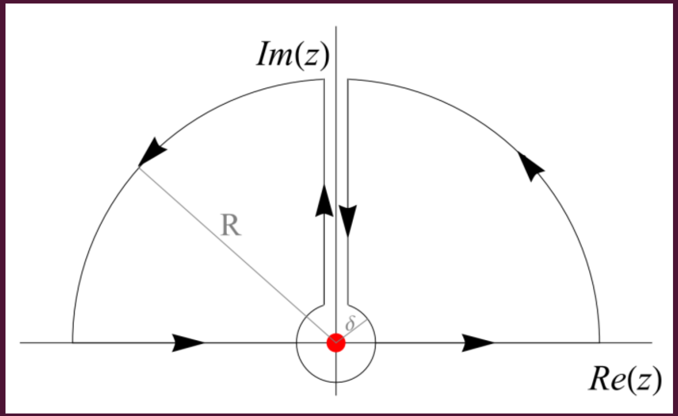
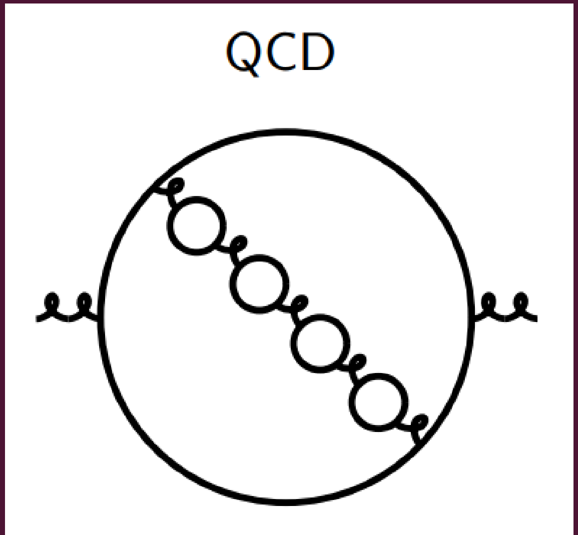
RENORMALON-BASED RESUMMATION OF THE BJORKEN SUM RULE IN HOLOMORPHIC QCD

CAMILO CASTRO ARRIAZA
(NAGOYA UNIVERSITY)





NOV. 13, 2025

BARYONS 2025, JEJU

BASED ON

- This presentation is based on:

Ayala, C., Castro-Arriaza, C., & Cvetič, G. (2023). Evaluation of Bjorken Polarised Sum Rule with a Renormalon-motivated Approach. Physics Letters B, 848, 138386. https://doi.org/10.1016/j.physletb.2023.138386

Ayala, C., Castro-Arriaza, C., & Cvetič, G. (2024). Renormalon-based Resummation of Bjorken Polarised Sum Rule in Holomorphic QCD. Nuclear Physics B, 1007, 116668. https://doi.org/10.1016/j.nuclphysb.2024.116668

QCD COUPLING AND PQCD

$$\mathcal{L} = \sum_q \bar{\psi}_{q,a}(i\gamma^\mu \partial_\mu \delta_{ab} - \sqrt{4\pi\alpha_s}\gamma^\mu t^C_{ab}\mathcal{A}^C_\mu - m_q \delta_{ab})\psi_{q,b} - \frac{1}{4}F^A_{\mu\nu}F^{A\ \mu\nu}$$

- $\alpha_s(\mathrm{Q}^2)$ is the QCD coupling. It encodes the strength of interactions. It decreases for large Q^2 due to asymptotic freedom.
- When using perturbation theory (pQCD), the coupling is renormalised as $\alpha_s(\mu^2)$, a function of renormalisation scale μ from a renormalisation scheme. It satisfies the RGE:

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -(b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \dots)$$

- pQCD predicts the appearance of (unphysical) Landau singularities at low Q^2 where the coupling becomes infinite, which spoils the expected holomorphic behaviour of observables.

DIVERGENT SERIES

- In QFTs, observables can be expressed as series expansion in terms of the coupling such as:

$$R = \sum_n r_n \alpha^n$$

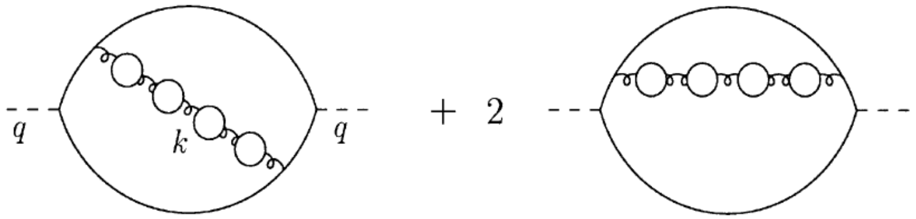
- These series are usually divergent independently of the coupling itself: $r_n \overset{n\to\infty}{\sim} K a^n n!\, n^b$

- As we do not know *R* non-perturbatively, we can use this expansion as a good approximation for the perturbative regime, but simply adding more terms does not give better predictions past a certain point and thus it is required to introduce *summations* of the series, such as the Borel Transform.

$$R \sim \sum_{n=0}^{\infty} r_n \alpha^{n+1} \Rightarrow B[R](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} \longrightarrow \tilde{R} = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t/\alpha} B[R](t)$$

RENORMALONS

- The Borel integral may not always be well defined, and can have singularities in its correspondent (complex) plane, which reflect and encode the inherent divergence of the original series.
- These singularities in the Borel plane that can be found in observables are called **renormalons**. Originally, they were found in contributions to correlators (for the Adler function in the case of this example) in diagrams such as:



$$B[D](u) = \frac{3C_F}{2\pi}\left(\frac{Q^2}{\mu^2}e^{-5/3}\right)^{-2}\frac{1}{2-u} \quad \text{(first IR renormalon)}$$

$$+\frac{C_F Q^2}{3\pi\,\mu^2}e^{-5/3}\left[\frac{1}{(1+u)^2}+\frac{5}{6}\frac{1}{1+u}\right] \quad \text{(first UV renormalon)}$$

M. Beneke, 1999.

- Renormalons *do not always give* contributions of this kind, but these are illustrative.
- It's the knowledge of the location of the renormalons for different observables what motivates their use in phenomenological methods such as the Sum Rule approach to obtain more information that what can be had from simple use of the truncated series.

