

Nonlocal effective theory for heavy meson from the instanton vacuum

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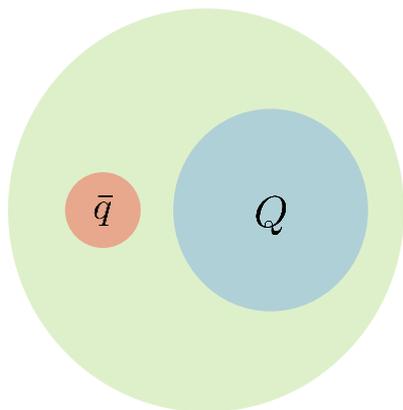
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Outline

- **Introduction**
- **QCD partition function from the instanton vacuum**
- **Half nonlocal model vs nonlocal model**
- **Application**
- **Summary**

Introduction



$$m_q \ll \Lambda_{QCD} \ll m_Q$$

The heavy quark is relatively static state in comparison with the light quark.



Heavy quark effective theory (HQET) is applicable.

The internal dynamics is determined by the light quark.



Scale $\mu \sim \Lambda_{QCD}$ gives $\alpha_s(\mu) \gg 1$

The nonperturbative contribution is dominant to describe the heavy-light mesons!

Nonperturbative structure instanton affects to construct the heavy-light composite particles.

QCD partition function from the instanton vacuum

- QCD partition function from the instanton vacuum[1] can be rewritten as

$$\mathcal{Z}_{\text{QCD}} \sim \int D[\text{fermion}] \exp \left(-S_q - S_Q + \int d^4x h^\dagger i\lambda \sum_{\pm} \Delta_{H,\pm}[\psi^\dagger, \psi] h \right)$$

$$S_{\text{int}} \sim \sum_{\pm} \int d[\xi_{\pm}, k, l, x, y] \prod_f^{N_{lf}} \sqrt{M_q(k_f) M_q(l_f)} \psi_f^\dagger(k_f) \psi_f(l_f) h^\dagger(x) \langle x | \theta^{-1} (w_{\pm} - \theta) \theta^{-1} | y \rangle h(y)$$

$M_q(k) = MF^2(k)$

[2] Diakonov et al, Phys. Lett. B 226, 372 (1989)

- After Fierz transformation and color-orientation integral for $N_f = 1$ case, the interaction action is

$$S_{qQ} = g^2 \int \frac{d^4k_1 d^4k_2 d^4p_1 d^4p_2}{(2\pi)^{12}} \delta^{(4)}(k_1 - k_2 + p_1 - p_2) F(k_1) F(k_2) T_Q(\vec{p}_1 - \vec{p}_2) \sum_i (\psi^\dagger(k_1) \Gamma_i h(p_1)) (h^\dagger(p_2) \Gamma_i \psi(k_2)),$$

where $\Gamma_i = (\mathbf{1}, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5, \sigma_{\mu\nu}/\sqrt{2})$ indicate the Dirac matrices.

Half nonlocal vs nonlocal

- The effective action can be induced from QCD action for 1 light quark and 1 heavy quark ($N_f = 1 + 1$)

$$S_{qQ} = g^2 \int \frac{d^4 k_1 d^4 k_2 d^4 p_1 d^4 p_2}{(2\pi)^{12}} \delta^{(4)}(k_1 - k_2 + p_1 - p_2) F(k_1) F(k_2) T_Q(\vec{p}_1 - \vec{p}_2) \sum_i (\psi^\dagger(k_1) \Gamma_i h(p_1)) (h^\dagger(p_2) \Gamma_i \psi(k_2)),$$

- Above action has a non-local property of light quark sector from the instanton form factor $F(k)$
- In the case of the heavy quark sector, T_Q plays the role of nonlocal structure

Heavy quark static state

$$T_Q(\vec{q}) = \int d^3 r e^{-i\vec{q}\cdot\vec{r}} T_Q(\vec{r})$$

$$T_Q(\vec{r}) = \frac{4N}{VN_c \Delta M_Q} \cos^2 \frac{\pi r}{2\sqrt{r^2 + \rho^2}}$$

$\vec{q} = \vec{p}_1 - \vec{p}_2 \sim 0$

$T_Q(0) = 1.4392$: Half nonlocal

: Nonlocal

Keep the momentum dependence

$$\rho = 1/3 \text{ fm}$$

$$N/V = (200 \text{ MeV})^4$$

$$\Delta M_Q = 67.5 \text{ MeV}$$

Why do we need non-local?

- In the half nonlocal model, we obtained the overestimated leading order and next-leading order of physical quantities

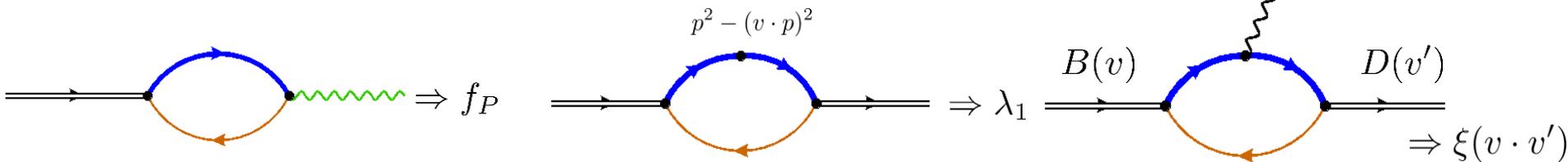
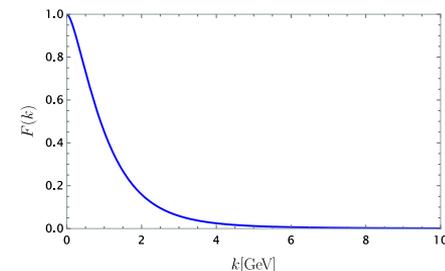
Ex) Decay constants, Isgur-Wise function, and $1/m_Q$ correction of heavy-meson mass



Because of the insufficient wavefunction for describing the heavy-meson composite operator: $\Phi_I(v, p) \propto F(p)q^\dagger(p)\Gamma_I h_v(p_\phi - p)$

- Each Feynman vertex function has the wavefunction

$$\mathcal{V}(p_q) = \frac{G}{\sqrt{N_c}} F(p_q) \Gamma_I$$



Nonlocal effective action for heavy meson

After integration over heavy and light quark fields

$$S_{\text{eff}}^{(2)} = \int \frac{d^4 P}{(2\pi)^4} \frac{d^3 k d^3 k'}{(2\pi)^6} \Phi_i^\dagger(P, \vec{k}) \left[T_Q^{-1}(\vec{k} - \vec{k}') - g^2 (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \Pi_i(P, \vec{k}) \right] \Phi_i(P, \vec{k}') S_H^{-1}$$

$$\Pi_i(P, \vec{p}_q) = N_c \int \frac{dp_{q4}}{2\pi} F^2(p_q) \text{tr}_D [S_q(p_q) \Gamma_i S_h(P - p_q) \Gamma_i]$$

BSE from EoM $T_Q \rightarrow K$

Heavy meson field: $\Phi_i(P, \vec{k}) \sim \phi(\vec{k}) \int \frac{dk_4}{2\pi} h^\dagger(P - k) \Gamma_i F(k) q(k)$

$$\phi(\vec{k}) = \beta^2 \int \frac{d^3 k'}{(2\pi)^3} K(\vec{k} - \vec{k}') \Pi(i\Lambda, \vec{k}') \phi(\vec{k}')$$

$$K(\vec{k} - \vec{k}') \equiv 4\pi \int_0^\infty dr r^2 j_0(|\vec{k} - \vec{k}'|r) \cos^2 \frac{\pi r}{2\sqrt{r^2 + \rho^2}}$$

K has a rotational symmetry \rightarrow Legendre polynomial

$$K(|\vec{k} - \vec{k}'|) = \sum_{l=0}^{\infty} \frac{2l+1}{2} K_l(k, k') P_l(\cos \theta), \quad K_l(k, k') = \int_{-1}^1 d\mu P_l(\mu) K\left(\sqrt{k^2 + k'^2 - 2kk'\mu}\right) \quad \phi(\vec{k}, \mu) = \sum_{l=0}^{\infty} \phi_l(\vec{k}) P_l(\mu)$$

BSE Solution

$$\phi_l(\vec{k}) = \beta^2 \int_0^\infty \frac{k'^2 dk'}{(2\pi)^2} K_l(k, k') U(k', \Lambda) \phi_l(\vec{k}')$$

$$U(p_q, \Lambda) \equiv \frac{F^2(p_q)[2iM(p_q) + 2v \cdot p_q]}{p_q^2 + M^2(p_q)} \frac{v \cdot p_q + i\Lambda}{(v \cdot p_q)^2 + \Lambda^2}$$

To solve the BSE, the wavefunction must satisfy the following two conditions:

1. Pole condition: $S_H^{-1} = 1 - G_l^2 \langle \phi_l | \Pi_i(i\Lambda_l) | \phi_l \rangle = 0$
 $\equiv \tilde{\Sigma}_P(i\Lambda_l)$

$$\langle \phi | K^{-1} | \phi \rangle = \alpha^2, \quad G^2 = g^2 / \alpha^2$$

2. Compositeness condition: (Taylor expansion around $P_4 = i\Lambda v_4 = i\Lambda$)

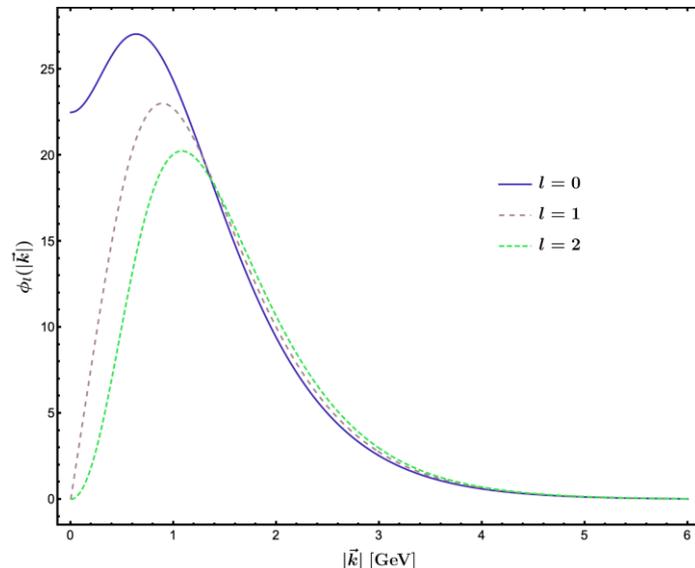
$$S_H^{-1} = 1 - G_l^2 \tilde{\Sigma}_P(v \cdot P) \approx -G_l^2 \tilde{\Sigma}'_P(i\Lambda_l)(v \cdot P - i\Lambda_l) = -2i(v \cdot P - i\Lambda_l)$$

