NANJING
UNIVERSITY

Pion and Kaon Fragmentation Functions

Hui-Yu Xing （邢惠瑜）

2025.11.13

In collaboration with: Prof. Craig D. Roberts

QCD: Emergent Phenomena

Craig D. Roberts, David G. Richards, Tanja Horn, Lei Chang
Prog. Part. Nucl. Phys. 120 (2021) 103883

➢ Two fundamental phenomena in QCD

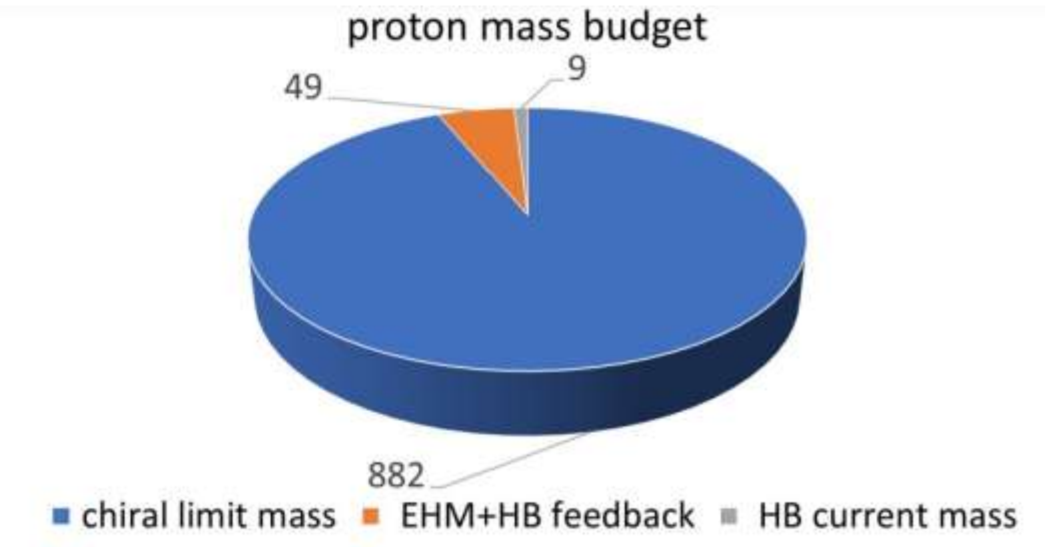
- Emergent Hadron Mass (EHM)
- Confinement

➢ Proton mass budget $\quad M_{A=N+Z} \approx N^{*}m_n + Z^{*}m_p$

Only 9 MeV/939 MeV is directly from Higgs

➢ Evidently, there is another phenomenon in Nature that is extremely effective in producing mass:

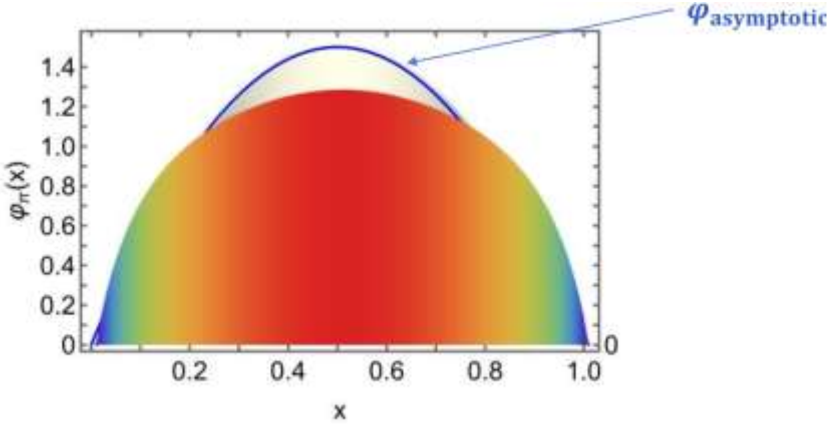
Emergent Hadron Mass (EHM)



