begin{cases} 1.26 > 2 \times 0.5 & (m = 0.5) \\ 2.27 > 2 \times 1.0 & (m = 1.0) \end{cases}$$

Hayata, KN and Hidaka, (2024)

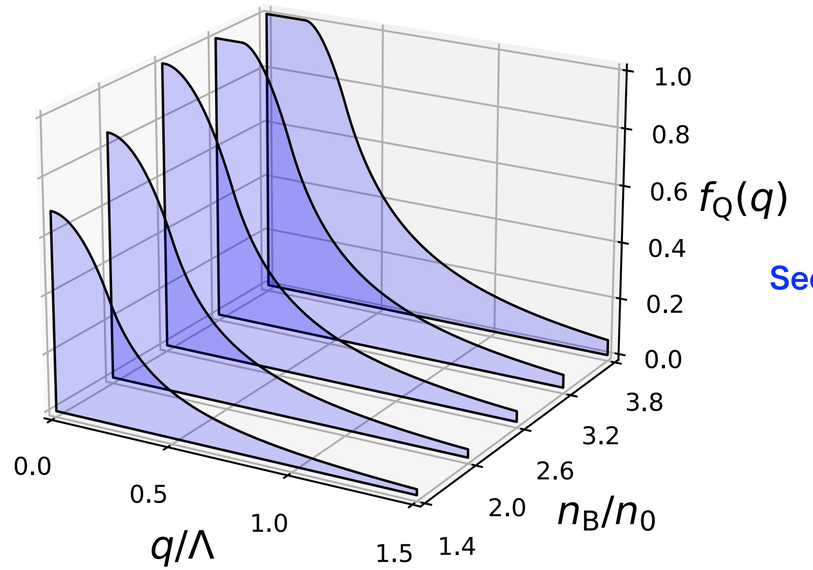
Quark distribution function

- Whether the system behaves as quark matter or not.



Hayata, KN and Hidaka, (2024)

- At low density, n(p) has a peak at p=0.
- As n_B increases, maximum of n(p) increases and forms Fermi sea.
- Similar behavior in Quarkyonic picture:



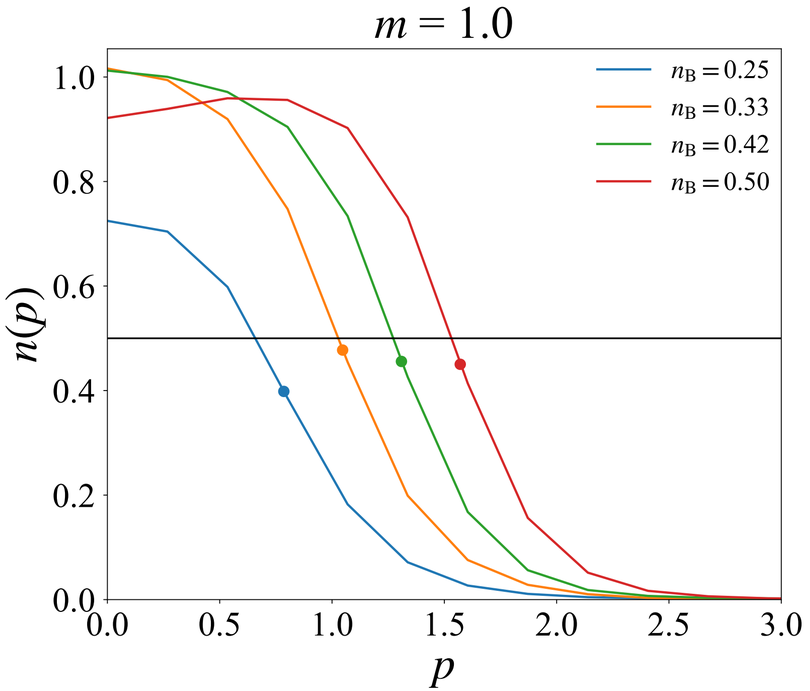
See talk by Kojo's talk

cf) BCS-BEC crossover of ultracold atom gases and QC_2D

Astrakharchik et.al., (2005); Regal et.al., (2005); Kojo and Suenaga (2022)

Summary and future work

- Variational numerical calculation based on Hamiltonian formalism.
  - Fermi surface forms about n_B~0.2.
  - We obtain similar behavior in N_c=3 case.



- Calculation of Baryon-Baryon correlation function
  - Related to verification of Quarkyonic matter and baryon mass
- Add flavor degrees of freedom

Thank you for your attention!

Back up

Solving gauss law

- In (1+1) dim, the gauge field are non-dynamical and can be eliminated from the Hamiltonian in open boundary condition.

- We choose an appropriate unitary matrix to eliminate the link variables in hopping term.

$$\Theta\tilde{H}_{\rm hop}\Theta^\dagger = \tilde{H}'_{\rm hop} = \frac{1}{2a}\sum_{n=1}^{N-1}(\chi^\dagger(n+1)\chi(n) + \chi^\dagger(n)\chi(n+1))$$

- Solving the Gauss law, we can represent the hopping term in terms of KS fermions.

$$\Theta\tilde{H}_{\rm E}\Theta^\dagger = \tilde{H}'_{\rm E} = \frac{ag^2}{2}(\sum_{n=1}^{N-1}(N-n)Q_i^2(n) + 2\sum_{n=1}^{N-2}\sum_{m=n+1}^{N-1}(N-m)\underbrace{Q_i(n)}_{\chi^\dagger(n)T_i\chi(n)}Q_i(m))$$

- The mass term and baryon number do not change under the unitary transformation.

$$\Theta\tilde{H}_{\rm m}\Theta^\dagger = \tilde{H}'_{\rm m} = \tilde{H}_{\rm m}\,, \quad \Theta N_{\rm B}\Theta^\dagger = N'_{\rm B} = N_{\rm B}$$

Wigner Jordan transformation

- For numerical calculations, we map the Hamiltonian to the spin system.
  - Using Pauli matrices, construct something that satisfies the fermionic anti-commutation relations.