Here Λ is a residual mass of the heavy meson in HQET

$$p_H = m_Q v + \Lambda v \quad \text{Almost on-shell in } \Lambda \ll m_Q$$

At s-wave $l = 0$ (D, B mesons), we get

$$\Lambda_0 = 696 \text{ MeV}, \quad G_0 = 1.92 \text{ GeV}^{-1/2}$$



Application

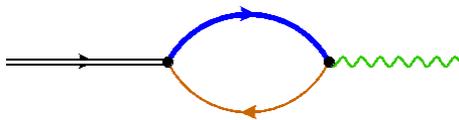
Feynman rules

Meson-quark-quark Feynman vertex : $\mathcal{V}_l(p_q, \mu = \cos \theta) = \frac{G_l}{\sqrt{N_c}} F(p_q) \phi_l(\vec{p}_q) P_l(\mu) \Gamma_{ll}$

Propagators : $S_q(p_q) = \frac{\not{p}_q + iM(p_q)}{p_q^2 + M^2(p_q)}$, and $S_h(p_q) = \frac{1 + \not{v}}{2(i\Lambda - v \cdot p_q)}$

Pseudoscalar D meson,

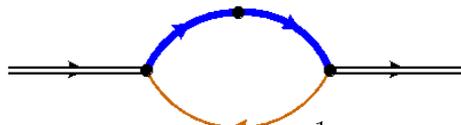
$$\langle 0 | q^\dagger \Gamma_\mu^A h | \Phi_P \rangle = -f_P \sqrt{m_P} v_\mu$$



Leading order of the decay constant f_D

$$\lambda_1 = \frac{1}{2} \langle \Phi_P | h^\dagger \partial_\perp^2 h | \Phi_P \rangle$$

$$p^2 - (v \cdot p)^2$$



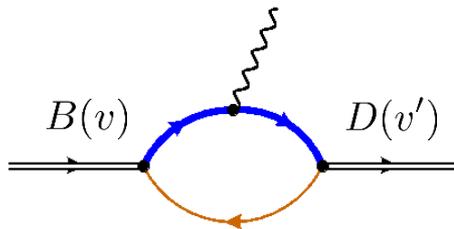
$$m_D = m_Q + \Lambda_0 + \frac{1}{2m_Q} \lambda_1 + \dots$$

Kinetic term of $1/m_Q$ corrections: $\lambda_1/2m_Q$

$$\mathcal{L}_{\text{kin}} = \frac{1}{2m_Q} h^\dagger D_\perp^2 h$$

$$D_{\perp\mu} \equiv D_\mu - v_\mu (v \cdot D)$$

$$-i \langle D(v') | c_{v'}^\dagger \Gamma_\mu b_v | B(v) \rangle = -\xi(w) \text{Tr} \left[\gamma_5 \frac{1 + \not{v}'}{2} \gamma_\mu \frac{1 + \not{v}}{2} \gamma_5 \right]$$

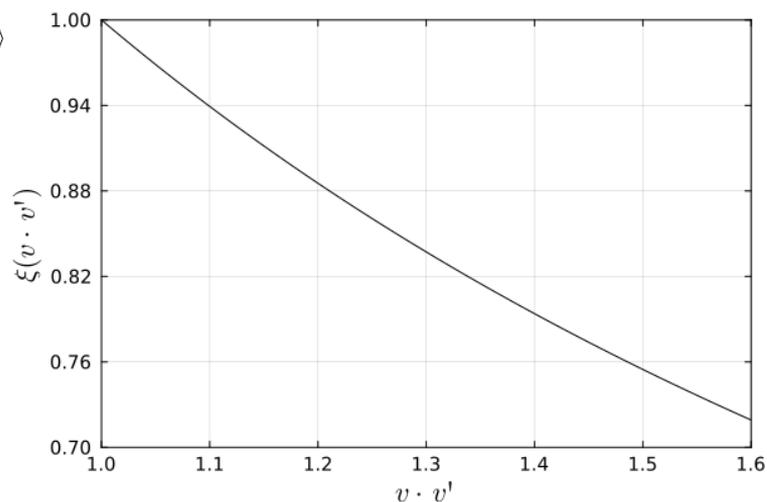


Isgur-Wise function from $B \rightarrow D$
(Universal form factor at $m_Q \rightarrow \infty$)

Results (Preliminary)

	Half nonlocal model	Nonlocal model	
f_D (leading order)	527 MeV	270 MeV	204 MeV (PDG)
λ_1	-1.6 GeV ²	-1.0 GeV ²	-0.3~0.5 GeV ²

$$\rho^2 \equiv -\xi'(1) = \frac{\Lambda^2}{3} \langle r^2 \rangle$$



HQET expectation:

$$\xi(v \cdot v' \sim 1.58) \approx 0.5 - 0.6$$

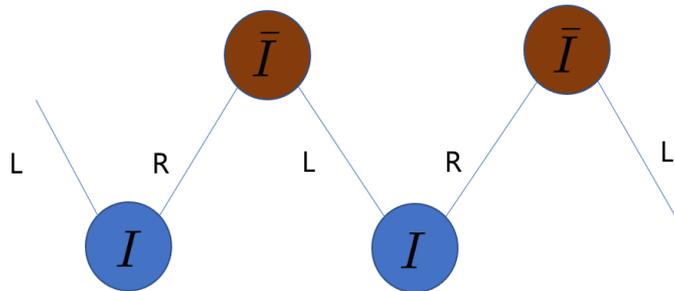
Summary

- We saw how to distinguish between half nonlocal and nonlocal from the form factor T_Q .
- The momentum dependent T_Q gives the additional wavefunction.
- The wavefunction and residual mass of the pseudoscalar heavy-meson are induced from BSE with the two conditions. $\Lambda_0 = 696 \text{ MeV}$
- In the instanton vacuum approach, it can be interpreted as the sum of $\Lambda_{QCD} = 284 \text{ MeV}$, $M = 345 \text{ MeV}$ (dynamical quark mass), and $M_Q = 67.5 \text{ MeV}$ (instanton contribution to the heavy quark mass).
- The present results f_D, λ_1 of this nonlocal model are compared with the previous work (half nonlocal model).
- In the next talk, Dr. Rakhimov will present the general N_f theory (half nonlocal) and the gluon operator from the instanton vacuum.

Thank you for your attention!

Instanton vacuum

- It properly shows the QCD vacuum properties: vacuum-to-vacuum transition, topological structure.
- Naturally, the instanton vacuum leads to the spontaneously chiral symmetry breaking, the light quark moves while being continuously changed helicity by the instanton and anti-instanton.
- This infinite helicity hopping process generates the dynamical quark mass.



Heavy-light quark interaction

- The light quark sector obviously gives the instanton vertex function
- The heavy quark propagator must be generated by two-point correlation function

$$\bar{w} = \frac{1}{\mathcal{Z}[0]} \prod_{\pm} \int d\lambda_{\pm} \int D\psi^{\dagger} D\psi \exp \int d^4x \left\{ \psi_f^{\dagger} i \hat{\partial} \psi_f + \lambda_{\pm} \left\langle \prod_f^{N_f} V_{\pm}[\psi^{\dagger}, \psi] \right\rangle \right\} \\ \times \int Dh^{\dagger} Dh h h^{\dagger} \exp \int d^4x \left\{ h^{\dagger} \left(\theta^{-1} - \sum_{\pm} \lambda_{\pm} \Delta_{H,\pm} \right) h \right\}.$$

$$\Delta_{H,\pm}[\psi^{\dagger}, \psi] = \left\langle \prod_f^{N_f} V_{\pm}[\psi_f^{\dagger}, \psi_f] \theta^{-1} (w_{\pm} - \theta) \theta^{-1} \right\rangle$$

This provides the interaction between heavy and light quarks (key function)

- λ_{\pm} is determined by the saddle point approximation. (Appendix C.2.1) (Simple example in C.2.2)
- From the Wilson line approach, one can reproduce the corresponding instanton effects on heavy quark

$$\Delta M_Q = \frac{N}{2VN_c} \sum_{\pm} \int d^3z_{\pm} \text{tr}_c \left(1 - P \exp \left(i \int d\tau A_{4\pm}(\tau) \right) \Big|_{z_{4,\pm}=0} \right) = 67.5 \text{ MeV}$$

Heavy-light quark interaction ($N_f = 1 + 1$)

- QCD partition function can be rewritten as

$$\mathcal{Z}_{\text{QCD}} \sim \int D[\text{fermion}] \exp \left(-S_q - S_Q + \int d^4x h^\dagger i\lambda \sum_{\pm} \Delta_{H,\pm}[\psi^\dagger, \psi] h \right)$$

$$S_{\text{int}} \sim \sum_{\pm} \int d[\xi_{\pm}, k, l, x, y] \prod_f^{N_{lf}} \sqrt{M_q(k_f) M_q(l_f)} \psi_f^\dagger(k_f) \psi_f(l_f) h^\dagger(x) \langle x | \theta^{-1} (w_{\pm} - \theta) \theta^{-1} | y \rangle h(y)$$

- After Fierz transformation and color-orientation integral for $N_f = 1$ case, the interaction action is

$$S_{\text{int}} = - \int \frac{d^4k_1 d^4k_2 d^4p_1 d^4p_2}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(k_1 - k_2 + p_1 - p_2) \sqrt{M_q(k_1) M_q(k_2)} \frac{\Delta M_Q}{N/V} \\ \times \left[(\psi^\dagger(k_1) \psi(k_2)) (Q_+^\dagger(p_1) Q_+(p_2)) \right. \\ \left. + \frac{1}{8} \sum_i (\psi^\dagger(k_1) \Gamma_i Q_+(p_1)) (Q_+^\dagger(p_2) \Gamma_i \psi(k_2)) \right],$$

where $\Gamma_i = (\mathbf{1}, \gamma_5, \gamma_\mu, i\gamma_\mu \gamma_5, \sigma_{\mu\nu} / \sqrt{2})$ indicate the Dirac matrices.