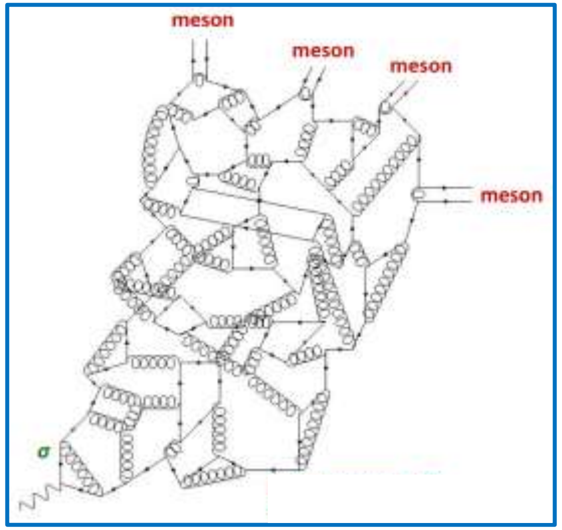
➢ EHM are expressed in every strong interaction observables.

➢ EHM generates broadening in parton distribution function (DF) & distribution amplitude (DA).

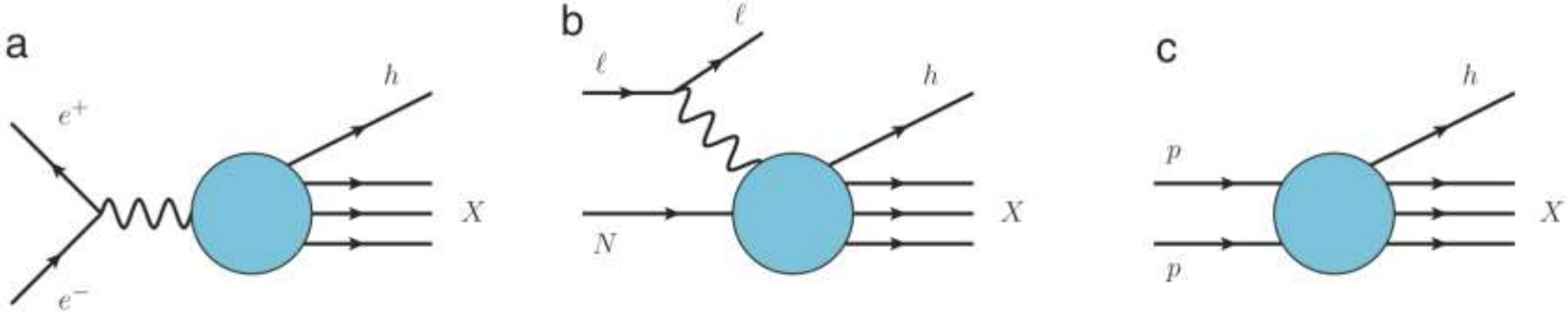
Hui-Yu Xing et al., Phys. Lett. B 849 (2024) 138462



Fragmentation Function



- High energy interaction often produce jets of energetic hadrons
  - nearly parallel longitudinal momenta & relative small transverse momenta
- Such jets are normally understood to originate with gluon and quark partons
  - produced in the initial collision
  - Escape interaction region
  - Driven by "confinement forces", fragment into a shower of colourless hadrons
- Hadronisation processes are described by fragmentation functions (FFs)
  - $D_1^{h/i}(z)$ is the number of hadrons h inside parton i in the light-front momentum-fraction range $[z, z+dz]$
  - it plays a crucial rule in the following processes:



$e^+ + e^- \to h + X$: $\sigma \sim \hat{\sigma} \otimes FF$

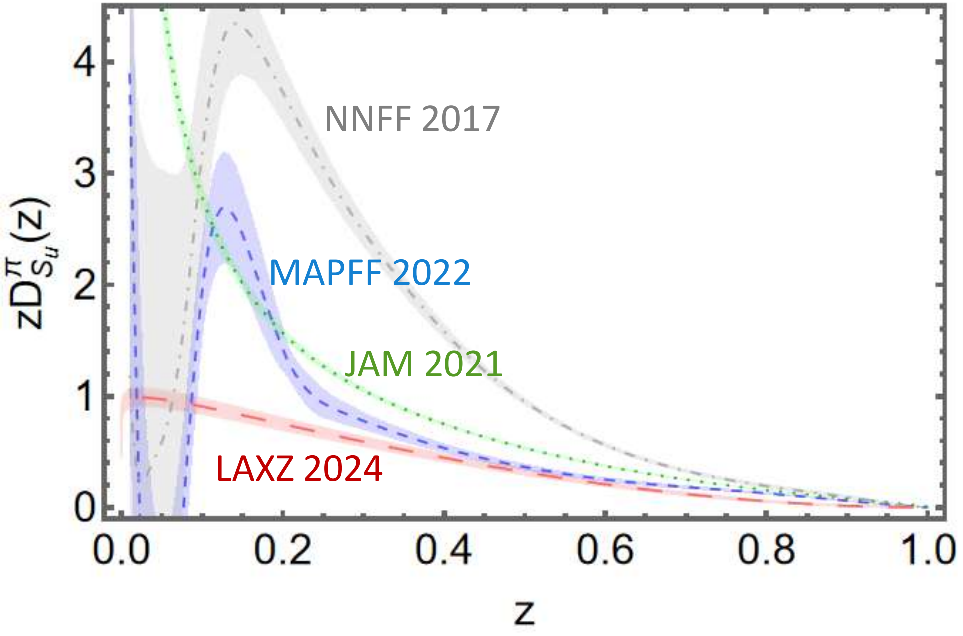
$\ell + N \to \ell + h + X$: $\sigma \sim \hat{\sigma} \otimes PDF \otimes FF$

$p + p \to h + X$: $\sigma \sim \hat{\sigma} \otimes PDF \otimes PDF \otimes FF$

Current development

- What is known about FFs? *Still limited*

- Experiment: TASSO 1982, JADE 1985, TPC 1988, OPAL 1994, DELPHI 1998, ALEPH 2000, SLD 2004, Belle 2013, BaBar 2013, BESIII 2023 ……
  - Data are used in global fits for FFs

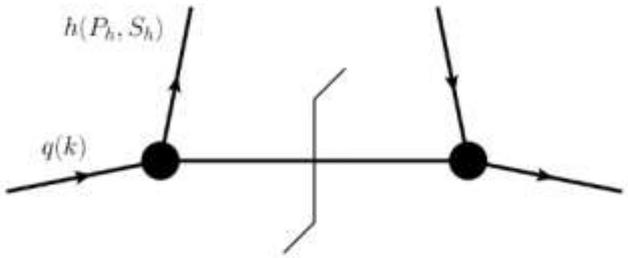


- Fit: HKNS 2007, DSS 2007, NNFF 2017, JAM 2021, MAPFF 2022, LAXZ 2024, NPC 2024 ……
  - Model dependent: mutually inconsistent

Which is correct? What should the FFs look like?

- Theory:
  - Like DFs, FFs are nonperturbative objects
  - Simplest version = spectator models