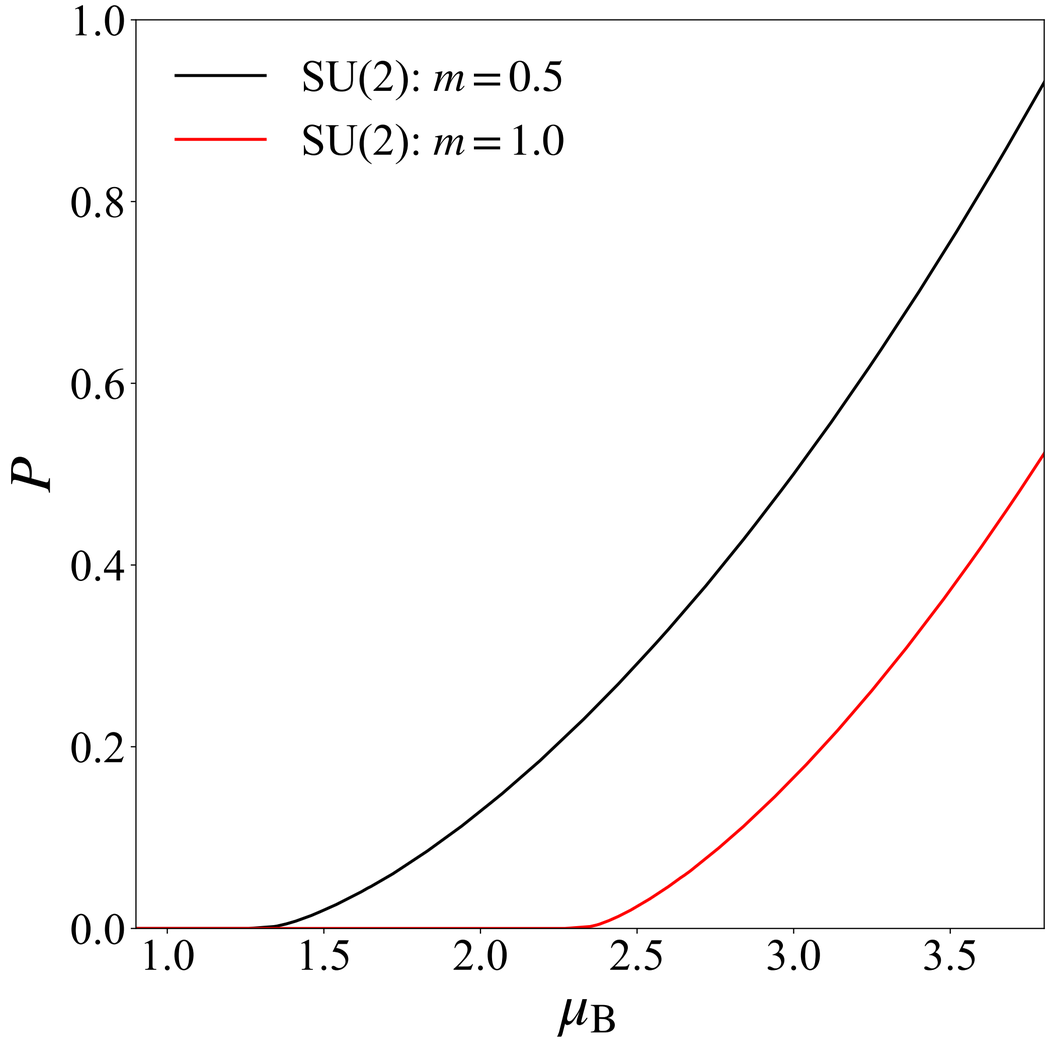
$$\chi_{f,c}(n) = \left[\prod_{k<\ell(n,f,c)} (-\sigma_k^z)\right] \sigma^-_{\ell(n,f,c)}\,,\ \chi^\dagger_{f,c}(n) = \left[\prod_{k<\ell(n,f,c)} (-\sigma_k^z)\right] \sigma^+_{\ell(n,f,c)}$$

  - The Hamiltonian in terms of fermions can be expressed using Pauli matrices (below is an example of the hopping term).

$$\tilde{H}_{\mathrm{hop}}/g_0 = \epsilon \sum_{n=1}^{N-1} \sum_{f=1}^{N_{\mathrm{f}}} \sum_{c=1}^{N_{\mathrm{c}}} (\sigma^+_{\ell(n+1,f,c)} \Sigma_{f,c}(n) \sigma^-_{\ell(n,f,c)} + \sigma^+_{\ell(n,f,c)} \Sigma^\dagger_{f,c}(n) \sigma^-_{\ell(n+1,f,c)})$$

Back up for SU(2) results

Pressure



Hayata, KN and Hidaka, (2024)

- **N=160, N_c=2, and w=2.0**
- **Grand potential and Pressure:**

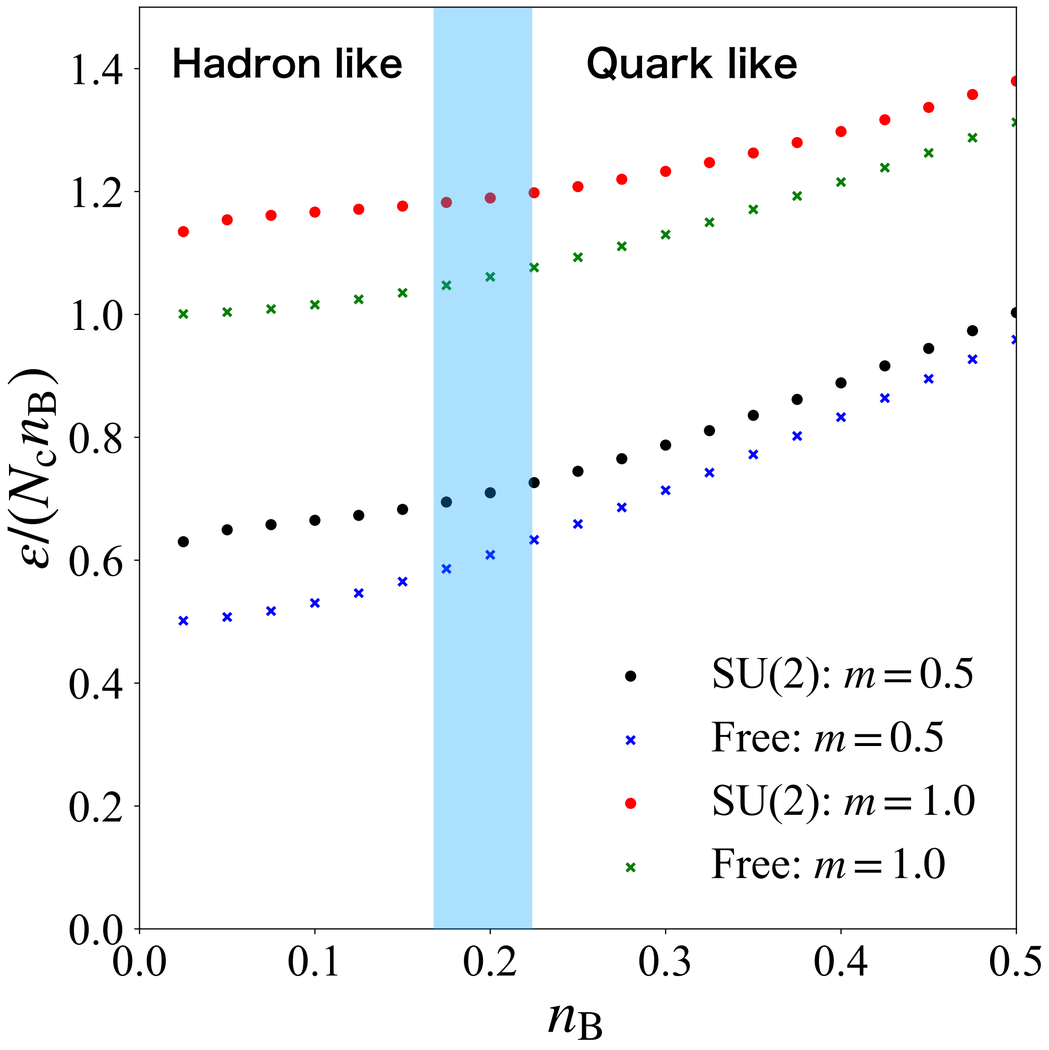
$$J = \langle \mathrm{GS} | (H - \mu_{\mathrm{B}} N_{\mathrm{B}}) | \mathrm{GS} \rangle$$

$$P = -J/V$$

- **First derivative = baryon number:**

$$n_{\mathrm{B}} = \frac{\mathrm{d}P(\mu_{\mathrm{B}})}{\mathrm{d}\mu_{\mathrm{B}}} = \frac{1}{V} \langle N_{\mathrm{B}} \rangle$$