Heavy-light quark interaction ($N_f = 1 + 1$)

- For $N_f = 1 + 1$ (four fermion interaction), the heavy-light interaction term can be rewritten as

$$\begin{aligned}
 S_{qQ} &= -g^2 \int \frac{d^4 k_1 d^4 k_2 d^4 p_1 d^4 p_2}{(2\pi)^{16}} (2\pi)^4 \delta^4(k_1 - k_2 + p_1 - p_2) F(p_1) F(p_2) [q^\dagger(p_1) \Gamma_H Q_+(k_1)] [Q_+^\dagger(k_2) \Gamma_H q(p_2)] \\
 &= -g^2 \int d^4 x [q_I^\dagger \Gamma_H Q_+] [Q_+^\dagger \Gamma_H q_I]
 \end{aligned}$$

$$\begin{aligned}
 q_I(x) &\equiv F(\partial)\psi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} F(p)\psi(p) & F(p) &= \sqrt{\frac{M_q(p)}{M_q(0)}} \\
 g^2 &= \frac{N_c}{8(N_c^2 - 1)} \left(1 - \frac{2}{N_c}\right) \frac{\Delta M_Q M(0)}{2n} & n &= \frac{N}{2VN_c} \sim \frac{(200 \text{ MeV})^4}{2N_c}
 \end{aligned}$$

- Bosonization process leads to the quark-meson interaction

$$\mathcal{L}_{\text{eff}} = -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h + m_\Phi^2 \Phi_i^\dagger \Phi_i - G Q_+^\dagger \Gamma_i F(\partial)\psi \Phi_i - G \Phi_i^\dagger F(\partial)\psi^\dagger \Gamma_i Q_+$$

$G = gm_\Phi > 0$, The positive G ensures the production of the composite particle composed of the heavy and light quarks.

Equivalent Lagrangian

$$\mathcal{L}_{\text{eff}} = -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h + m_\Phi^2 \Phi_i^\dagger \Phi_i - GQ_+^\dagger \Gamma_i F(\partial)\psi\Phi_i - G\Phi_i^\dagger F(\partial)\psi^\dagger \Gamma_i Q_+$$

- The auxiliary field Φ cannot be directly interpreted as a physical heavy meson field (bosonization does not impose physical requirements.)
- To establish a proper meson field with the physical mass, renormalization is necessary.

Ken-ichi Shizuya, Phys. Rev. D, 21:2327, 1980.

- If the compositeness condition $Z_\Phi = 0$ is satisfied, the kinetic term is allowed.

Renormalization transform:

$$\begin{aligned} h &= \sqrt{Z_Q} h_R, \quad \Phi = \sqrt{Z_\Phi} \Phi_R, \\ G &= Z_G Z_Q^{-1/2} Z_\Phi^{-1/2} G_R, \\ Z_\Phi m_\Phi^2 &= m_{\Phi_R}^2 + \delta m_\Phi^2, \end{aligned}$$

With kinetic:

$$\begin{aligned} \mathcal{L}'_{\text{eff}} &= -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h + \partial_\mu \Phi_i^\dagger \partial_\mu \Phi_i \\ &+ m_\Phi^2 \Phi_i^\dagger \Phi_i - G\Phi_i^\dagger F(\partial)\psi^\dagger \Gamma_i Q_+ - GQ_+^\dagger \Gamma_i F(\partial)\psi\Phi_i \end{aligned}$$



$$\begin{aligned} \mathcal{L}'_{R,\text{eff}} &= -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - Z_Q h_R^\dagger i(v \cdot \partial)h_R \\ &+ Z_\Phi \partial_\mu \Phi_{Ri}^\dagger \partial_\mu \Phi_{Ri} + (m_{\Phi_R}^2 + \delta m_\Phi^2) \Phi_{Ri}^\dagger \Phi_{Ri} \\ &- Z_G G_R \Phi_{Ri}^\dagger F(\partial)\psi^\dagger \Gamma_i Q_{R+} \Phi_{Ri} - Z_G G_R Q_{R+}^\dagger \Gamma_i F(\partial)\psi \end{aligned}$$

Our Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h \\ &+ m_\Phi^2 \Phi_i^\dagger \Phi_i - GQ_+^\dagger \Gamma_i F(\partial)\psi\Phi_i - G\Phi_i^\dagger F(\partial)\psi^\dagger \Gamma_i Q_+ \end{aligned}$$



$$\begin{aligned} \mathcal{L}_{R,\text{eff}} &= -\psi^\dagger(i\cancel{\partial} + iM_q(\partial))\psi - Z_Q h_R^\dagger i(v \cdot \partial)h_R \\ &+ (m_{\Phi_R}^2 + \delta m_\Phi^2) \Phi_{Ri}^\dagger \Phi_{Ri} \\ &- Z_G G_R \Phi_{Ri}^\dagger F(\partial)\psi^\dagger \Gamma_i Q_{R+} - Z_G G_R Q_{R+}^\dagger \Gamma_i F(\partial)\psi \Phi_{Ri} \end{aligned} \quad 17$$

Equivalent Lagrangian

- To check the compositeness condition, let us start from

$$\begin{aligned} \mathcal{L}'_{\text{eff}} = & -\psi^\dagger (i\not{\partial} + iM_q(\partial)) \psi - h^\dagger i(v \cdot \partial)h \\ & + \partial_\mu \Phi_i^\dagger \partial_\mu \Phi_i + m_\Phi^2 \Phi_i^\dagger \Phi_i - G \Phi_i^\dagger F(\partial) \psi^\dagger \Gamma_i Q_+ - G Q_+^\dagger \Gamma_i F(\partial) \psi \Phi_i \end{aligned}$$

- In $m_Q \rightarrow \infty$, the kinetic and mass terms of Φ can be represented to zeroth-order of $1/m_Q$ given as

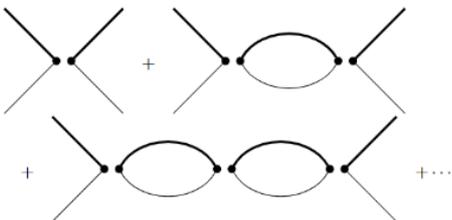
$$\begin{aligned} \mathcal{L}'_{\text{eff}} = & -\psi^\dagger (i\not{\partial} + iM_q(\partial)) \psi - h^\dagger i(v \cdot \partial)h \\ & + \phi_{vi}^\dagger (-2v \cdot \partial - 2\Lambda) \phi_{vi} - G_0 \phi_{vi}^\dagger F(\partial) \psi^\dagger \Gamma_i h - G_0 h^\dagger \Gamma_i F(\partial) \psi \phi_{vi} \end{aligned} \quad \Phi^\pm \equiv \frac{1}{\sqrt{m_Q}} e^{\mp m_Q v \cdot x} \phi_v^\pm \text{ with } \phi^+ = \phi, \phi^- = \phi^\dagger$$

- The effective coupling $gF(p_q)$ for the heavy-light quark vertex in momentum space suppresses the divergence ($gF(p_q) \rightarrow$ expansion parameter).

$$P_\Phi = m_\Phi v = m_Q v + \underline{p_\Phi} = m_Q v + \underline{\Lambda v} = p_q + p_Q = m_Q v + \underline{k} + \underline{p_q}$$

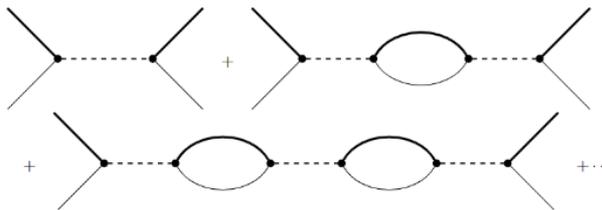
Residual momentum is predicted around $\Lambda_{QCD} \ll m_Q$
In the large m_Q limit, on-mass shell condition becomes $P_\Phi^2 = m_\Phi^2 \sim m_Q^2$

Quark-quark(four fermion) coupling picture



$$\mathcal{A}_{Qq}^a = \frac{-F(p_q)F(p'_q)}{(v \cdot p_\Phi - i\Lambda)\Pi'(i\Lambda) + \Pi r(i\Lambda)}$$

$\mathcal{L}'_{\text{eff}}$ meson-quark coupling picture



$$\mathcal{A}_{Qq}^b = \frac{iG^2 F(p_q)F(p'_q)}{2(v \cdot p_\Phi - i\Lambda) - iG^2 \Pi r(i\Lambda)}$$

$$G^2 = G_0^2 \left(1 + \frac{iG^2}{2} \Pi'(i\Lambda) \right) \equiv Z_\Phi G_0^2 \quad 18$$

Equivalent Lagrangian

$$\mathcal{A}_{Qq}^a = \mathcal{A}_{Qq}^b \longrightarrow G^2 = \frac{2i}{\Pi'(i\Lambda)} \longrightarrow Z_\Phi = 1 + \frac{iG^2}{2}\Pi'(i\Lambda) = 0$$

- The $Z_\Phi = 0$ has the following meanings: The fundamental bare field does not exist, meaning the physical particle field is always a composite operator form.

- Hence, the meson-quark Lagrangian is equivalent with

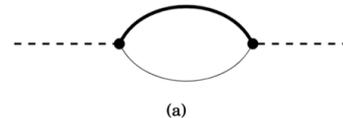
$$\begin{aligned} \mathcal{L} = & -\psi^\dagger (i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h \\ & + \phi_{vi}^\dagger (-2v \cdot \partial - 2\Lambda)\phi_{vi} - G\phi_{vi}^\dagger F(\partial)\psi^\dagger \Gamma_i h - Gh^\dagger \Gamma_i F(\partial)\psi\phi_{vi} \end{aligned}$$

- The compositeness condition gives $\Lambda \approx 10$ GeV, which is unexpected scale.
- The composite operator composed of light and heavy quark fields must be normalized.