BJORKEN SUM RULE (BSR)

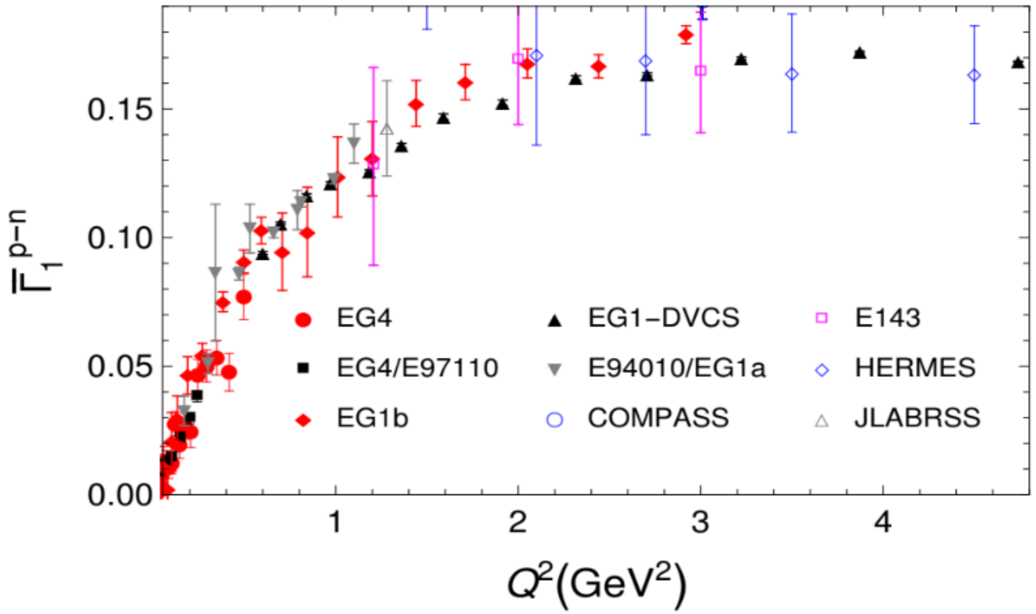
- It is the difference between the polarised structure functions g of the proton and neutron integrated over the Bjorken scaling variable *x*. Of particular interest is the inelastic part:

$$\overline{\Gamma}_1^{p-n}(Q^2) = \int_0^1 dx \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right]$$

- It has a known theoretical expression in OPE which is:

$$\overline{\Gamma}_1^{\rm p-n,OPE}(Q^2) = \left|\frac{g_A}{g_V}\right|\frac{1}{6}\left[1 - d(Q^2) - \delta d(Q^2)_{m_c}\right] + \frac{M_N^2}{9}\frac{\left[(a_2^{\rm (p-n)}(Q^2) + 4d_2^{\rm (p-n)}(Q^2)) + 4\bar{f}_2\, a(Q^2)^{k_1}\right]}{Q^2} + \frac{\mu_6}{(Q^2)^2}$$

- It has been measured by many experiments from JLab, CERN, DESY and SLAC.



C. Ayala, C.C.A. & G. Cvetič (2023)

BJORKEN SUM RULE (BSR) II

- Data are available down to very low Q^2 values.

- The canonical perturbative part of the OPE (first term) is known:

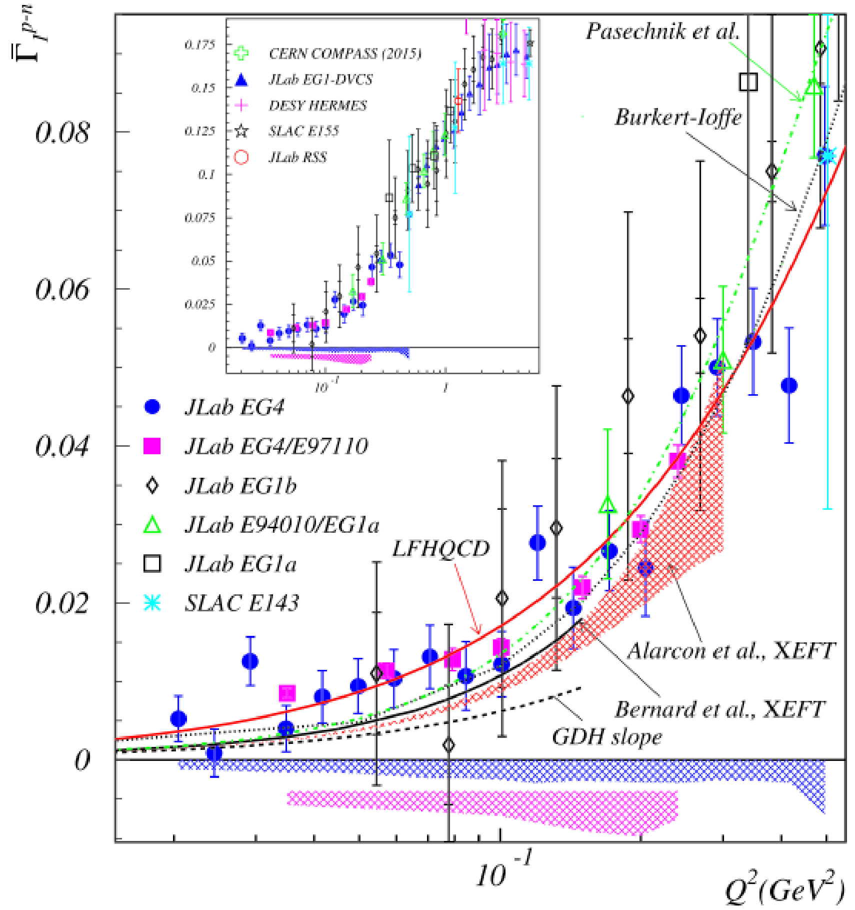
$$d(Q^2)_{\rm pt} \;=\; a + d_1 a^2 + d_2 a^3 + d_3 a^4 + \mathcal{O}(a^5)$$

$$a(Q^2) \equiv \alpha_s(Q^2)/\pi$$

- The coefficients in this quantity are known exactly up to *d_3*, but there are very good estimates for *d_4* which we used.