From hadron to quark



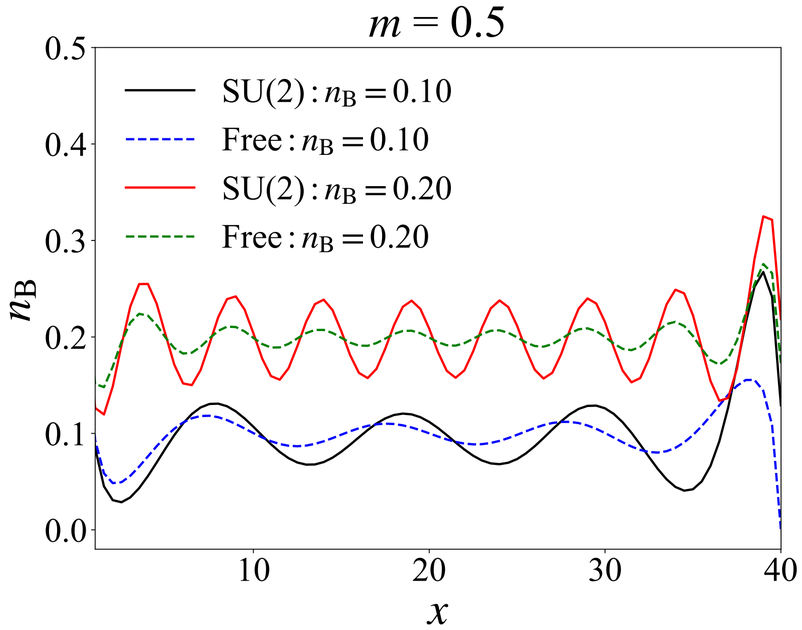
Hayata, KN and Hidaka, (2024)

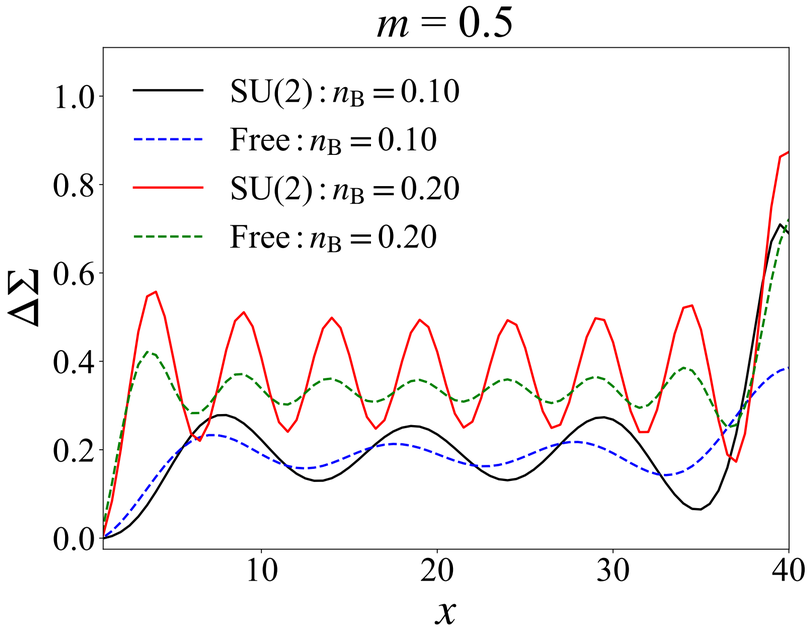
- Energy per one quark
- At high density, ε/n_q behaves like the free theory.
- At low density, ε/n_q is larger due to the confining energy.
- The behavior changes to free theory around $n_B \sim 0.2$.
- If interaction can be negligible, baryons will degenerate to the lowest energy state.

$$\epsilon/(N_c n_B) = \text{const}$$

Inhomogeneous phase (m=0.5)

- The explicit translational symmetry leads to spatial dependence of n_B and Σ.
  - This explicit breaking is induced by the open boundary condition and finite lattice spacing.

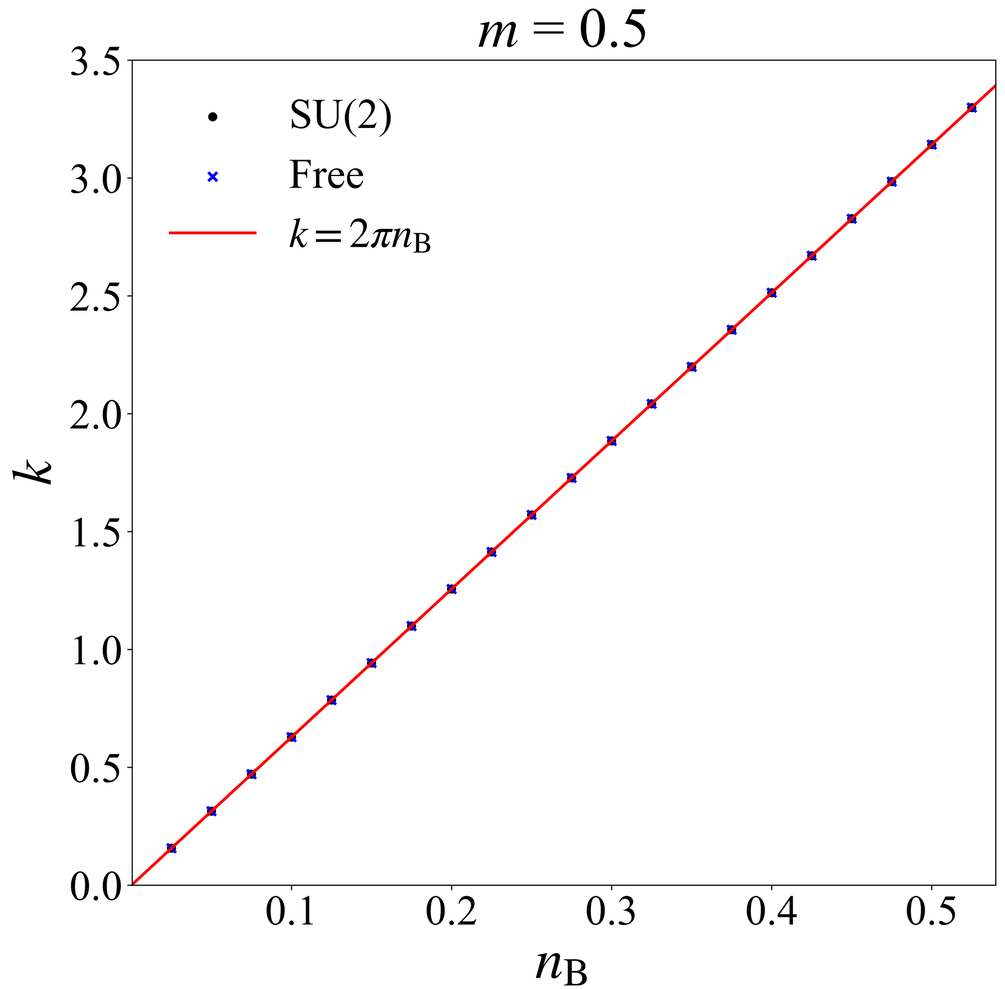




- SSB of continuous symmetry is prohibited.
  - Taking large volume and continuum limit, the above inhomogeneous state will vanish,
  - Important direction: calculation with larger site, investigation of quasi-long range order
  - In free theory, the modulations vanish proportionally to 1/L.

Wave number (period)

- Fourier transform on $n_B(x)$ and wave number of the largest modulation.