    a time-like off-shell parton fragments into a hadron



Hitherto, no realistic results & QCD prediction have been available

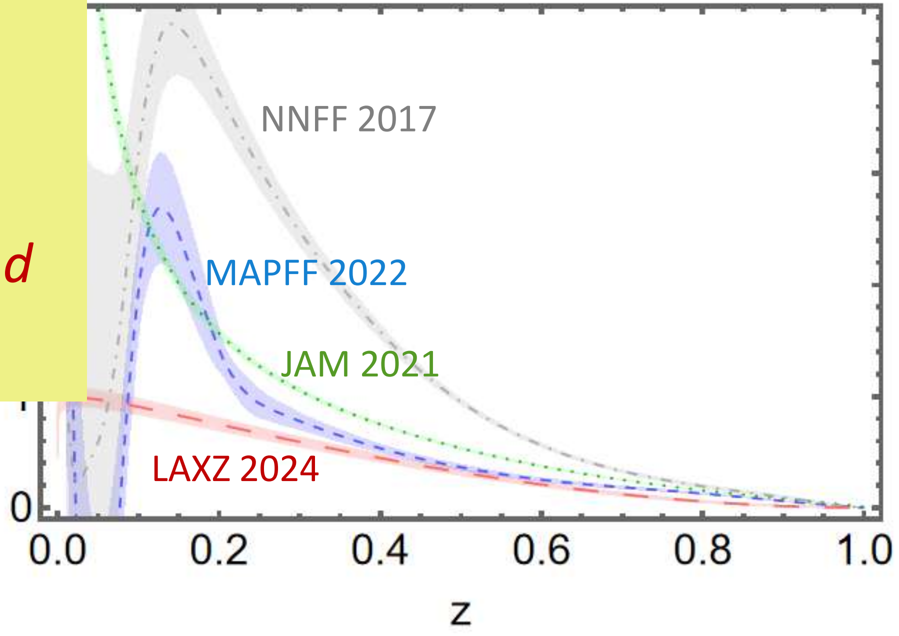
Current development

- What is known about FFs? *Still limited*

- Experiment: TASSO 1982, JADE 1985, TPC 1988, OPAL 1994, DELPHI 1998, ALEPH 2000, SLD 2004,

- *Uncertain information in*

  ⇒ uncertain information out
- *Difficult to learn anything about DFs and hadron structure*
- *Serious impediment to interpretation of data from modern and anticipated facilities*



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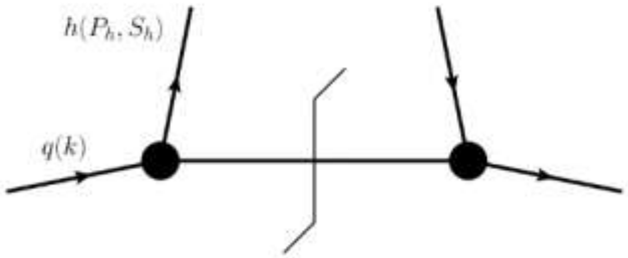
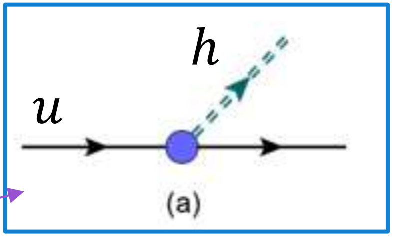


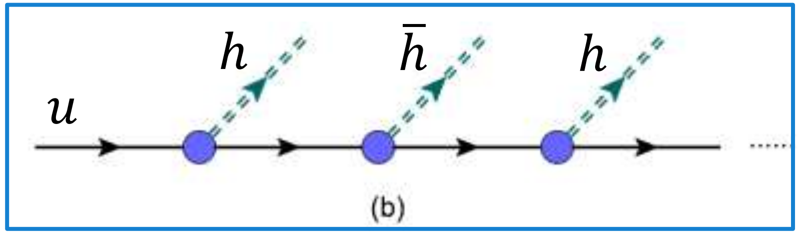
Hitherto, no realistic results & QCD prediction have been available

DLY relation and jet equation

<u>Models</u>

- Field + Feynman Jet Fragmentation approach
  - Readily handles multiple hadron production





- Basic need is elementary fragmentation function (EFF):

$$d_p^h(z;\zeta)$$

which is mathematical probability that initial parton, p, produces hadron, h in one single process

$$\boxed{D_p^h(z)} = \boxed{d_p^h(z)} + \int_z^1 (dy/y)\, d_p^h(1-z/y) D_p^h(y)$$

- Using $d_q^h(z;\zeta)$, one solves coupled set of linear cascade equations to determine complete FF, $D_q^h(z;\zeta)$, which accounts for all hadrons produced from the initial quark

- Sum rule: $\sum_h \int_0^1 dz\, z D_p^h(z;\zeta) = 1$
  - The hadron jet generated by parton p contains all momentum of initial state

DLY relation and Elementary Fragmentation Functions

- FF is timelike twin of the spacelike process

$$\gamma^*(Q^2) + h \to X \Leftrightarrow \gamma^*(Q^2) \to h + X$$

- Crossing symmetry ⇒ relation between DFs and FFs

  Drell-Levy-Yan (DLY) Relation

  – Compute elementary FFs by analytic continuation of DFs onto domain $x > 1$

$$d_q^h(z;\zeta) \propto z q^h(\frac{1}{z};\zeta)$$

  – DLY was proved using operator definitions and crossing symmetry

S. D. Drell, D. J. Levy, T.-M. Yan, A Theory of Deep Inelastic Lepton Nucleon Scattering and Lepton Pair Annihilation Processes. 2. Deep Inelastic electron Scattering,

Phys. Rev. D 1 (1970) 1035–1068.