- For the *D*=2 term in the OPE we know its structure:

$$\frac{\mu_4(Q^2)}{Q^2} = \frac{M_N^2}{9}\,\frac{\left[(a_2^{p-n} + 4d_2^{p-n}) + 4\bar{f}_2\; a(Q^2)^{k_1}\right]}{Q^2}$$



A. Deur *et al.*, Phys. Lett. B 825 136878 (2022)

THE CVETIC METHOD: RECONSTRUCTING QUANTITIES

- To obtain the characteristic function of $d(Q^2)$ to generate a resummation, we first need to change our d expansion to use logarithmic derivatives instead of the simple series as before:

$$\tilde{a}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{(n-1)!\beta_0^{n-1}} \left(\frac{d}{d\ln\mu^2}\right)^{n-1} a(\mu^2) \quad (n = 0, 1, 2, \ldots) \longrightarrow d(Q^2) = a(Q^2) + \widetilde{d}_1\, \widetilde{a}_2(Q^2) + \ldots + \widetilde{d}_n\, \widetilde{a}_{n+1}(Q^2) + \ldots .$$

- Through the use of the Renormalisation Group Equation (RGE) we can relate the new expansion coefficients to the original ones and thus construct an auxiliary expression where:

$$\widetilde{d}(Q^2) = a(Q^2) + \widetilde{d}_1\, a(Q^2)^2 + \ldots + \widetilde{d}_n\, a(Q^2)^{n+1} + \ldots$$

- Its Borel transform is thus in the form:

$$\mathcal{B}[\widetilde{d}](u) \equiv 1 + \frac{\widetilde{d}_1}{1!\beta_0} u + \ldots + \frac{\widetilde{d}_n}{n!\beta_0^n} u^n + \ldots .$$

BSR CASE: RENORMALON STRUCTURE

- This transform turns out to be very similar to the Borel transform of the original $d(Q^2)$ that is well known, so from its well known renormalon structure we can construct a reasonable *ansatz* for this version:

$$\mathcal{B}[\tilde{d}](u) = \exp\left(\tilde{K}u\right)\pi\left[\tilde{d}_1^{\mathrm{IR}}\frac{1}{(1-u)^{\xi}} + \tilde{d}_2^{\mathrm{IR}}\frac{1}{(2-u)} + \tilde{d}_1^{\mathrm{UV}}\frac{1}{(1+u)} + \tilde{d}_2^{\mathrm{UV}}\frac{1}{(2+u)}\right]$$

- The parameters in this expression can easily be found from the known original coefficients of *d(Q^2)*.
- It can be demonstrated that this *ansatz* has the same structure as the corresponding form of the original BSR but conserving its simple structure not only for the large *b_0* approximation. Very useful!

$$\mathcal{B}[d](u) \propto \frac{1}{(p-u)^{\xi+p\beta_1/\beta_0^2}}(1+\mathcal{O}(p-u)), \quad \frac{1}{(p+u)^{\xi-p\beta_1/\beta_0^2}}(1+\mathcal{O}(p+u)) \quad \mathcal{B}[\tilde{d}](u) \propto \frac{1}{(p-u)^{\xi}}(1+\mathcal{O}(p-u)), \quad \frac{1}{(p+u)^{\xi}}(1+\mathcal{O}(p-u))$$

$$\beta_0 \to \infty$$

NOTE: BOREL TRANSFORM κ-DEPENDENCE

- Here, a very important show of the consistency of the method is that the *k*-dependence (which reflects renormalisation scale dependence) can be disregarded, as it should be for physical quantities:

$$\frac{d}{\ln \kappa}\mathcal{B}[\tilde{d}](u;\kappa) = u\mathcal{B}[\tilde{d}](u;\kappa)$$

$$\Longrightarrow \mathcal{B}[\tilde{d}](u;\kappa) = \kappa^u \mathcal{B}[\tilde{d}](u;\kappa=1)$$

$$\mathcal{B}[\tilde{d}](u;\kappa) = \int_0^\infty \frac{dt}{t} F_d(t) \sum_{n=0}^{\infty} \frac{1}{n!}\left(-u\ln\left(t/\kappa\right)\right)^n$$

$$\Longrightarrow \mathcal{B}[\tilde{d}](u;\kappa) = \int_0^\infty \frac{dt}{t} F_d(t) t^{-u}\kappa^u.$$

$$\mathcal{B}[\tilde{d}](u,\kappa=1) = \mathcal{B}[\tilde{d}](u) = \int_0^\infty \frac{dt}{t} F_d(t) t^{-u}$$

RENORMALON STRUCTURE CONSEQUENCES

$$d(Q^2)_{\rm res} = \int_0^\infty \frac{dt}{t} G_d(t) a(te^{-\widetilde{K}} Q^2)$$

- The beta coefficients from before are those arising from the RGE:

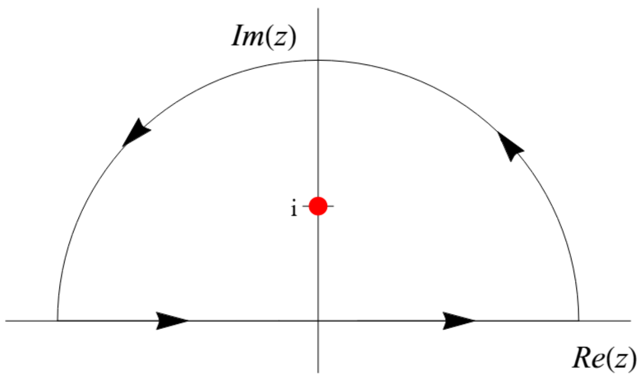
$$\begin{aligned} \frac{da(Q^2)}{d\ln Q^2} &= -\beta_0 a(Q^2)^2 - \beta_1 a(Q^2)^3 - \beta_2 a(Q^2)^4 - \ldots \\ &= -\beta_0 a(Q^2)^2 \left[1 + c_1 a + c_2 a^2 + \ldots\right]. \end{aligned}$$

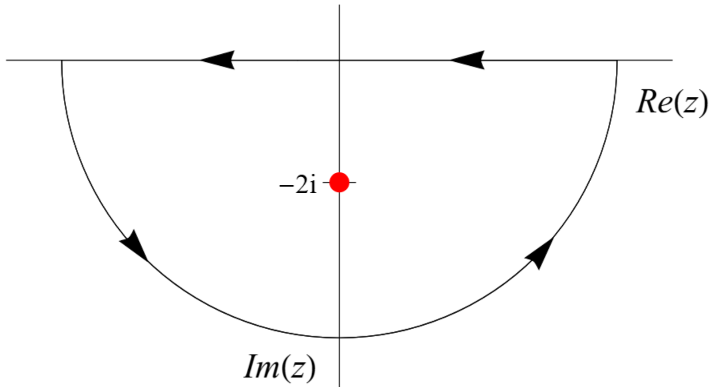
- It turns out that it is the structure of our special *ansatz* what permits a simple resummation since its inverse Mellin transform coincidentally turns out to be exactly the characteristic function we need:

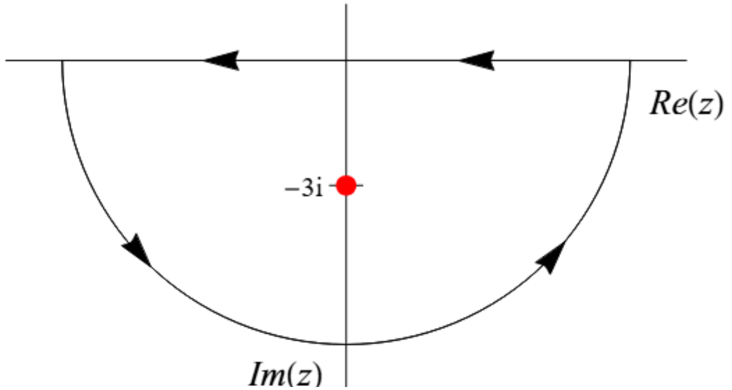
$$G_d(t) = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du \mathrm{B}[\widetilde{d}](u) t^u \longrightarrow G_d(t) = \Theta(1-t)\pi \left[\frac{\widetilde{d}_1^{\rm IR} t}{\Gamma(1-k_1)\ln^{k_1}(1/t)} + \widetilde{d}_2^{\rm IR} t^2 \right] + \Theta(t-1)\pi \left[\frac{\widetilde{d}_1^{\rm UV}}{t} + \frac{\widetilde{d}_2^{\rm UV}}{t^2} \right]$$

COMMENTS ON THE COMPUTATION OF THE CHARACTERISTIC FUNCTION INTEGRALS

- Requires some "clever" integration of each renormalon term in the *ansatz:*



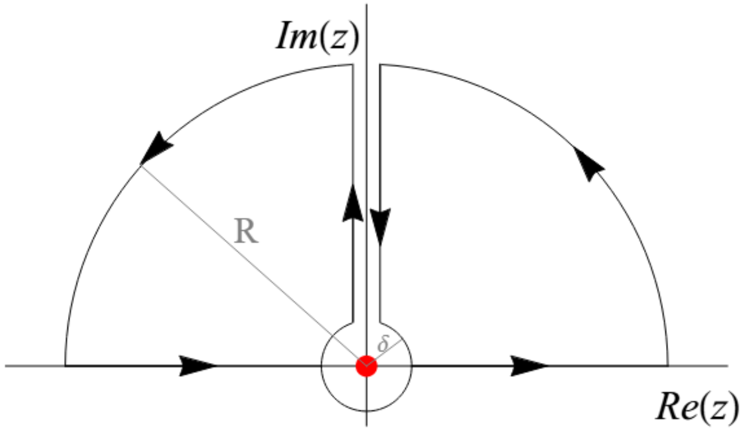




$$G_d(t) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\ \mathcal{B}[\tilde{d}](u)\Big|_{\tilde{K}\mapsto 0} t^u$$

$$-1 < u_0 < +1 \implies G_d(t) = \frac{t}{2\pi}\int_{-\infty}^{+\infty} dz\ \mathcal{B}[\tilde{d}](u = 1 - iz)\Big|_{\tilde{K}\mapsto 0} e^{-iz\ln(t)}$$

$$G_d(t) = \frac{t}{2}\int_{-\infty}^{+\infty} dz \left(\frac{\tilde{d}_1^{\mathrm{IR}}}{(iz)^{\xi}} + \frac{\tilde{d}_2^{\mathrm{IR}}}{(1+iz)} + \frac{\tilde{d}_1^{\mathrm{UV}}}{(2-iz)} + \frac{\tilde{d}_2^{\mathrm{UV}}}{(3-iz)}\right) e^{-iz\ln(t)}.$$



FINALLY, RESUMMATION

- With the found expression for *G_d* we can proceed to the resummation in the form of:

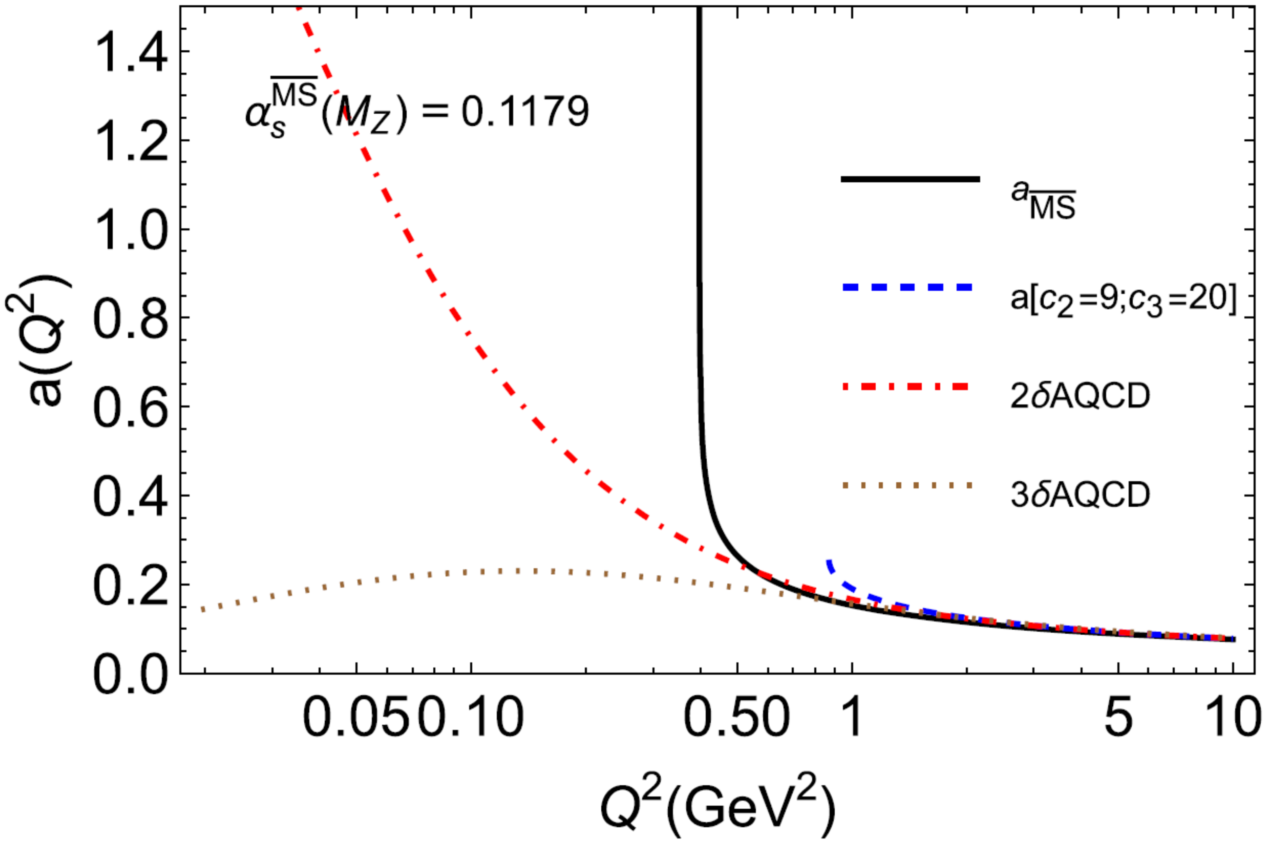
$$d(Q^2)_{\mathrm{res}} = \int_0^\infty \frac{dt}{t} G_d(t) a(te^{-\widetilde{K}} Q^2)$$

- Consistence here can be tested by re-expanding in terms of the coupling around *ln(Q^2)*.
- **BUT!:** The pQCD coupling has Landau singularities at low Q^2, so this integral for the whole range has ambiguities we need to go around of:

$$d(Q^2)_{\mathrm{res}} = \mathrm{Re}\left[\int_0^\infty \frac{dt}{t} G_d(t) a(te^{-\widetilde{K}} Q^2 + i\varepsilon)\right]$$

- This is completely skipped if, for instance, we use a different coupling constructed not to have such singularities but to conserve the pQCD behaviour at high *Q^2*.

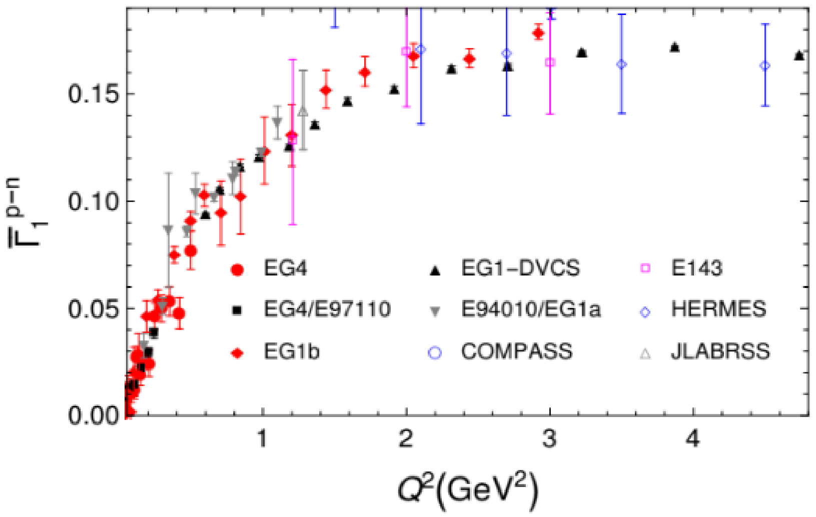
HOLOMORPHIC QCD VARIANTS (AQCD)