\longrightarrow standard nonrelativistic normalization condition: $\langle \phi(v') | \phi(v) \rangle = (2\pi)^3 2p_4 / m_\Phi \delta^{(3)}(\Lambda \vec{v}' - \Lambda \vec{v}) = (2\pi)^3 2v_4 \delta^{(3)}(\Lambda \vec{v}' - \Lambda \vec{v})$

- The composite operator can be induced by the equation of motion

$$\Phi_{vi}(p_\Phi) = \langle p_{\Phi 4} | \phi_i(v) \rangle = \sqrt{2v_4} i g \mathcal{N} \int \frac{d^4 p_q d^4 k}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(p_\Phi - k - p_q) F(p_q) \psi^\dagger(p_q) \Gamma_i h(k)$$



- The one-particle state $|\phi_i(v)\rangle$ of the heavy meson is defined as $|\phi(v)\rangle \equiv \phi_v|0\rangle$.

$$\begin{aligned} \langle \phi_i(v') | \phi_j(v) \rangle &= \frac{d p_{\Phi 4}}{2\pi} \text{tr}_D \Phi_{vj}(p_\Phi) \Phi_{v'i}^\dagger(p'_\Phi) \\ &= 2v_4 (2\pi)^3 \delta^3(\Lambda v' - \Lambda v) \delta_{ij} \end{aligned} \longrightarrow g^2 \mathcal{N}^2 \Pi(i\Lambda) = 1 \quad \Pi(v \cdot p_\Phi) = \int \frac{d^4 p_q}{(2\pi)^4} F^2(p_q) \text{tr}_D \left[\Gamma_P \frac{(1 + \not{v})}{2v \cdot (p - p_q) + i\epsilon} \Gamma_P \frac{-(\not{p}_q - iM(p_q))}{p_q^2 + M^2(p_q)} \right]$$

Residual mass and heavy-meson masses

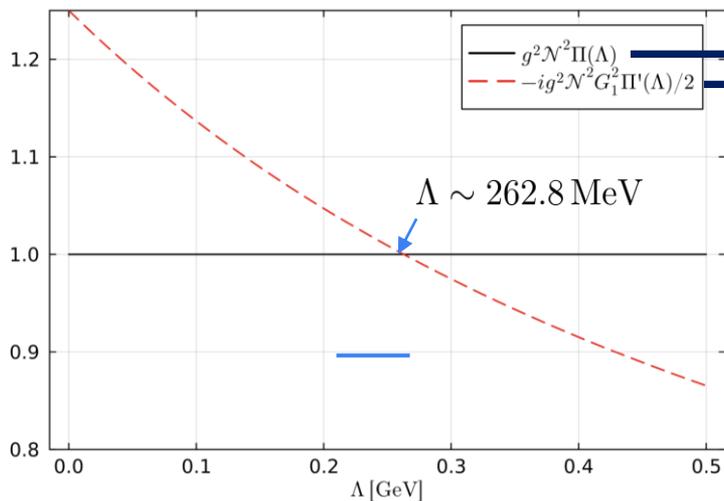


그림 V.3.2: The black solid line depicts the normalization condition, while the red dashed curve represents the renormalization condition with the function $\Pi(\Lambda)$ rescaled by the normalization constant $\mathcal{N}(\Lambda)$. The point where these two lines intersect determines the physical residual mass, $\Lambda = 0.2628$, GeV.

Arisen by instanton effects

$$m_\Phi = m_Q + M_q + \Delta M_Q + \Lambda$$

$$M_q \sim 340 \text{ MeV}, \Delta M_Q \sim 68 \text{ MeV}$$

$$m_D = 1.942 \text{ GeV} \quad (m_c = 1.27 \text{ GeV}), \quad \frac{1.867 + 2.010}{2} = 1.940 \text{ GeV}$$

$$m_B = 5.322 \text{ GeV} \quad (m_b = 4.65 \text{ GeV}), \quad \frac{5.279 + 5.325}{2} = 5.302 \text{ GeV}$$

PDG values:
(pseudoscalar+vector)/2

Heavy meson masses (Heavy-quark spin symmetry)

- To break the heavy quark spin symmetry, we need to consider $1/m_Q$ corrections.

$$\mathcal{L}_Q^{1/m_Q} = \frac{1}{2m_Q} \bar{h}(i\vec{D})^2 h - \frac{1}{m_Q} \bar{h} \vec{S} \cdot \vec{B} h$$

- The second term gives spin-spin interaction term

$$\begin{aligned} \frac{1}{4m_Q} \langle \phi_{\vec{j}} | h^\dagger \sigma_{\mu\nu} h G_{\mu\nu} | \phi_{\vec{j}} \rangle &= \frac{1}{4m_Q} \langle h; \vec{S}_Q | h^\dagger \sigma_{\mu\nu} h | h; \vec{S}_Q \rangle \langle q; \vec{S}_q | G_{\mu\nu} | q; \vec{S}_q \rangle \\ &= \frac{\lambda_2}{m_Q} \langle 2\vec{S}_Q \cdot \vec{S}_q \rangle, \end{aligned}$$

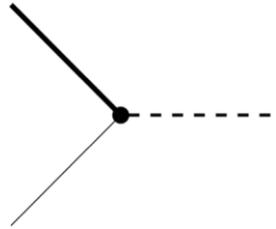
Ref.[1,2] have calculated the quark-gluon operator from instanton vacuum.

$$\begin{aligned} s_{Qq} \equiv \langle 2\vec{S}_Q \cdot \vec{S}_q \rangle &= J(J+1) - S_Q(S_Q+1) - S_q(S_q+1) \\ &= \begin{cases} -3/2 & \text{pseudoscalar} \\ 1/2 & \text{vector} \end{cases} \end{aligned}$$

[1] Maxim V. Polyakov et al, Phys. Lett. B, 387:841–847, 1996,

[2] Dmitri Diakonov et al, Nucl. Phys. B, 461:539–580, 1996

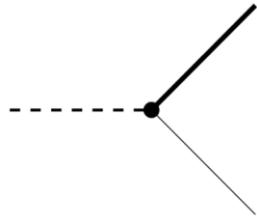
Weak decay constants f_D and f_B



A Feynman diagram showing a vertex where a solid line (representing a quark) enters from the bottom-left, a solid line (representing an antiquark) enters from the top-left, and a dashed line (representing a meson) exits to the right. The vertex is marked with a black dot.

$$= -ig\mathcal{N}F(p_q)\Gamma_i$$

The effective coupling constant $gF(p_q)$ in momentum space can be used as the expansion parameter.



A Feynman diagram showing a vertex where a dashed line (representing a meson) enters from the left, a solid line (representing a quark) enters from the bottom-right, and a solid line (representing an antiquark) enters from the top-right. The vertex is marked with a black dot.

$$= -ig\mathcal{N}F(p_q)\Gamma_i$$

It presumes that the nonperturbative effects from the instanton vacuum modify the Feynman diagram.

$$\Gamma_V = -\gamma_\mu \epsilon_\mu \quad \text{For vector meson}$$

$$\Gamma_P = i\gamma_5 \quad \text{For pseudoscalar meson}$$

Weak decay constants f_D and f_B

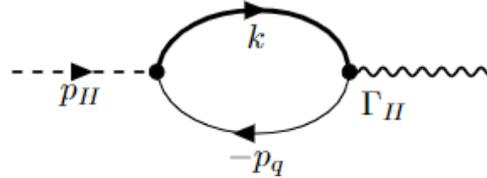
- The axial-vector and vector currents are given as

$$J_\mu^A = h^\dagger \gamma_\mu \gamma_5 q$$

$$J_\mu^V = h^\dagger \gamma_\mu q$$

- In the heavy quark limit, the heavy meson decay constants are defined as

$$\langle 0 | J_\mu^A | P \rangle = \langle 0 | h^\dagger \gamma_\mu \gamma_5 q | P \rangle = f_P \sqrt{m_P} v_\mu, \quad \langle 0 | J_\mu^V | V \rangle = \langle 0 | h^\dagger \gamma_\mu q | V \rangle = i f_V \sqrt{m_V} \epsilon_\mu$$



- $1/m_Q$ corrections must be considered the following transforms:

$$Q(x) = e^{-im_Q v \cdot x} \left(1 - \frac{1}{2m_Q} \not{D}_\perp + \mathcal{O}(m_Q^{-2}) \right) h(x)$$

$$|\phi_v\rangle \rightarrow \left(1 - \frac{1}{2m_Q} \int d^4x T \{ \mathcal{L}_Q^{1/m_Q}(x) \} + \dots \right) |\phi_v\rangle$$

- They give the matrix elements of axial-vector and vector currents:

$$\langle 0 | J_\mu^{V(A)} | V(P)(v) \rangle = \langle 0 | h^\dagger \gamma_\mu (\gamma_5) q | V(P)(v) \rangle - \frac{1}{2m_Q} \langle 0 | h^\dagger \gamma_\mu (\gamma_5) \not{D}_\perp q | V(P)(v) \rangle - \frac{1}{2m_Q} \int d^4y \langle 0 | T \{ h^\dagger \gamma_\mu (\gamma_5) q(0), h^\dagger i D_\perp^2 h(y) \} | V(P)(v) \rangle + \dots$$

Weak decay constants f_D and f_B

- Unlike the real QCD, the currents are not conserved because of large heavy quark mass limit. So, QCD matching is needed:

$$J_\mu^{V(A)} = \sum_i C_i(\mu_0) \left(J_\mu^{V(A)(i)} - \frac{1}{2m_Q} \left(O_\mu^{V(A)(i)} + T_\mu^{V(A)(i)} \right) \right) + \mathcal{O} \left(\frac{1}{m_Q^2} \right)$$

$$\text{Wilson coefficients: } C_1(\mu_0) = 1 + \frac{\alpha_s(\mu_0)}{\pi} \left(\ln \frac{\mu}{\mu_0} - \frac{4}{3} \right), \quad C_2(\mu_0) = -\frac{2\alpha_s(\mu_0)}{3\pi} \quad C_{12} \equiv C_1(\mu_0) - C_2(\mu_0)$$

$$\begin{aligned} \langle 0 | J_\mu^A | P(v) \rangle &= C_{12} F_P^{(0)} v_\mu - \frac{C_{12}}{2m_Q} (F_{P,1}^{(1/m_Q)} - F_{P,2}^{(1/m_Q)}) v_\mu \\ &= C_{12} F_P v_\mu, \end{aligned}$$

$$\begin{aligned} \langle 0 | J_\mu^V | V(v) \rangle &= i C_{12} F_V^{(0)} \epsilon_\mu - i \frac{C_{12}}{2m_Q} (F_{V,1}^{(1/m_Q)} + F_{V,2}^{(1/m_Q)}) \epsilon_\mu \\ &= i C_{12} F_V \epsilon_\mu \end{aligned}$$

Appendix E.5

$$\begin{aligned} J_\mu^{V(A)(1)} &= h^\dagger \gamma_\mu (\gamma_5) q, & J_\mu^{V(A)(2)} &= h^\dagger v_\mu (\gamma_5) q \\ O_\mu^{V(A)(1)} &= h^\dagger \gamma_\mu (\gamma_5) \not{D}_\perp q, & O_\mu^{V(A)(2)} &= h^\dagger v^\mu (\gamma_5) \not{D}_\perp q, \\ T_\mu^{V(A)(1)} &= \int d^4 y \left\{ J_\mu^{V(A)(1)}(0), h^\dagger(y) i D_\perp^2 h(y) \right\}, \\ T_\mu^{V(A)(2)} &= \int d^4 y \left\{ J_\mu^{V(A)(2)}(0), h^\dagger(y) i D_\perp^2 h(y) \right\} \end{aligned}$$

⚡ V.1: The first column shows f_D and f_B in the present approach, which are in comparison with recent results from lattice QCD and QCD sum rules. The last column indicates PDG average values. The normalization point is used in the range of $\mu_0 = (0.6^{(+)} - 1.4^{(-)})$ GeV and the matching scale $\mu = m_Q$.