S. D. Drell, D. J. Levy, T.-M. Yan, A Theory of Deep Inelastic Lepton Nucleon Scattering and Lepton Pair Annihilation Processes. 3. Deep Inelastic electron-positron Annihilation,

Phys. Rev. D 1 (1970) 1617-1639.

- Fragmentation Functions and Confinement

  – FFs express how a shower of coloured partons coalesce into colour singlet final states … this is the empirical expression of confinement

  – Overlap representation of parton DFs

$$q^h(x;\zeta) \sim \int d^2k_\perp \left|\Psi_q^h(x,k_\perp^2)\right|^2$$

✓ *Seeds of confinement, expressed in hadronisation, are already present in wave functions of the hadrons involved*

✓ *EHM, expressed in DF and LFWFs, modulates the FFs, so that the hadronisation and confinement process*

Predictions for Fragmentation Functions using cascade equations

- Empirically, at energies below ~ 100 GeV,
  π, K dominate (98%) particle production
  in SIA: $e^+ + e^- \to h + X$
- So, illustrate procedure via predictions for π, K FFs
- Exploit G-parity symmetry, one has set of 9 coupled Volterra integral equations of 2nd kind

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Kaon and Pion Fragmentation Functions

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Accepted by Eur. Phys. J. C

favoured

$$D_u^{\pi^+}(z) = d_u^{\pi^+}(z) + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{\pi^+}(y)\,,$$

$$D_u^{K^+}(z) = d_u^{K^+}(z) + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{K^+}(y)\,,$$

$$D_s^{K^-}(z) = d_s^{K^-}(z) + \int_z^1 \frac{dy}{y} \sum_{q=u,d} d_s^q(\frac{z}{y}) D_q^{K^-}(y)\,,$$

unfavoured

$$D_u^{\pi^-}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{\pi^-}(y)\,,$$

$$D_u^{K^-}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{K^-}(y)\,,$$

$$D_u^{K^0}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{K^0}(y)\,,$$

$$D_u^{\bar{K}^0}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d,s} d_u^q(\frac{z}{y}) D_q^{\bar{K}^0}(y)\,,$$

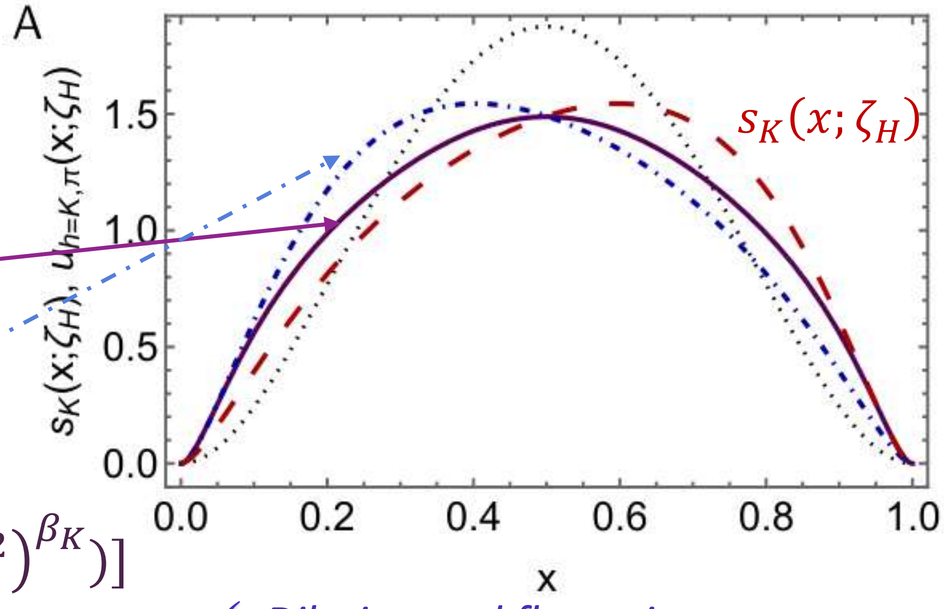
$$D_s^{\pi^+}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d} d_s^q(\frac{z}{y}) D_q^{\pi^+}(y)\,,$$