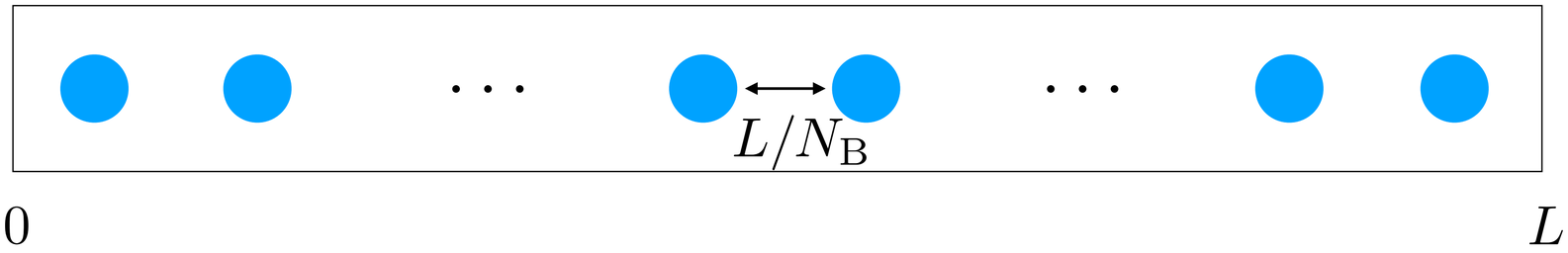


Hayata, KN and Hidaka, (2024)

Baryons crystal and chiral density wave

- If I assume repulsive force, the baryons are periodically aligned.

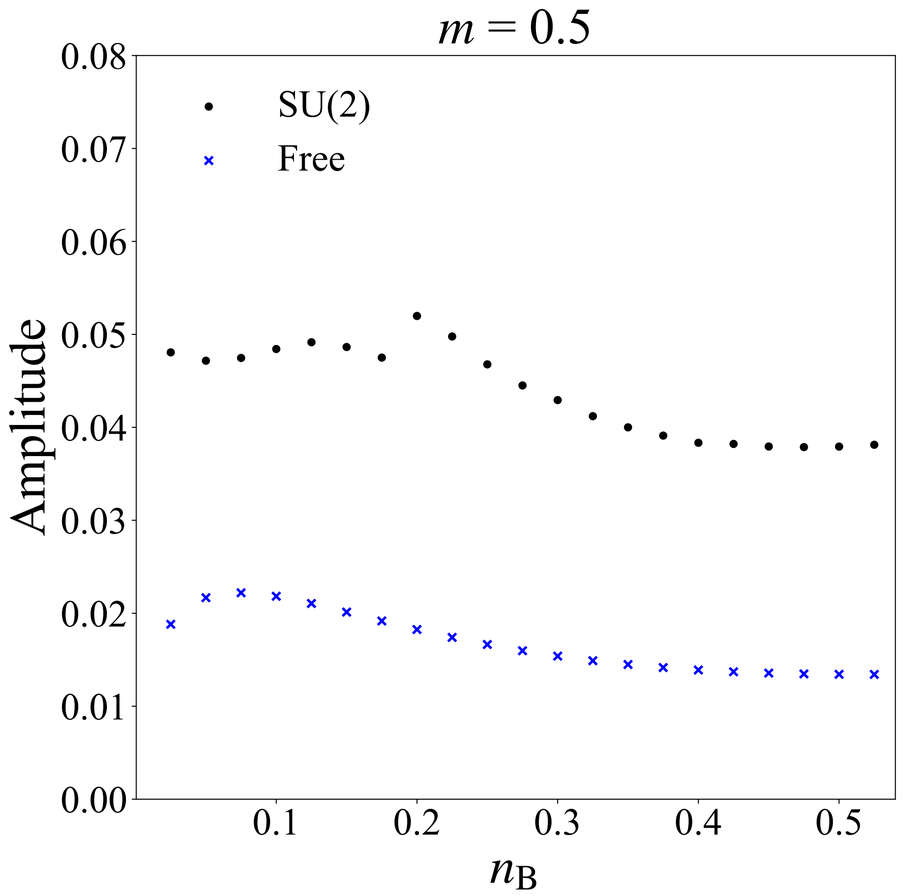
cf) Bosonization with large-N_c expansion Kojo, Nucl.A (2012)



- $k=2\pi/(\text{period})=2\pi/(L/N_B)=2\pi n_B$
- Due to the Peierls instability, the particle and hole form the condensate.
- Both of the particle and hole have p_F, then the total momentum is $2p_F$.
- $k=2p_F=2\pi n_B$ -In (1+1) dim, the fermi momentum is πn_B.