	This work	Lattice QCD [96]	QCD sum rule [97]	PDG average [1]
f_D [MeV]	$204.6_{-4.5}^{+13.6}$	209.0(2.4)	190(15)	203.8(4.7)(0.6)(1.4)
f_B [MeV]	$191.2_{-8.4}^{+30.0}$	192.0(4.3)	192_{-15}^{+20}	190.0(1.3)

[1] S. Navas et al. Review of particle physics. Phys. Rev. D, 110(3):030001, 2024.

[96] Y. Aoki et al. FLAG Review 2021. Eur. Phys. J. C, 82(10):869, 2022

[97] Ben Pullin et al, JHEP, 09:023, 2021.

Summary & Outlook

□ The instanton contributions to the heavy and light quark systems are reviewed.

□ Heavy-meson is the complicated nonperturbative QCD structure

→ Instanton well explained this nonperturbative effects.

□ Instanton vacuum clearly traces the effective Lagrangian for Qq interaction ($N_f = 1 + 1$)

$$\mathcal{L} = -\psi^\dagger (i\cancel{\partial} + iM_q(\partial))\psi - h^\dagger i(v \cdot \partial)h \\ + \phi_{vi}^\dagger (-2v \cdot \partial - 2\Lambda)\phi_{vi} - G\phi_{vi}^\dagger F(\partial)\psi^\dagger \Gamma_i h - Gh^\dagger \Gamma_i F(\partial)\psi\phi_{vi}$$

• The model provides the weak decay constants, which agree nicely with other determinations (experiment, lattice...) without any fitting parameters.

	[1]	Lattice QCD [4]	QCD sum rule [5]	PDG average [6]
f_D [MeV]	$204.6^{+13.6}_{-4.5}$	209.0(2.4)	190(15)	203.8(4.7)(0.6)(1.4)
f_B [MeV]	$191.2^{+30.0}_{-8.4}$	192.0(4.3)	192^{+20}_{-15}	190.0(1.3)

Summary & Outlook

□ The higher flavor number ($N_f = 2 + 1$) gives much more complicated interaction actions

$$\begin{aligned}
 S_{\text{int}}^{(S,P)} &= \int d\mathcal{R} \frac{1}{4} M^2(0) \Delta M_Q R^8 \\
 &\times \left\{ [(\psi^\dagger \psi)^2 + (\psi^\dagger \gamma_5 \psi)^2 - (\psi^\dagger \tau^A \psi)^2 - (\psi^\dagger \tau^A \gamma_5 \psi)^2] (Q_+^\dagger Q_+) \right. \\
 &- \frac{1}{4} [(\psi^\dagger \psi) (\psi_f^\dagger Q_+) (Q_+^\dagger \psi^f) - (\psi^\dagger \tau^A \psi) (\psi_f^\dagger [\tau^A]_g^f Q_+) (Q_+^\dagger \psi^g) \\
 &+ (\psi^\dagger \psi) (\psi_f^\dagger \gamma_5 Q_+) (Q_+^\dagger \gamma_5 \psi^f) - (\psi^\dagger \tau^A \psi) (\psi_f^\dagger [\tau^A]_g^f \gamma_5 Q_+) (Q_+^\dagger \gamma_5 \psi^g) \\
 &+ (\psi^\dagger \gamma_5 \psi) (\psi_f^\dagger Q_+) (Q_+^\dagger \gamma_5 \psi^f) - (\psi^\dagger \tau^A \gamma_5 \psi) (\psi_f^\dagger [\tau^A]_g^f Q_+) (Q_+^\dagger \gamma_5 \psi^g) \\
 &\left. + (\psi^\dagger \gamma_5 \psi) (\psi_f^\dagger \gamma_5 Q_+) (Q_+^\dagger \psi^f) - (\psi^\dagger \tau^A \gamma_5 \psi) (\psi_f^\dagger [\tau^A]_g^f \gamma_5 Q_+) (Q_+^\dagger \psi^g) \right\}
 \end{aligned}$$

$$\begin{aligned}
 S_{\text{int}}^{(V,A)} &= - \int d\mathcal{R} \frac{1}{16} M^2(0) \Delta M_Q R^8 \\
 &\times \left\{ (\psi^\dagger \psi) (\psi_f^\dagger \gamma_\mu Q_+) (Q_+^\dagger \gamma_\mu \psi^f) - (\psi^\dagger \tau^A \psi) (\psi_f^\dagger \gamma_\mu [\tau^A]_g^f Q_+) (Q_+^\dagger \gamma_\mu \psi^g) \right. \\
 &- (\psi^\dagger \psi) (\psi_f^\dagger \gamma_\mu \gamma_5 Q_+) (Q_+^\dagger \gamma_\mu \gamma_5 \psi^f) + (\psi^\dagger \tau^A \psi) (\psi_f^\dagger \gamma_\mu \gamma_5 [\tau^A]_g^f Q_+) (Q_+^\dagger \gamma_\mu \gamma_5 \psi^g) \\
 &+ (\psi^\dagger \gamma_5 \psi) (\psi_f^\dagger \gamma_\mu Q_+) (Q_+^\dagger \gamma_\mu \gamma_5 \psi^f) - (\psi^\dagger \gamma_5 \tau^A \psi) (\psi_f^\dagger \gamma_\mu [\tau^A]_g^f Q_+) (Q_+^\dagger \gamma_\mu \gamma_5 \psi^g) \\
 &- (\psi^\dagger \gamma_5 \psi) (\psi_f^\dagger \gamma_\mu \gamma_5 Q_+) (Q_+^\dagger \gamma_\mu \psi^f) \\
 &\left. + (\psi^\dagger \gamma_5 \tau^A \psi) (\psi_f^\dagger \gamma_\mu \gamma_5 [\tau^A]_g^f Q_+) (Q_+^\dagger \gamma_\mu \psi^g) \right\}.
 \end{aligned}$$

□ It will offer the following hadronic and leptonic decay process

$$\begin{aligned}
 \psi &\rightarrow J/\psi + \pi\pi, \\
 D^{*+} &\rightarrow D^+ + \pi^0, & B^+ &\rightarrow \tau^+ \nu \\
 D^{*+} &\rightarrow D^0 + \pi^+, & B^+ &\rightarrow \mu^+ \nu \\
 B^{*+} &\rightarrow B^+ + \pi^0, & D^+ &\rightarrow \tau^+ \nu \\
 B^{*+} &\rightarrow B^0 + \pi^+, \text{ etc.} & D^+ &\rightarrow \mu^+ \nu, \text{ etc.}
 \end{aligned}$$

✘ D^* and B^* must be considered the heavy-quark symmetry breaking.

Thank you for your attention

Instanton vacuum: condensate and dynamical quark mass

- QCD partition function:
$$Z_{\text{QCD}} = \int DB_\mu \exp \left\{ -\frac{1}{2g^2(M)} \int d^4x \text{Tr} F_{\mu\nu}^2(\mathcal{A}) \right\} \\ \times \int D\psi^\dagger D\psi \exp \left\{ \int d^4x \psi^\dagger (i\mathcal{D}(\mathcal{A}) + im)\psi \right\}$$

Classical background field

Quantum fluctuation

$$\mathcal{A} = \bar{A} + B,$$

$$\bar{A} = \sum_I A_I + \sum_{\bar{I}} A_{\bar{I}}$$

- After regularization and integration over ψ and B fields

• After regularization and integration over ψ and B fields

Gives quark zero modes

Gives nonzero eigenvalues

$$\text{Det}_{\text{low}} \equiv \frac{\det(i\mathcal{D} + im)\det(i\mathcal{D} + iM_1)}{\det(i\mathcal{D} + im)\det(i\mathcal{D} + iM_1)}, \quad \text{Det}_{\text{high}} \equiv \frac{\det(i\mathcal{D} + iM_1)\det(i\mathcal{D} + iM)}{\det(i\mathcal{D} + iM_1)\det(i\mathcal{D} + iM)}$$

M_1 is intermediate mass

- In the single instanton configuration, one can write the fermion propagator

zero-mode eigenvalues $\lambda_0 \sim 0$

$$i\mathcal{D}|\Phi_n\rangle = \lambda_n|\Phi_n\rangle$$

$$S_{I(\bar{I})}(x-y) = \langle 0|\psi(x)\psi^\dagger(y)|0\rangle \\ = -\left\langle x \left| \frac{1}{i\mathcal{D}(\bar{A}_I + im)} \right| y \right\rangle \\ = -\sum_n \langle x | \frac{|\Phi_n\rangle\langle\Phi_n|}{i\mathcal{D}(\bar{A}_I + im)} | y \rangle \\ = -\sum_n \frac{\Phi_n(x)\Phi_n^\dagger(y)}{\lambda_n + im},$$



$$S_{I(\bar{I})} \simeq S_0 - \frac{\Phi_{I(\bar{I})}(x)\Phi_{I(\bar{I})}^\dagger(y)}{im}$$

By solving Dirac equation in Appendix C

$S_0 = (i\mathcal{D})^{-1}$: free quark propagator

Instanton vacuum: condensate and dynamical quark mass

- Using the fermion propagator, one can rewrite the effective partition function for $N = N_+ + N_-$ instantons

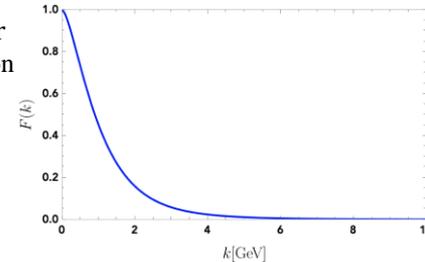
$$\mathcal{Z}_{\text{QCD}} = \int D\psi^\dagger D\psi \exp \left(\int d^4x \psi_f^\dagger i \not{\partial} \psi_f \right) \left(\frac{Y_{N_f}^{(+)}}{VM_1^{N_f}} \right)^{N_+} \left(\frac{Y_{N_f}^{(-)}}{VM_1^{N_f}} \right)^{N_-}$$

+: instanton

-: antiinstanton

$$Y_{N_f}^{(+)} = \int dU \prod_f \int \frac{d^4k_f}{(2\pi)^4} [2\pi\rho F(k_f)] \frac{d^4l_f}{(2\pi)^4} [2\pi\rho F(l_f)] U_{\gamma_f}^{\alpha_f} U_{\beta_f}^{\dagger\delta_f} \epsilon^{i_f\gamma_f} \epsilon_{j_f\delta_f} \\ \times (2\pi)^4 \delta^{(4)} \left(\sum_f k_f - \sum_f l_f \right) \left[i\psi_{L_f\alpha_f i_f}^\dagger(k_f) \psi_{L_f\beta_f j_f}^f(l_f) \right].$$