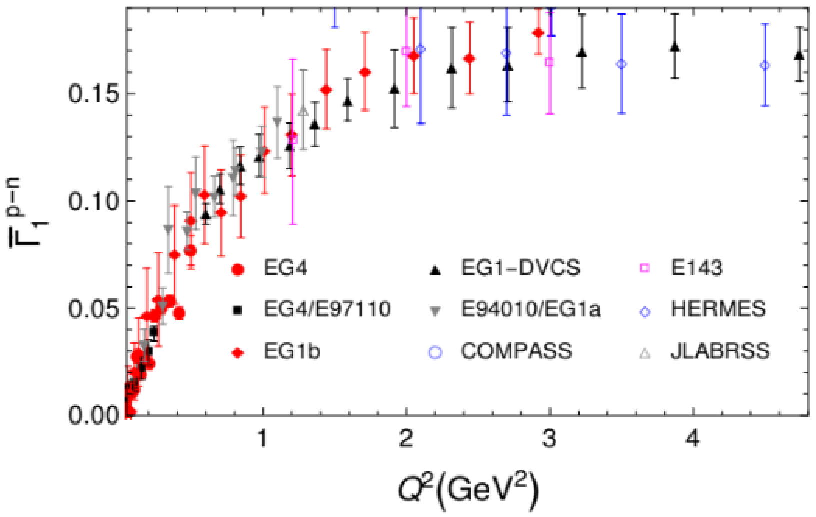


C. Ayala, C.C.A. & G. Cvetič (2024)

APPLICATION: SCHEME VARIATION AND FITTING TO DATA

- Having our renormalon-based resummmation of the BSR, we can tune it to qualitative expectations by varying the renormalisation scheme (RS) to be used, as there is still RS dependence.
- With a chosen RS that reproduces the expected behaviour, we can fix the parameters and fit the next BSR term(s) to experimental data while using our resummed *d(Q^2)*, thus complementing the perturbative part.





C. Ayala, C.C.A. & G. Cvetič (2023)

SCHEME VARIATION

- The five *ansatz* coefficients change depending on the Renormalisation Scheme (RS) prescription used:

c_2	c_3	$\widetilde{K}$	$\widetilde{d}_1^{\rm IR}$	$\widetilde{d}_2^{\rm IR}$	$\widetilde{d}_1^{\rm UV}$	$\widetilde{d}_2^{\rm UV}$	$d(Q^2)$	$d(Q^2)_{\rm IR1}$	$d(Q^2)_{\rm IR2}$	$d(Q^2)_{\rm UV1}$	$d(Q^2)_{\rm UV2}$
9.	20.	0.450041	0.331813	0.231437	-0.0809782	0.0964868	0.1816	0.1631	0.0597	-0.0247	-0.0164
7.6	20.	0.896252	0.210843	0.137235	-0.158441	0.394581	0.1843	0.1196	0.0421	-0.0551	0.0777
11.0	20.	0.12894	0.422324	0.301175	-0.015641	-0.477922	0.1807	0.1904	0.0701	-0.0044	-0.0752
9.	5.	0.359327	0.474866	-0.026115	-0.080151	-0.126694	0.1737	0.2243	-0.0064	-0.0236	-0.0207
9.	35.	0.484948	0.237256	0.431189	-0.067963	-0.133156	0.1888	0.1188	0.1143	-0.0212	-0.0232
$c_2^{\overline{\rm MS}}$	$c_3^{\overline{\rm MS}}$	-1.83223	7.86652	-14.9299	-0.0444416	-0.0776748	0.1632	1.9466	-1.7675	-0.0082	-0.0076

$$[1 - a_1 c_1^2 a(Q^2)^2]\,[1 + (a_0 - 1) c_1 a(Q^2) + a_1 c_1^2 a(Q^2)^2]$$

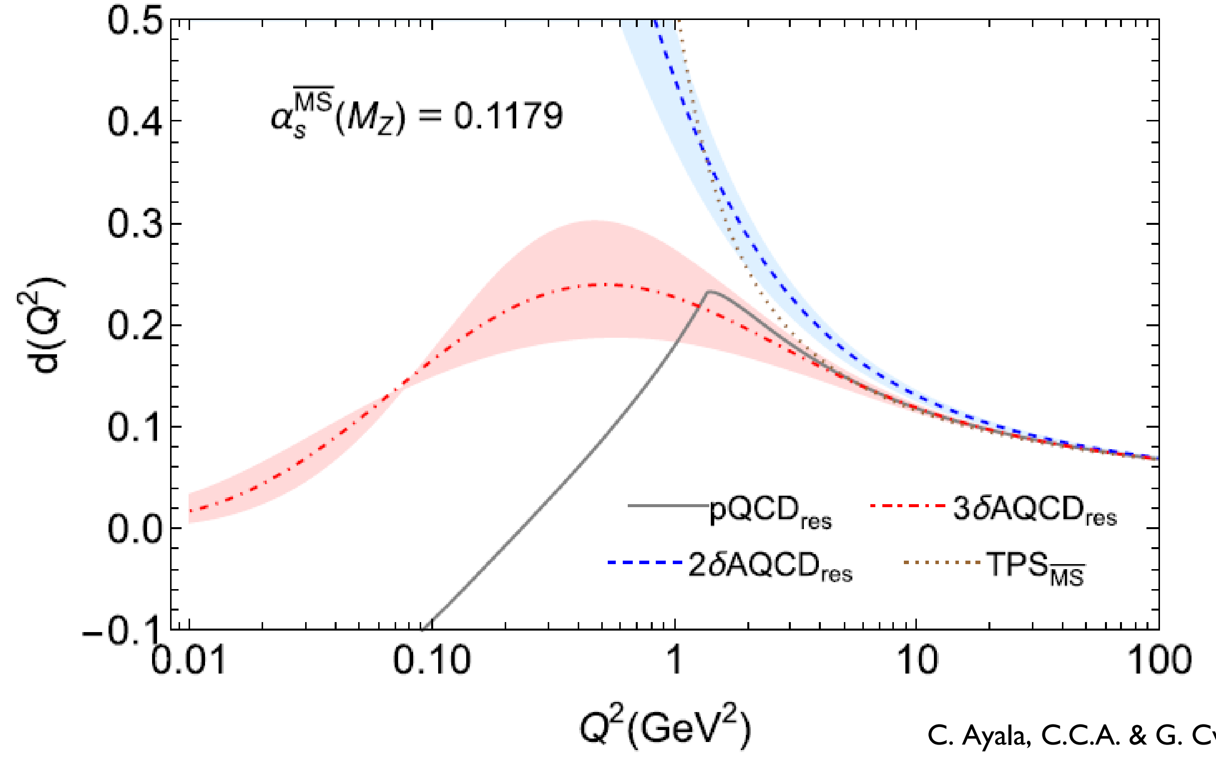
$$a_0 = 1 + \sqrt{c_3/c_1^3}, \quad a_1 = c_2/c_1^2 + \sqrt{c_3/c_1^3}$$