$$D_s^{K^+}(z) = 0 + \int_z^1 \frac{dy}{y} \sum_{q=u,d} d_s^q(\frac{z}{y}) D_q^{K^+}(y)\,.$$

Pion and Kaon DFs ⇒ EFFs

- Continuum (CSMs) predictions for hadron scale pion and kaon valence-quark DFs

$$u_\pi(x;\zeta_H) = n_\pi \ln[1 + (\frac{1}{\rho^2})x^2(1-x)^2$$
$$(1+\gamma_\pi^2\left[([1-x)^2]^{\beta_\pi} + (x^2)^{\beta_\pi}\right)]$$
$$n_\pi = 0.858, \rho_\pi = 0.116, \gamma_\pi = 1.967, \beta_\pi = 5.938$$
$$u_K(x;\zeta_H) = n_K \ln[1 + (\frac{1}{\rho_K^2})x^2(1-x)^2 \ (1+\gamma_K^2(x^2)^{\alpha_K}([1-x]^2)^{\beta_K})]$$
$$s_K(x;\zeta_H) = u_K(1-x;\zeta_H)$$
$$n_K = 0.444, \rho_K = 0.0746, \gamma_K = 6.276, \alpha_K = 0.710, \beta_K = 1.650$$



- ✓ *Dilation and flattening are expressions of EHM*
- ✓ *Skewing for kaon is expression of EHM + Higgs-Boson interference*

Kaon and pion parton distributions, Zhu-Fang Cui *et al.*, Eur. Phys. J. C 80 (2020) 11, 1064

- In functional form is perfect for using DLY relation

✓ *satisfy the QCD constrain* $q^\pi(x \simeq 1;\zeta) \propto (1-x)^{2+\gamma(\zeta)}$

⇒ *EFF for mesons:* $d_p^h(z \simeq 1;\zeta) \propto (1-z)^{2+\gamma(\zeta)}$ through DLY

✓ *vanish at the endpoints*

✓ *no divergence for EFF*

are not satisfied by some fits ex: HKNS

TABLE IV: Parameters determined for the pion.