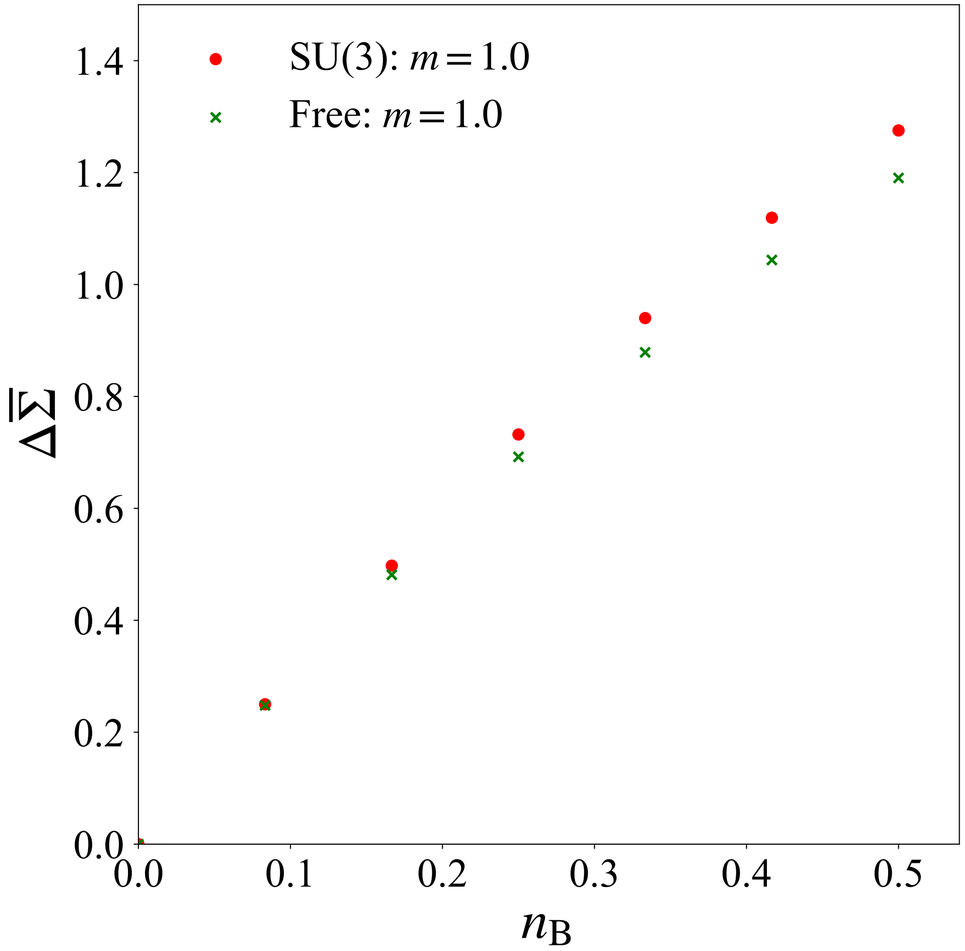
Amplitude

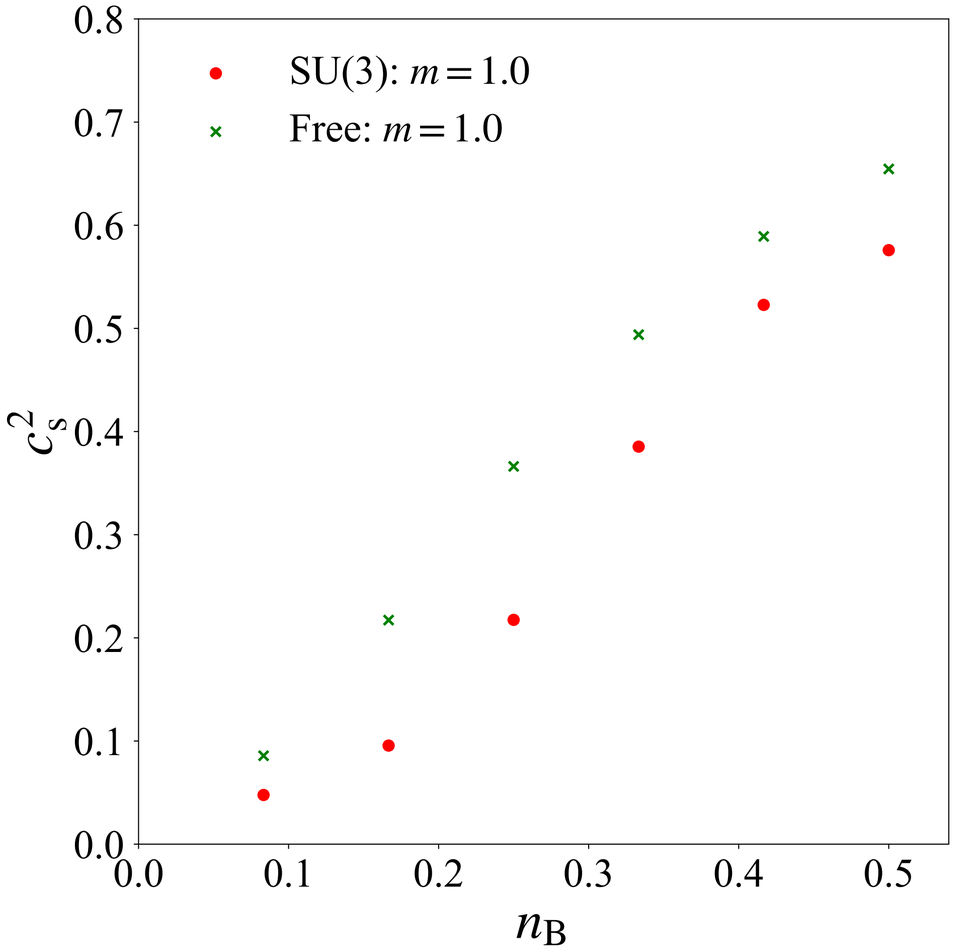
- The largest amplitude of the modulation with k in the previous slide.



- It is difficult to ascertain numerically whether the modulation persist in large V.