- Doing so for our expected hierarchy thus means our preferred RS parameters are in the range:

$$c_2 = 9^{+2}_{-1.4}, \quad c_3 = 20 \pm 15.$$

RESULTS FOR THE RESUMMED TERM

- With this scheme and comparing different couplings used for the resummed term:



C. Ayala, C.C.A. & G. Cvetič (2024)

- For pQCD the method clearly stops working at Q^2 < 1.44 GeV^2 due to Landau singularities in $a\left(Q^2\right)_{pQCD}$, although it still outperforms using Truncated Series (TPS).

RESULTS FOR THE FULL OBSERVABLE

- For adjusting the higher twist terms of the OPE we can either choose to truncate at D=2 or D=4, which gives:

$$\bar{f}_2^{(\mathrm{pQCD})} = -0.107^{-0.001}_{+0.007}(c_2)^{+0.022}_{-0.029}(c_3) \pm 0.020(\alpha_s) \mp 0.009(d_4) \mp 0.067(\mathrm{ren}) \\ {}^{+0.012}_{-0.029}(Q^2_{\mathrm{min}}) \pm 0.033(\mathrm{exp.u.}) \pm 0.154(\mathrm{exp.c.}),$$

$$\bar{f}_2^{(2\delta)} = -0.0082^{+0.0124}_{-0.0087}(c_2)^{+0.0130}_{-0.0176}(c_3)^{+0.0283}_{-0.0274}(\alpha_s)^{-0.0001}_{+0.0003}(d_4)^{-0.0763}_{+0.0541}(M_1)^{-0.0209}_{+0.0210}(a_2 d_2) \\ {}^{-0.0006}_{+0.0035}(Q^2_{\mathrm{min}}) \pm 0.0038(\mathrm{exp.u.}) \pm 0.0207(\mathrm{exp.c.}) \pm 0.0001(k).$$

$$\bar{f}_2^{(3\delta)} = -0.2534^{+0.0110}_{-0.0109}(\alpha_s) \pm 0.0001(d_4)^{+0.0447}_{-0.0427}(M_1)^{-0.0187}_{+0.0186}(a_2 d_2) \\ {}^{-0.0044}_{+0.0019}(Q^2_{\mathrm{min}}) \pm 0.0042(\mathrm{exp.u.}) \pm 0.0202(\mathrm{exp.c.}) \mp 0.0013(k).$$

$$\bar{f}_2^{(\mathrm{pQCD})} = -0.160^{-0.007}_{+0.025}(c_2)^{+0.054}_{-0.039}(c_3)^{+0.044}_{-0.041}(\alpha_s)^{-0.012}_{+0.016}(d_4) \mp 0.043(\mathrm{ren}) \\ {}^{+0.016}_{+0.119}(Q^2_{\mathrm{min}}) \pm 0.160(\mathrm{exp.u.}) \pm 0.297(\mathrm{exp.c.}),$$

$$\mu_6^{(\mathrm{pQCD})} = +0.022^{+0.003}_{-0.008}(c_2)^{-0.013}_{+0.004}(c_3)^{-0.010}_{+0.008}(\alpha_s)^{+0.002}_{-0.003}(d_4) \mp 0.010(\mathrm{ren}) \\ {}^{-0.006}_{-0.053}(Q^2_{\mathrm{min}}) \pm 0.062(\mathrm{exp.u.}) \mp 0.059(\mathrm{exp.c.})\ [\mathrm{GeV}^4],$$

$$\bar{f}_2^{(2\delta)} = +0.0041^{+0.0151}_{-0.0109}(c_2)^{+0.0083}_{-0.0148}(c_3)^{+0.0489}_{-0.0466}(\alpha_s) \pm 0.0034(d_4)^{-0.0770}_{+0.0557}(M_1)^{-0.0129}_{+0.0128}(a_2 d_2) \\ {}^{+0.0388}_{-0.0238}(Q^2_{\mathrm{min}}) \pm 0.0233(\mathrm{exp.u.}) \pm 0.1314(\mathrm{exp.c.}) \mp 0.0125(k),$$