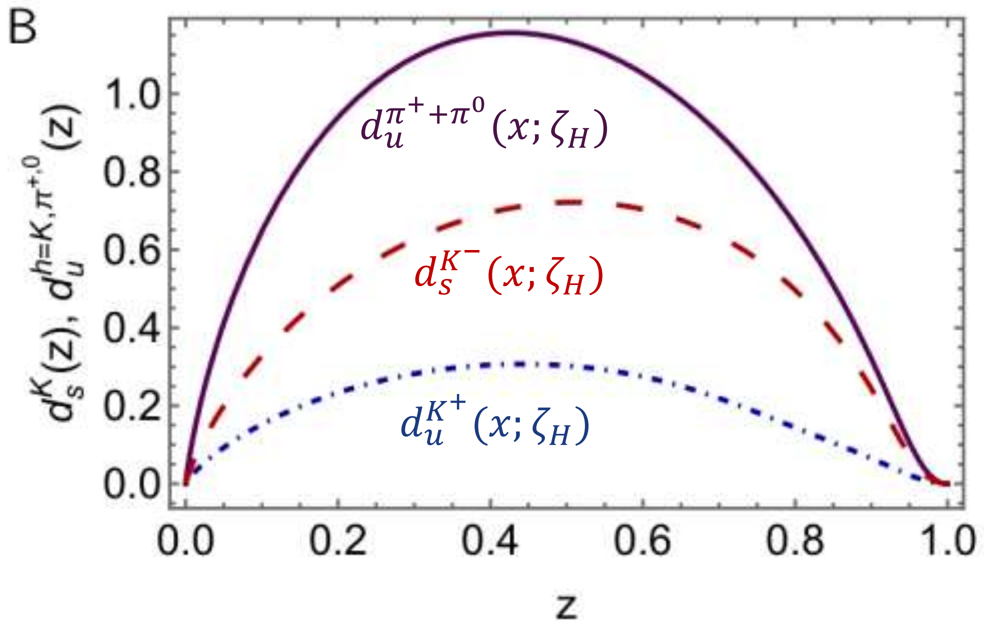
function	M	α	β
(LO)			
$D_u^{\pi^+}$	0.546 ± 0.085	−1.100 ± 0.183	$\boxed{1.282 \pm 0.140}$
$D_{\bar u}^{\pi^+}$	0.250 ± 0.068	−0.500 ± 0.301	5.197 ± 0.576
$D_c^{\pi^+}$	0.305 ± 0.046	−1.007 ± 0.123	3.918 ± 0.236
$D_b^{\pi^+}$	0.302 ± 0.023	−1.176 ± 0.045	5.805 ± 0.188
$D_g^{\pi^+}$	0.115 ± 0.111	1.405 ± 0.897	8.0 (fixed)

Elementary Fragmentation Functions

- DLY relation $d_q^h(z;\zeta) \propto z q^h(\frac{1}{z};\zeta)$
- CSMs EFFs are given at right
- Normalised

$$\int_0^1 dz\,\left[\tfrac{3}{2}d_u^{\pi^+}(z;\zeta_{\mathcal{H}}) + d_u^{K^+}(z;\zeta_{\mathcal{H}})\right] = 1$$

Elementary probability that u quark produces some kind of hadron is unity

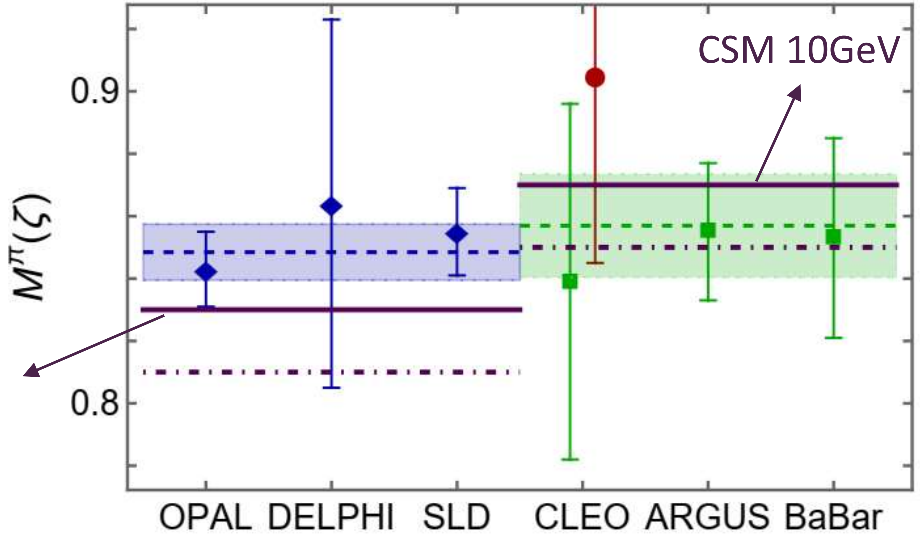


- Prediction: EFF multiplicities

$$m_u^\pi = \int_0^1 dz\,\tfrac{3}{2}d_u^{\pi^+}(z;\zeta_{\mathcal{H}}) = 0.80\,,$$

$$m_u^K = \int_0^1 dz\,d_u^{K^+}(z;\zeta_{\mathcal{H}}) = 0.20\,.$$

$\Rightarrow$ determine total multiplicities at all scales



Pion FF Results

- Solve cascade equations at $\zeta_H \Rightarrow$ complete FFs
- Use all-orders (AO) scheme to evolve solutions to scales relevant for measurements, *e.g.*, $\zeta = \zeta_2 := 2$ GeV
  - AO scheme is nonperturbative extension of DGLAP