Chiral condensate and sound velocity

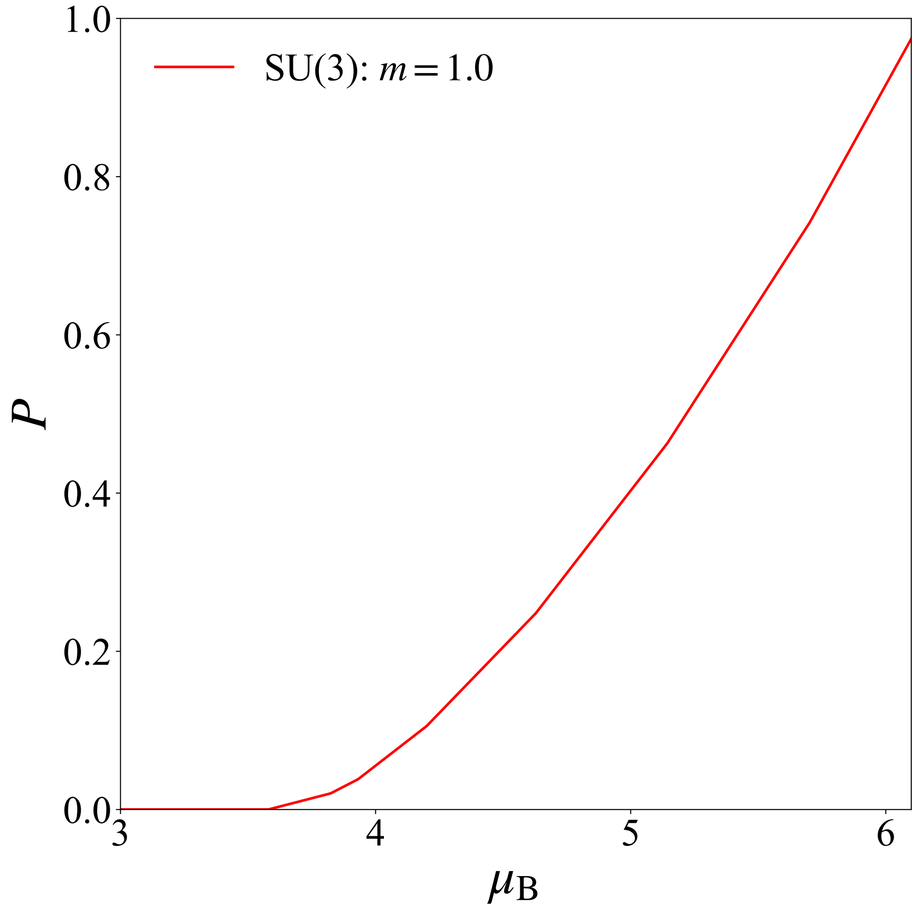


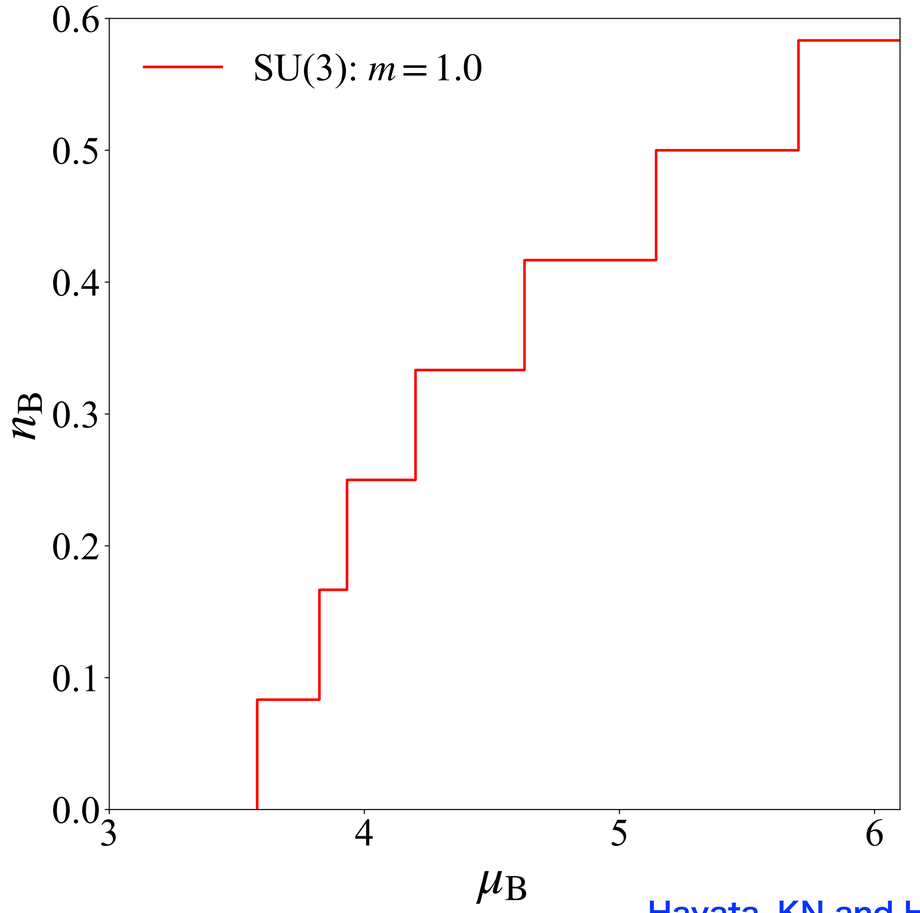


Back up for SU(3) results

Pressure and baryon number

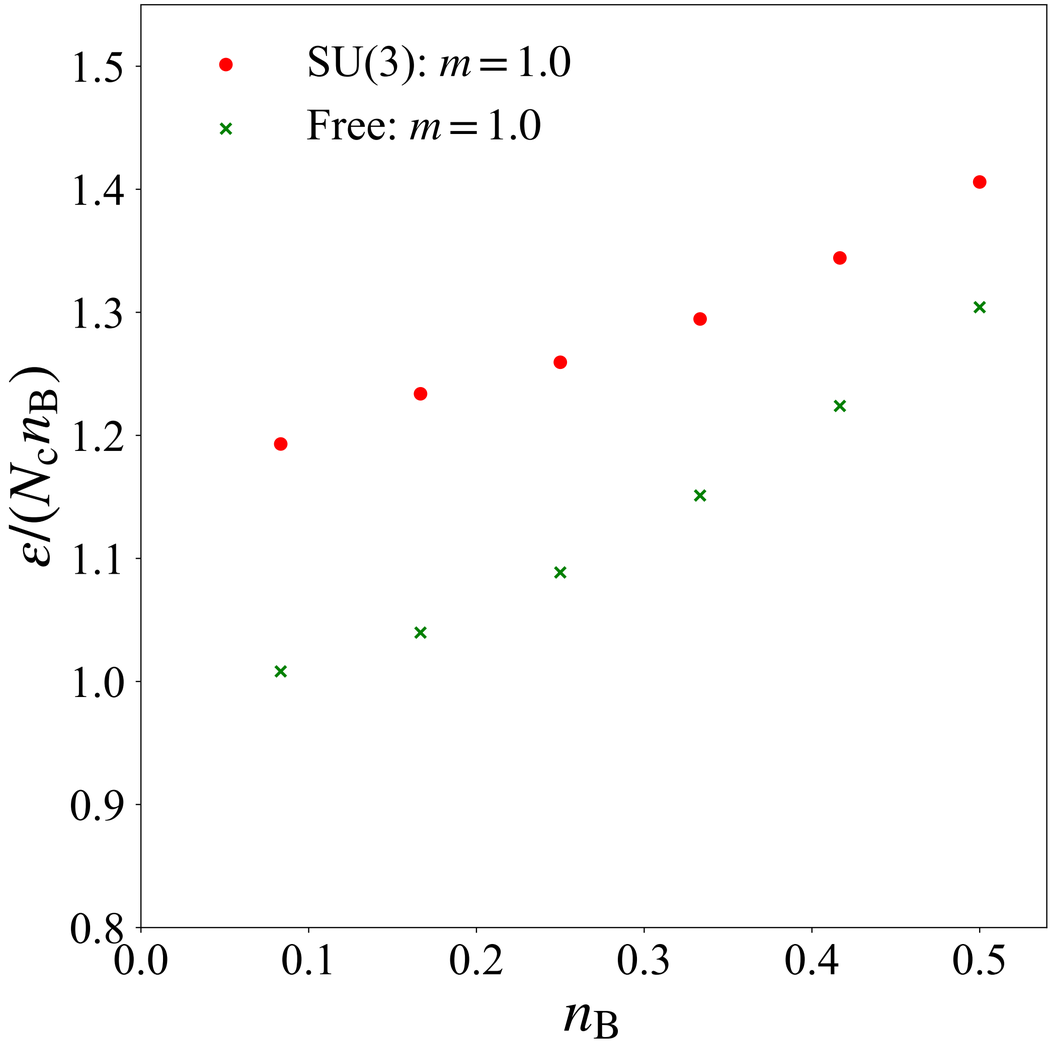
- N=48, w=2. Threshold value is 3.58.





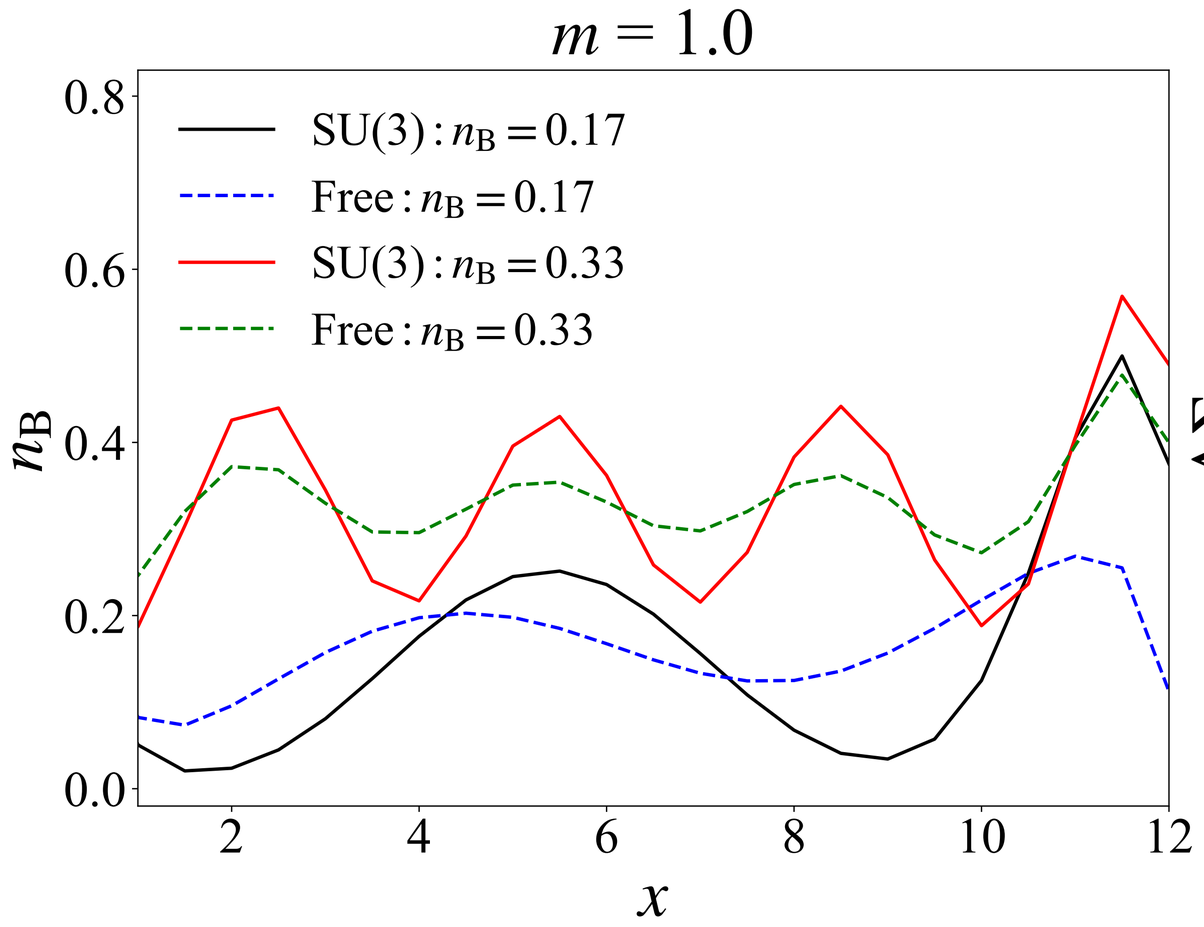
Hayata, KN and Hidaka, (2024)