$F(k)$: instanton form factor
 Y^\pm : instanton vertex function



Including momentum conservation

- For the simplest case $N_f = 1$, the effective action is obtained as

$$S_{eff}^{N_f=1} = - \int \frac{d^4k}{(2\pi)^4} \psi^\dagger(k) [k - iM(k)] \psi(k),$$

+: instanton
 -: antiinstanton

Related to the gluon condensate



Instanton distance

$\bar{R} \simeq \frac{1}{200 \text{ MeV}} = 1 \text{ fm}$

Phenomenological estimation: $\approx (200 \text{ MeV})^{-4} > 0$

$F(k)$: instanton form factor
 Y^\pm : instanton vertex function

$M \approx 345 \text{ MeV}$ with $\rho = 1/3 \text{ fm}$

QCD vacuum

- Not empty space! Complex medium filled with quantum fluctuations of both quark and gluon fields. (Non-trivial vacuum)
- Non-zero condensates (quark condensate, gluon condensate)

$$\langle 0 | \bar{q}q | 0 \rangle \neq 0 \quad \langle 0 | G_{\mu\nu}^a G_{\mu\nu}^a | 0 \rangle \neq 0$$

- The Banks-Casher relation provides a deep connection between the chiral symmetry breaking and the density of eigenvalues of the Dirac operator near zero.

$$|\langle \bar{q}q \rangle| = \pi \rho(0) \begin{cases} \rho(0) = 0 & \text{Unbroken symmetry, no dynamical mass} \\ \rho(0) \neq 0 & \text{Spontaneously symmetry broken, dynamical mass (300~400 MeV)} \end{cases}$$

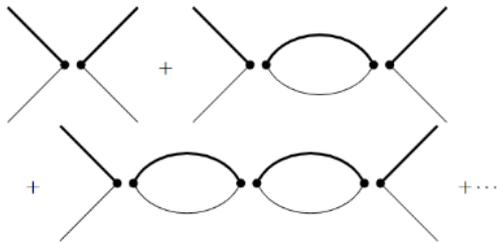
: condensate mixes left and right-handed quarks → can explain hadron masses

D. Diakonov et al, Nucl. Phys. B, 272:457–489, 1986.
C. D. Roberts et al, Prog. Part. Nucl. Phys., 33:477–575, 1994

Heavy-light quark scattering

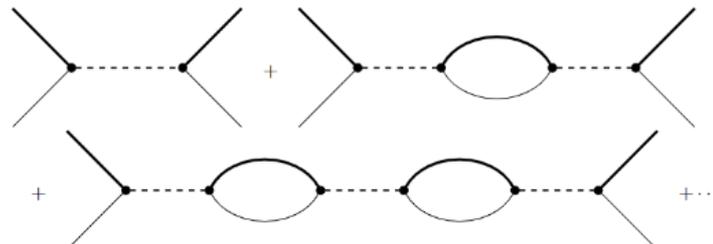
- The momentum distribution of the light quark inside a heavy meson is dominated by the instanton form factor $F(p_q)$.
- The average value of the effective coupling $gF(p_q)$ for the heavy-light quark vertex in momentum space approaches zero ($\langle gF(p_q) \rangle \approx 0$).
- Each external light-quark leg has one instanton form factor, and the light-quark propagator 2nd order of form factor  Each vertex has one instanton form factor
- This ensures the convergence of infinite quark-quark and meson-quark loop sums.

$-g^2[q_I^\dagger \Gamma_i h][h^\dagger \Gamma_i q_I]$ Quark-quark coupling picture



$$\mathcal{A}_{Qq}^a = \frac{-F(p_q)F(p'_q)}{(v \cdot p_\Phi - i\Lambda)\Pi'(i\Lambda) + \Pi^r(i\Lambda)}$$

$\mathcal{L}'_{\text{eff}}$ meson-quark coupling picture



$$\mathcal{A}_{Qq}^b = \frac{iG^2 F(p_q)F(p'_q)}{2(v \cdot p_\Phi - i\Lambda) - iG^2 \Pi^r(i\Lambda)}$$

$$G^2 = G_0^2 \left(1 + \frac{iG^2}{2} \Pi^r(i\Lambda) \right) \equiv Z_\Phi \mathfrak{G}_0^2$$

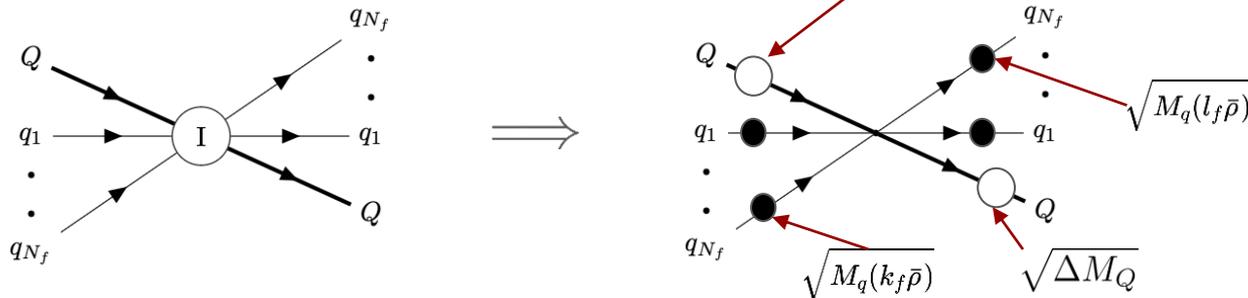
Heavy-light quark interaction ($N_f = 1 + 1$)

$$S_{\text{int}} = - \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} \frac{d^4 p_1 d^4 p_2}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(k_1 - k_2 + p_1 - p_2) \sqrt{M_q(k_1) M_q(k_2)} \frac{\Delta M_Q}{N/V}$$

$$\times \left[(\psi^\dagger(k_1) \psi(k_2)) (Q_+^\dagger(p_1) Q_+(p_2)) \right.$$

$$\left. + \frac{1}{8} \sum_i (\psi^\dagger(k_1) \Gamma_i Q_+(p_1)) (Q_+^\dagger(p_2) \Gamma_i \psi(k_2)) \right],$$

□ HQ and N_f light quarks interaction:



QCD vacuum

- Rich Topological Structure
 - Multiple degenerate vacuum states (n-vacua)
 - Charaterized by different winding numbers : topological properties of gauge field
 - Classical transitions between forbidden by energy barrier
 - QCD vacuum topology structure make U(1) amomaly
 - η' (958 MeV) meson much heavier than other pseudoscalar mesons (π, K, η)
- Shows importance of nontrivial vacuum structure

U(1) anomaly

- Classically, QCD Lagrangian has U(1)_A chiral symmetry (when massless limit)
- Axial vector current: $j_\mu^5 = \bar{q}\gamma_\mu\gamma_5q \longrightarrow \partial^\mu j_\mu^5 = 0$
- Because of the quantization process (fermion path integrals), we need regularization

→ cannot preserve all conservations

(gauge invariant and vector current conservation must be needed)

- Adler-Bell-Jackiw anomaly ($U(1)_A$ symmetry is broken) \longrightarrow Axial vector meson η' is not Nambu-Goldstone boson

$$\partial^\mu j_\mu^5 = \frac{N_f}{16\pi^2} F_{\mu\nu}^a \tilde{F}^{a,\mu\nu} \longrightarrow \rho_T = \frac{1}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a,\mu\nu}$$

$$\Delta Q_5 = 2N_f Q_T$$

Q_T is a topological charge, which must be integer numbers because of the topological properties.



QCD vacuum's topological properties solve the U(1) problem

Instanton contribution to light quark systems

- What about higher N_f ?
- Partition function is more and more complicated

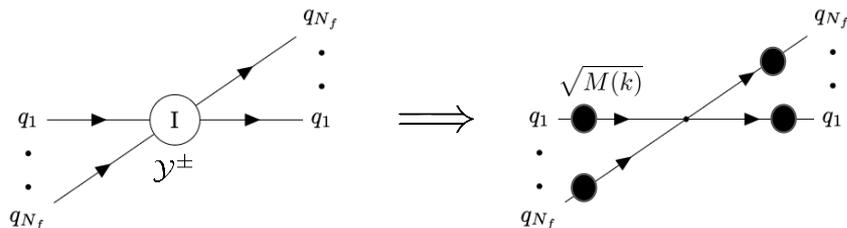
$$\mathcal{Z} = \int D\psi^\dagger D\psi \exp \left(\int d^4x \sum_f^{N_f} \psi_f^\dagger i\partial\psi_f + \mathcal{Y}_{N_f}^{(+)} + \mathcal{Y}_{N_f}^{(-)} \right)$$

$$\mathcal{Y}_{N_f}^{(+)} = \left(\frac{V}{N} \right)^{N_f-1} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_{N_f}}{(2\pi)^4} \int \frac{d^4l_1}{(2\pi)^4} \cdots \frac{d^4l_{N_f}}{(2\pi)^4} \\ \times (2\pi)^4 \delta^{(4)} \left(\sum_f^{N_f} k_f - \sum_f^{N_f} l_f \right)$$

$$\times \int dU \prod_f^{N_f} \sqrt{M(k_f)M(l_f)} U_{\gamma_f}^{\alpha_f} U_{\beta_f}^{\dagger\delta_f} \epsilon^{i_f\gamma_f} \epsilon_{j_f\delta_f} \left[i\psi_{Lf\alpha_f i_f}^\dagger(k_f) \psi_{Lf\beta_f j_f}^\dagger(l_f) \right]$$