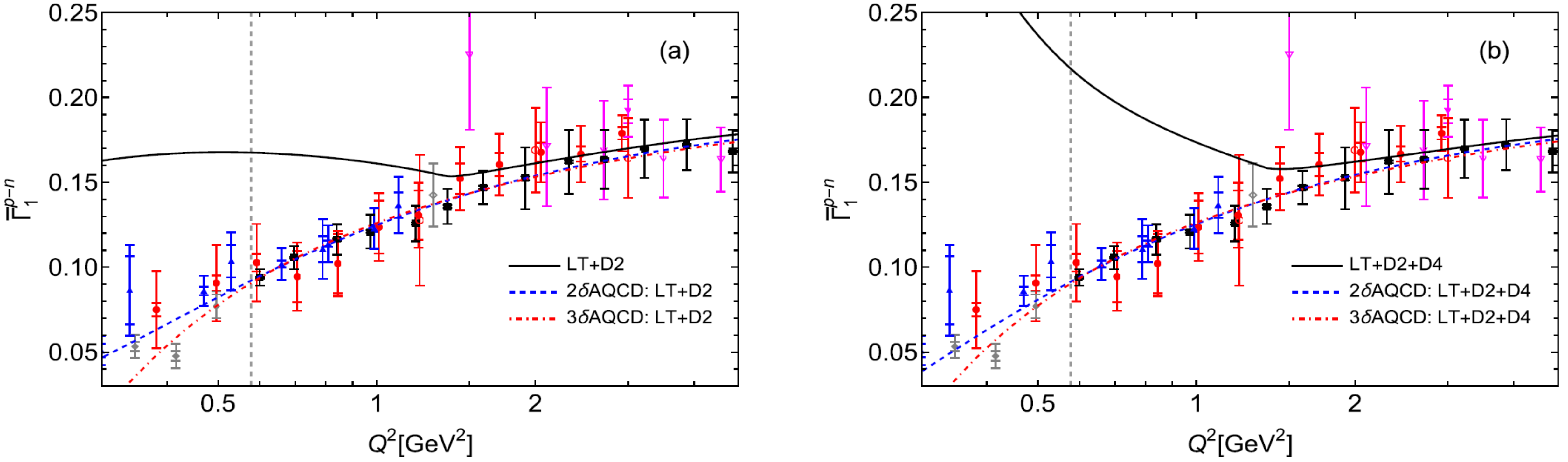
$$\mu_6^{(2\delta)} = -0.0016 \mp 0.0004(c_2)^{+0.0006}_{-0.0004}(c_3)^{-0.0029}_{+0.0025}(\alpha_s) \mp 0.0005(d_4)^{+0.0001}_{-0.0002}(M_1) \mp 0.0011(a_2 d_2) \\ {}^{-0.0076}_{+0.0035}(Q^2_{\mathrm{min}}) \pm 0.0031(\mathrm{exp.u.}) \mp 0.0148(\mathrm{exp.c.}) \pm 0.0016(k)\ [\mathrm{GeV}^4].$$

$$\bar{f}_2^{(3\delta)} = -0.2488^{+0.0244}_{-0.0237}(\alpha_s)^{-0.0014}_{+0.0013}(d_4)^{+0.0781}_{-0.0823}(M_1)^{-0.0148}_{+0.0149}(a_2 d_2) \\ {}^{+0.0229}_{-0.0132}(Q^2_{\mathrm{min}}) \pm 0.0247(\mathrm{exp.u.}) \pm 0.1193(\mathrm{exp.c.}) \mp 0.0201(k),$$

$$\mu_6^{(3\delta)} = -0.0006^{-0.0018}_{+0.0017}(\alpha_s) \pm 0.0002(d_4)^{-0.0045}_{+0.0051}(M_1) \mp 0.0005(a_2 d_2) \\ {}^{-0.0040}_{+0.0020}(Q^2_{\mathrm{min}}) \pm 0.0032(\mathrm{exp.u.}) \mp 0.0131(\mathrm{exp.c.}) \pm 0.0025(k)\ [\mathrm{GeV}^4].$$

- In all cases the theoretical uncertainties are lower than those from experiment.

RESULTS FOR THE FULL OBSERVABLE



C. Ayala, C.C.A. & G. Cvetič (2024)

SUMMARY OF RESULTS AND OUTLOOK

- As can be seen, the method generates a consistent resummation using the known renormalon structure of the BSR, which eliminates the problems of integrating directly in the Borel plane, or those of using a truncated perturbative expansion series.
- The nonperturbative information that is conceivably introduced by using the renormalon resummation of the sum rule is shown to improve the evaluation potential of these kind calculations down to lower Q^2 values below the usual pQCD regime, and an even better improvement is reached when using analytic versions of the QCD coupling (AQCD) which eliminate Landau singularities.
- It is interesting to apply this method to more observables, especially any of phenomenological interest for low-energy physics, as the combination of the Sum Rules approach and renormalon-resummation could be powerful for probing this region.

RECENT UPDATE!!!

(https://arxiv.org/abs/2509.09042)

Renormalon-based resummation for spacelike and timelike QCD quantities whose perturbation expansion has general form

César Ayala[a],* Gorazd Cvetič[b],† and Reinhart Kögerler[c‡]
[a] *Departamento de Ingeniería y Tecnologías, Sede La Tirana, Universidad de Tarapacá, Av. La Tirana 4802, Iquique, Chile*
[b] *Department of Physics, Universidad Técnica Federico Santa María, Avenida España 1680, Valparaíso, Chile and*
[c] *Department of Physics, Universität Bielefeld, 33501 Bielefeld, Germany*
(Dated: September 12, 2025)

We present a generalisation of our previous approach of a renormalon-motivated resummation of the QCD observables. Previously it was applied to the spacelike observables whose perturbation expansion was $\mathcal{D}(Q^2) = a(Q^2) + \mathcal{O}(a^2)$, where $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ is the running QCD coupling. Now we generalise the resummation to spacelike quantities $\mathcal{D}(Q^2) = a(Q^2)^{\nu_0} + \mathcal{O}(a^{\nu_0+1})$ and timelike quantities $\mathcal{F}(\sigma) = a(\sigma)^{\nu_0} + \mathcal{O}(a^{\nu_0+1})$, where ν_0 is in general a noninteger number ($0 < \nu_0 \leq 1$). We evaluate with this approach a timelike quantity, namely the scheme-invariant factor of the Wilson coefficient of the chromomagnetic operator in the heavy-quark effective Lagrangian, and related quantities.

- Now it has been generalised to also apply to timelike quantities and noninteger powers of the canonical term, so more observables can be probed. Also, six theorems are proven which more robustly show the validity of the approach.