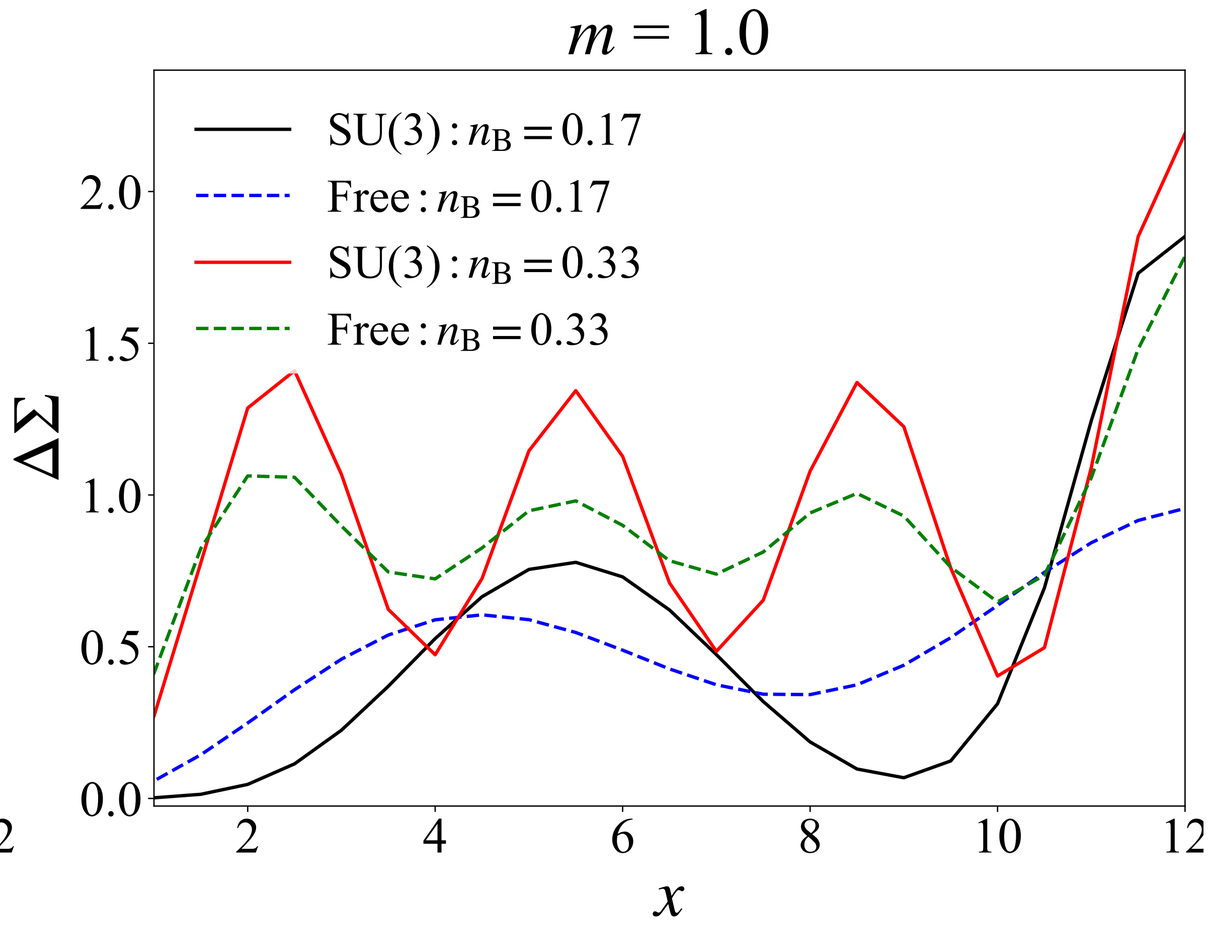
Energy per one quark



Hayata, KN and Hidaka, (2024)