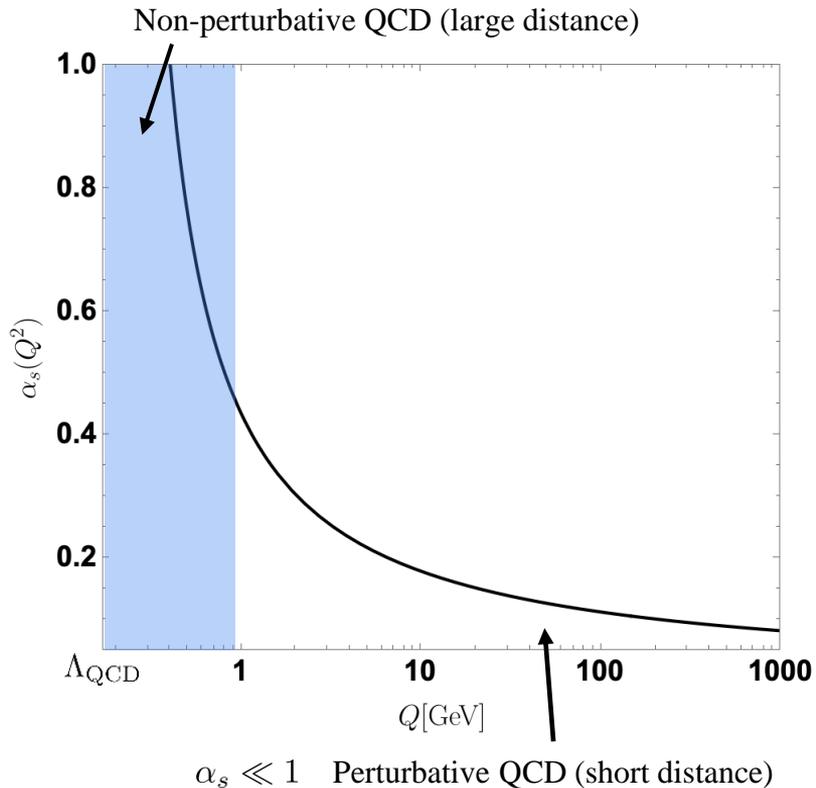
$$\frac{4VN_c}{N} \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)} = 1,$$

$$M(k) = u_0 F^2(k) = \frac{\lambda_0^{\frac{1}{N_f}} (2\pi\rho)^2}{N_c} \left(\frac{V}{N} \right)^{\frac{1-N_f}{N_f}} F^2(k).$$



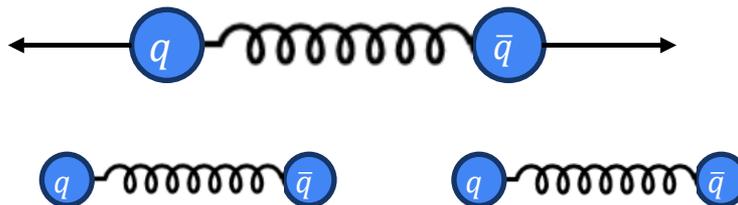
Depicted the instanton vertex imposes the dynamical quark mass to each quark legs

QCD properties



Two important QCD properties

- Asymptotic freedom (short distance):
Quarks and gluons behave as quasi-free particles.
- Confinement (large distance):
Quarks and gluons interact each other very strongly.



We cannot see the single quark

Instanton vacuum

□ Yang-Mills action :
$$S = \frac{1}{4g^2} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a = \frac{1}{2g^2} \int d^4x \text{tr} F_{\mu\nu} F_{\mu\nu}, \quad (F_{\mu\nu} = F_{\mu\nu}^a \tau^a / 2)$$

$$= \frac{1}{2g^2} \int d^4x \text{tr} \left[\frac{1}{2} (F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \mp F_{\mu\nu} \tilde{F}_{\mu\nu} \right] \geq \pm \frac{1}{2g^2} \int d^4x \text{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}.$$

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \quad (\text{anti}) \text{ Self duality} \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$$

□ Invariant under the gauge transformation :

$$A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + iU(x) \partial_\mu U^\dagger(x) \quad A_\mu(x) \underset{x \rightarrow \infty}{\rightarrow} ig(x) \partial_\mu g^{-1}(x), \quad g(x) = i \frac{\tau_\mu^+ x_\mu}{x}$$

□ Self(anti-self) duality condition gives a singular gauge instanton(anti-instanton) solution :

$$A_{\pm, \mu}(x, z_I) = \eta_{\mu\nu}^{\pm a} \tau^a \frac{\rho^2 (x - z_I)_\nu}{(x - z_I)^2 [(x - z_I)^2 + \rho^2]}$$

+ : instanton
- : antiinstanton



$$\eta_{\mu\nu\alpha}^\pm = \epsilon_{\alpha\mu\nu 4} \pm \delta_{\alpha\mu} \delta_{4\nu} \mp \delta_{\alpha\nu} \delta_{4\mu}$$

→ $D_\mu^{ab} F_{\mu\nu}^b = 0, D_\mu^{ab} \tilde{F}_{\mu\nu}^b = 0$

Satisfying YM EOM

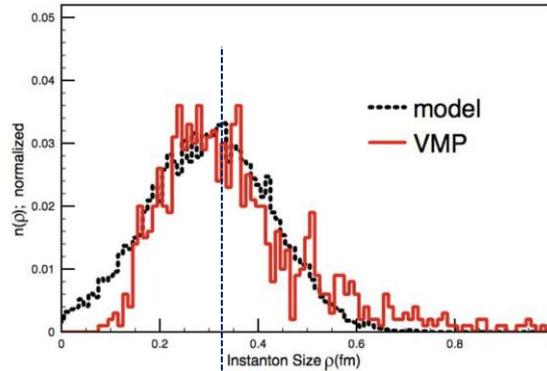
Instanton Parameters

- The instanton size distribution is predicted by phenomenological, Lattice and variation method

ρ : instanton size

R : inter-instanton distance

[Millo Raffaele and Faccioli Pietro, Phys. Rev. D 84, 034504 (2011)]



$\bar{\rho} \approx 0.33$ fm

Lattice : $R \approx 0.89$ fm, $\rho \approx 0.36$ fm
Phenomological : $R \approx 1$ fm, $\rho \approx 0.33$ fm
variational : $R \approx 0.76$ fm, $\rho \approx 0.32$ fm

They are used as a packing parameter : $\lambda = \left(\frac{\rho}{R}\right)^4 \sim 0.01 - 0.03$

This value means that the interaction is small, which gives the justification for using the semi-classical methods.

What is QCD?

- Compare QED and QCD

	QCD	QED
Interaction	Strong interactions	Electromagnetic interactions
Lagrangian	$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_{a,b,f=1}^{3,3,N_f} \bar{q}_{a,f}(i\not{D} - m_f)_b^a q_f^b$	$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$
Gauge Particle	Gluon	Photon
Charge	Color charge	Electric charge
Coupling constant	Running coupling constant α_s	$\alpha \approx 1/137$

What can we do from the interaction action?

□ We need to obtain the interaction action for any N_f .

For $N_f = 1 + 1$ (from heavy quark), the color-orientation integral is given as

$$\int dU U_{i_1'}^{a_1} U_{b_1}^{\dagger j_1'} U_{i_2'}^{a_2} U_{b_2}^{\dagger j_2'} = \frac{1}{N_c^2} \delta_{b_1}^{a_1} \delta_{j_1'}^{i_1'} \delta_{b_2}^{a_2} \delta_{j_2'}^{i_2'} + \frac{1}{4(N_c^2 - 1)} [\lambda^c]_{b_1}^{a_1} [\lambda^c]_{j_1'}^{i_1'} [\lambda^c]_{b_2}^{a_2} [\lambda^c]_{j_2'}^{i_2'}$$

$$= \frac{1}{N_c^2 - 1} \left[\delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \left(\delta_{i_1'}^{j_1'} \delta_{i_2'}^{j_2'} - \frac{1}{N_c} \delta_{i_2'}^{j_1'} \delta_{i_1'}^{j_2'} \right) + \delta_{b_2}^{a_1} \delta_{b_1}^{a_2} \left(\delta_{i_2'}^{j_1'} \delta_{i_1'}^{j_2'} - \frac{1}{N_c} \delta_{i_1'}^{j_1'} \delta_{i_2'}^{j_2'} \right) \right]$$

$$S_{Qq} = -iN_c V \int \frac{d^4 k_1 d^4 k_2 d^4 p_1 d^4 p_2}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(k_1 - k_2 + p_1 - p_2) \sqrt{M(k_1)M(k_2)} T_{\pm}(\vec{p}_1 - \vec{p}_2, \omega_2, \omega_1)$$

$$\times \left[\frac{1}{N_c^2 - 1} \left(1 - \frac{1}{2N_c} \right) (\psi^\dagger(k_1)\psi(k_2)) (Q^\dagger(p_1)Q(p_2)) \right. \\ \left. + \frac{1}{8(N_c^2 - 1)} \left(1 - \frac{2}{N_c} \right) \sum_i (\psi^\dagger(k_1)\Gamma_i Q(p_1)) (Q^\dagger(p_2)\Gamma_i \psi(k_2)) \right]$$

where $\Gamma_i = (\mathbf{1}, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5, \sigma_{\mu\nu}/\sqrt{2})$.

For $N_f = 2 + 1$ (from heavy quark)

$$\int dU U_{i_1'}^{a_1} U_{b_1}^{\dagger j_1'} U_{i_2'}^{a_2} U_{b_2}^{\dagger j_2'} U_{i_3'}^{a_3} U_{b_3}^{\dagger j_3'} = \frac{1}{N_c^3} \delta_{b_1}^{a_1} \delta_{j_1'}^{i_1'} \delta_{b_2}^{a_2} \delta_{j_2'}^{i_2'} \delta_{b_3}^{a_3} \delta_{j_3'}^{i_3'}$$

$$+ \frac{1}{4N_c(N_c^2 - 1)} \left([\lambda^i]_{b_1}^{a_1} [\lambda^i]_{b_2}^{a_2} [\lambda^j]_{i_1'}^{j_1'} [\lambda^j]_{i_2'}^{j_2'} \delta_{b_3}^{a_3} \delta_{i_3'}^{j_3'} + (3 \leftrightarrow 1) + (2 \leftrightarrow 1) \right)$$

$$+ \frac{N_c}{8(N_c^2 - 4)(N_c^2 - 1)} d_{ijk} d_{abc} [\lambda^i]_{b_1}^{a_1} [\lambda^j]_{b_2}^{a_2} [\lambda^k]_{b_3}^{a_3} [\lambda^i]_{i_1'}^{j_1'} [\lambda^j]_{i_2'}^{j_2'} [\lambda^k]_{i_3'}^{j_3'}$$

$$+ \frac{1}{8N_c(N_c^2 - 1)} f_{ijk} f_{abc} [\lambda^i]_{b_1}^{a_1} [\lambda^j]_{b_2}^{a_2} [\lambda^k]_{b_3}^{a_3} [\lambda^a]_{i_1'}^{j_1'} [\lambda^b]_{i_2'}^{j_2'} [\lambda^c]_{i_3'}^{j_3'}$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$S_{Qq}^{(0)} = -\lambda_s \int \frac{d^4 z}{V} \int dt_1 dt_2 d^3 x \text{tr}_c \langle S_Q^{-1} \rangle \sum_{\pm} \frac{1}{(N_c^2 - 4)(N_c^2 - 1)}$$

$$\times \left[\frac{(N_c - 2)(N_c + 1)}{N_c} \left\{ (u_{\pm}^\dagger(z)u_{\pm}(z))(d_{\pm}^\dagger(z)d_{\pm}(z)) - (u_{\pm}^\dagger(z)d_{\pm}(z))(d_{\pm}^\dagger(z)u_{\pm}(z)) \right\} (Q^\dagger(\vec{x}, t_1)Q(\vec{x}, t_2)) \right. \\ - \frac{N_c - 2}{N_c} \left\{ (u_{\pm,i}^\dagger(z)u_{\pm,j}(z))(d_{\pm}^{\dagger j}(z)d_{\pm}^i(z)) - (u_{\pm,i}^\dagger(z)d_{\pm,j}(z))(d_{\pm}^{\dagger j}(z)u_{\pm}^i(z)) \right\} (Q^\dagger(\vec{x}, t_1)Q(\vec{x}, t_2)) \\ - \left((u_{\pm,i}^\dagger(z)u_{\pm,j}(z))(Q^\dagger(\vec{x}, t_1)d_{\pm}^i(z)) - (u_{\pm,i}^\dagger(z)d_{\pm,j}(z))(Q^\dagger(\vec{x}, t_1)u_{\pm}^i(z)) \right) (d_{\pm}^{\dagger j}(z)Q(\vec{x}, t_2)) \\ + (u_{\pm,i}^\dagger(z)Q(\vec{x}, t_2)) \left((d_{\pm,j}^\dagger(z)u_{\pm}^i(z))(Q^\dagger(\vec{x}, t_1)d_{\pm}^j(z)) - (d_{\pm,j}^\dagger(z)d_{\pm}^i(z))(Q^\dagger(\vec{x}, t_1)u_{\pm}^j(z)) \right) \left. \right\} \\ - \frac{N_c - 2}{2} \left\{ \left((u_{\pm}^\dagger(z)u_{\pm}(z))(d_{\pm,i}^\dagger(z)Q(\vec{x}, t_2)) - (u_{\pm}^\dagger(z)d_{\pm}(z))(d_{\pm,i}^\dagger(z)Q(\vec{x}, t_2)) \right) (Q^\dagger(\vec{x}, t_1)d_{\pm}^i(z)) \right. \\ \left. - (u_{\pm,i}^\dagger(z)Q(\vec{x}, t_2)) \left((d_{\pm}^\dagger(z)u_{\pm}(z))(Q^\dagger(\vec{x}, t_1)d_{\pm}^i(z)) - (Q^\dagger(\vec{x}, t_1)u_{\pm}^i(z))(d_{\pm}^\dagger(z)d_{\pm}^i(z)) \right) \right\}$$

More complicated terms appear

So, we need to find the generalization form of the interaction action for any N_f .

Normalization

- The standard nonrelativistic normalization condition is defined as

$$\langle \phi(v') | \phi(v) \rangle = (2\pi)^3 2p_4 / m_\Phi \delta^{(3)}(\Lambda \vec{v}' - \Lambda \vec{v}) = (2\pi)^3 2v_4 \delta^{(3)}(\Lambda \vec{v}' - \Lambda \vec{v})$$

- The composite operator can be induced by the equation of motion

$$\Phi_{vi}(p_\Phi) = \langle p_{\Phi 4} | \phi_i(v) \rangle = \sqrt{2v_4} ig \mathcal{N} \int \frac{d^4 p_q d^4 k}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(p_\Phi - k - p_q) F(p_q) \psi^\dagger(p_q) \Gamma_i h(k)$$

- The one-particle state $|\phi_i(v)\rangle$ of the heavy meson is defined as $|\phi(v)\rangle \equiv \phi_v |0\rangle$.

$$\begin{aligned} \langle \phi_i(v') | \phi_j(v) \rangle &= \int \frac{dp_{\Phi 4}}{2\pi} \text{tr}_D \Phi_{vj}(p_\Phi) \Phi_{v'i}^\dagger(p'_\Phi) \\ &= 2v_4 g^2 \mathcal{N}^2 (2\pi)^3 \delta^3(\Lambda v' - \Lambda v) \\ &\quad \times \int \frac{d^4 p_q}{(2\pi)^4} F^2(p_q) \text{tr}_D [S_q(p_q) \Gamma_i S_Q(p_\Phi - p_q) \Gamma_j] \\ &= 2v_4 g^2 \mathcal{N}^2 \Pi(i\Lambda) (2\pi)^3 \delta^3(\Lambda v' - \Lambda v) \delta_{ij} \\ &= 2v_4 (2\pi)^3 \delta^3(\Lambda v' - \Lambda v) \delta_{ij}, \quad \color{red}{\longrightarrow} \quad g^2 \mathcal{N}^2 \Pi(i\Lambda) = 1 \end{aligned}$$



(a)

$$\begin{aligned} \overleftarrow{-p_q} &= \frac{\not{p}_q - iM(p_q)}{p_q^2 + M^2(p_q)} = S_q(p_q) \\ \overrightarrow{k} &= \frac{1 + \not{p}}{2v \cdot k} = S_Q(k) \end{aligned}$$

(b)

$$\Pi(v \cdot p_\Phi) = \int \frac{d^4 p_q}{(2\pi)^4} F^2(p_q) \text{tr}_D \left[\Gamma_P \frac{(1 + \not{p})}{2v \cdot (p - p_q) + i\epsilon} \Gamma_P \frac{-(\not{p}_q - iM(p_q))}{p_q^2 + M^2(p_q)} \right]$$

Light-quark zero mode

□ Pobylitsa representation

Gamma matrices in Euclidean space

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_\mu^C \equiv \gamma_\mu$$

$$\frac{1 + \gamma_5}{2}\psi = \psi_L = \psi_{\bar{I}}, \quad \frac{1 - \gamma_5}{2}\psi = \psi_R = \psi_I, \quad \text{with } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Unitary transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The Dirac equation in chiral limit

$$\begin{aligned} \mathcal{L}_q &= i\bar{\psi}(x)\not{D}\psi(x) \rightarrow i\psi^\dagger U^\dagger U \gamma_4^D U^\dagger U \gamma_\mu^D D_\mu U^\dagger U \psi(x) = i\bar{\phi} \gamma_\mu^C D_\mu \phi \\ &\Rightarrow -i\gamma_\nu^C D_\nu \phi = 0. \end{aligned}$$

Light-quark zero mode

- The zero-mode Dirac spinor is derived as

$$\psi_{cs}(x) = \frac{1}{\sqrt{2}} \phi(x) x_\mu (\tau_\mu^-)_{cc'} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \epsilon_{c's}$$

$$\phi(x) = \frac{\rho}{\pi x(x^2 + \rho^2)^{3/2}} \xrightarrow{\text{Fourier transformation}} \phi(k) = 2\pi\rho \left\{ I_0\left(\frac{k\rho}{2}\right) K_0\left(\frac{k\rho}{2}\right) - I_1\left(\frac{k\rho}{2}\right) K_1\left(\frac{k\rho}{2}\right) \right\}$$

$$\phi_I(x)_{cs} \equiv \phi(x) x_\mu (\tau_\mu^- \epsilon)_{cs} \xrightarrow{\text{Fourier transformation}} \phi_I(k)_{cs} = \frac{2\pi i \rho}{k^2} F(k) k_\mu (\tau_\mu^- \epsilon)_{cs}$$

$$F(k) = -\frac{k}{2\pi\rho} \phi'(k) \Big|_{t=k\rho/2} = -t \frac{d}{dt} [I_0(t)K_0(t) - I_1(t)K_1(t)]$$

- The rotational zero mode density functions

$$\psi_I(k_1)_{c'i} \psi_{I'}^\dagger(k_2)_{ci} = \frac{1}{8} (2\pi\rho)^2 \frac{F(k_1)F(k_2)}{k_1^2 k_2^2} \left(\not{k}_1 \gamma_\rho \gamma_\sigma \not{k}_2 \frac{1 - \gamma_5}{2} \right)_{i'i} (U_I \tau_\rho^- \tau_\sigma^+ U_{I'}^\dagger)_{c'c}$$

$$\psi_{\bar{I}}(k_1)_{c'i} \psi_{\bar{I}'}^\dagger(k_2)_{ci} = \frac{1}{8} (2\pi\rho)^2 \frac{F(k_1)F(k_2)}{k_1^2 k_2^2} \left(\not{k}_1 \gamma_\rho \gamma_\sigma \not{k}_2 \frac{1 + \gamma_5}{2} \right)_{i'i} (U_{\bar{I}} \tau_\rho^+ \tau_\sigma^- U_{\bar{I}'}^\dagger)_{c'c}$$

$$\psi_I(k_1)_{c'i} \psi_{\bar{I}'}^\dagger(k_2)_{ci} = -\frac{i}{2} (2\pi\rho)^2 \frac{F(k_1)F(k_2)}{k_1^2 k_2^2} \left(\not{k}_1 \gamma_\rho \not{k}_2 \frac{1 + \gamma_5}{2} \right)_{i'i} (U_I \tau_\rho^- U_{\bar{I}'}^\dagger)_{c'c}$$

$$\psi_{\bar{I}}(k_1)_{c'i} \psi_{I'}^\dagger(k_2)_{ci} = \frac{i}{2} (2\pi\rho)^2 \frac{F(k_1)F(k_2)}{k_1^2 k_2^2} \left(\not{k}_1 \gamma_\rho \not{k}_2 \frac{1 - \gamma_5}{2} \right)_{i'i} (U_{\bar{I}} \tau_\rho^+ U_{I'}^\dagger)_{c'c}$$

Heavy quark propagator in an instanton ensemble

□ HQ propagator in instanton ensemble: $\bar{w} = \left\langle \left(\theta^{-1} - \sum_I a_I \right)^{-1} \right\rangle$

□ In terms of single instanton propagators: $\bar{w} = \theta + \sum_I \langle w_I - \theta \rangle + \sum_{I \neq J} \langle w_I - \theta \rangle \theta^{-1} \langle w_J - \theta \rangle + \dots$

$$\begin{aligned}
 \bar{w} - \theta = & \sum_I \textcircled{I} + \sum_{I \neq J} \textcircled{I} - \textcircled{J} + \sum_{I \neq J \neq K} \textcircled{I} - \textcircled{J} - \textcircled{K} \\
 & + \sum_{I \neq J} \textcircled{I} - \textcircled{J} - \textcircled{I} + \sum_{I \neq J \neq K \neq L} \textcircled{I} - \textcircled{J} - \textcircled{K} - \textcircled{L} \\
 & + \sum_{I \neq J \neq K} \left[\textcircled{I} - \textcircled{J} - \textcircled{K} - \textcircled{I} + \textcircled{I} - \textcircled{J} - \textcircled{K} - \textcircled{J} \right. \\
 & \left. + \textcircled{I} - \textcircled{J} - \textcircled{I} - \textcircled{K} \right] + \sum_{I \neq J} \textcircled{I} - \textcircled{J} - \textcircled{I} - \textcircled{J}
 \end{aligned}$$

additional $1/N_c$ factor