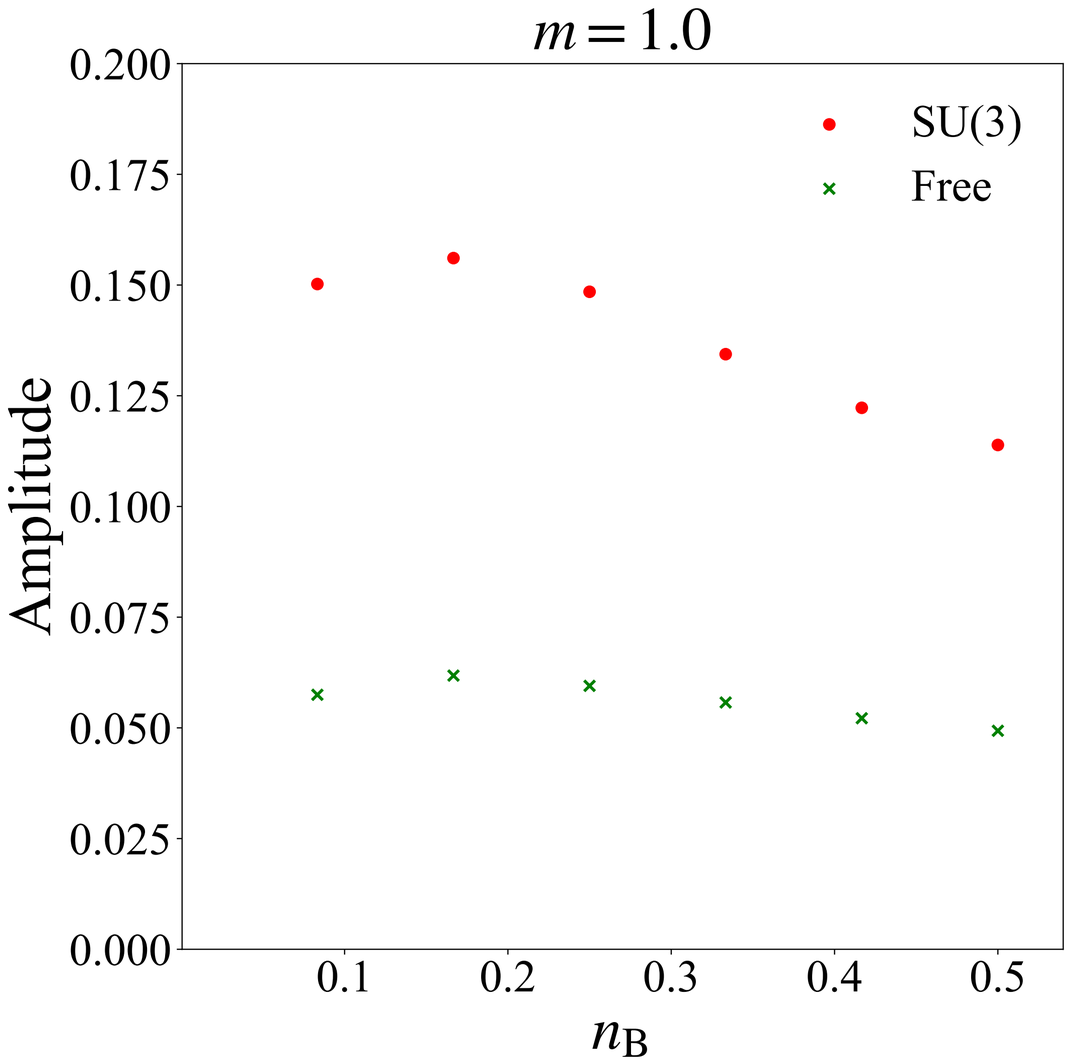
Spatial modulation

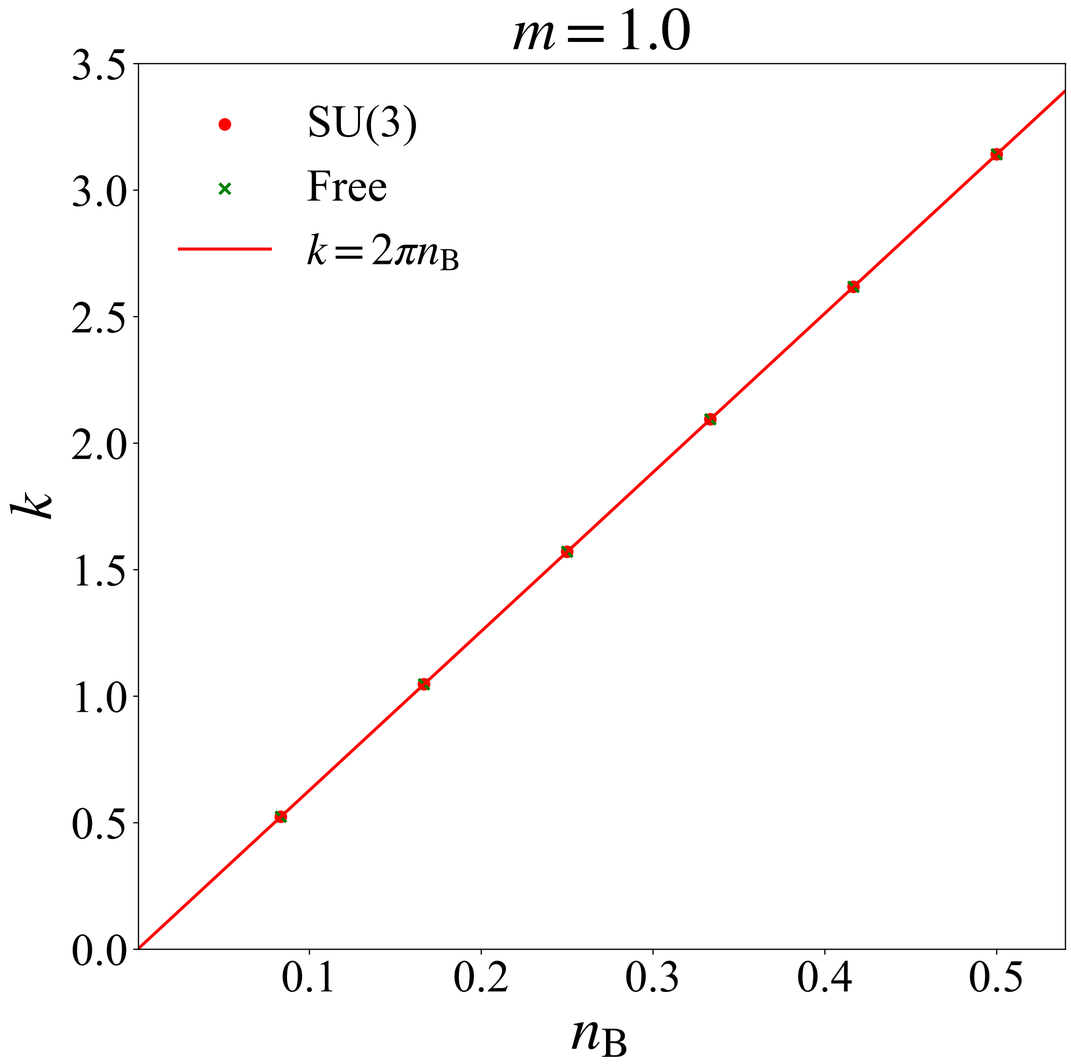




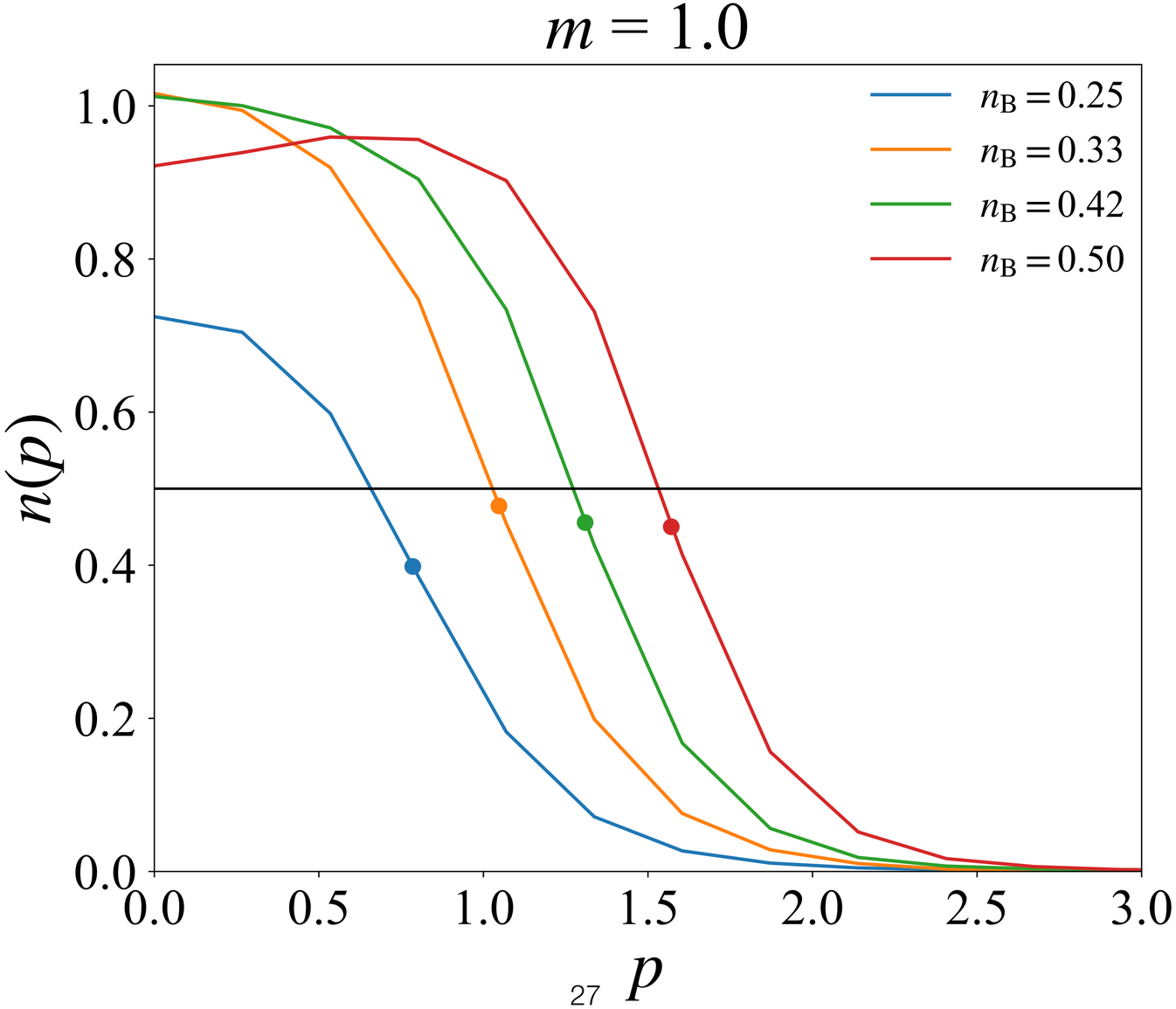
Hayata, KN and Hidaka, (2024)

Wave number and amplitude





Quark distribution function



Hayata, KN and Hidaka, (2024)