See José Rodríguez-Quintero's talk tomorrow

- Approach guarantees that particle and momentum sum rules are preserved

$$\sum_h \int_0^1 dz\, z D_p^h(z;\zeta) = 1$$

Table 2 SCI FF momentum fractions obtained from solutions of the cascade equations at the hadron scale and after evolution to $\zeta = \zeta_2 := 2\,\text{GeV}$, following the prescription described in Sect. 4. (No entry means the fraction is zero. $c \to q \to h$ contributions are negligible in all cases.)

h	$\pi^+ + \pi^0 + \pi^-$		K^+	
	$\zeta_{\mathcal{H}}$	ζ_2	$\zeta_{\mathcal{H}}$	ζ_2
$\langle z\rangle^h_{D_{S_u^u}}$	0.664	0.433	0.182	0.119
$\langle z\rangle^h_{D_{S_u^d}}$		0.115		0.032
$\langle z\rangle^h_{D_{S_u^s}}$		0.085		0.023
$\langle z\rangle^h_{D_{S_u^c}}$		0.031		0.009
$\langle z\rangle^h_{D_{S_d^u}}$		0.115		0.007
$\langle z\rangle^h_{D_{S_d^d}}$	0.664	0.443	0.042	0.028
$\langle z\rangle^h_{D_{S_d^s}}$		0.085		0.005
$\langle z\rangle^h_{D_{S_d^c}}$		0.031		0.002
$\langle z\rangle^h_{D_{S_s^u}}$		0.017		0.069
$\langle z\rangle^h_{D_{S_s^d}}$		0.017		0.069
$\langle z\rangle^h_{D_{S_s^s}}$	0.098	0.059	0.396	0.239
$\langle z\rangle^h_{D_{S_s^c}}$		0.005		0.019
$\langle z\rangle^\zeta_{D^h_{g_u}}$	0.083	0.083	0.023	0.023
$\langle z\rangle^\zeta_{D^h_{g_d}}$	0.083	0.083	0.005	0.005
$\langle z\rangle^\zeta_{D^h_{g_s}}$	0.012	0.012	0.050	0.050

Pion FF Results

- Approach guarantees that particle and momentum sum rules are preserved

$$\sum_h \int_0^1 dz\, zD_p^h(z;\zeta) = 1$$

- Momentum sum rule is typically *not enforced* in global fitting schemes because …
  - No experiment can reach $z = 0$.
  - For a given set of data, the empirical hadron final-state space is unknown because hadrons in addition to $\pi, K, p, \bar{p}$ might be produced and yet go undetected.
- Contemporary FF fit procedures are typically only able to ensure that the results are consistent with some set of practitioner-dependent momentum sum rule bounds or inequalities.

Table 2 SCI FF momentum fractions obtained from solutions of the cascade equations at the hadron scale and after evolution to $\zeta = \zeta_2 := 2\,\text{GeV}$, following the prescription described in Sect. 4. (No entry means the fraction is zero. $c \to q \to h$ contributions are negligible in all cases.)

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Pion FF Results

- Approach guarantees that particle and momentum sum rules are preserved

$$\sum_h \int_0^1 dz\, z D_p^h(z;\zeta) = 1$$

$$\sum_{\text{all h}} \int_0^1 dz\, z \left[\sum_q D^h_{S_u^q}(z;\zeta_2) + D^h_{g_u}(z;\zeta_2) \right]$$

$$= \overset{u\to u\to\pi}{0.433} + \overset{u\to d\to\pi}{0.115} + \overset{u\to s\to\pi}{0.085} + \overset{u\to c\to\pi}{0.031}$$
$$+ \overset{u\to u\to K^{+-}}{0.119} + \overset{u\to d\to K^{+-}}{0.032} + \overset{u\to s\to K^{+-}}{0.023} + \overset{u\to c\to K^{+-}}{0.009}$$
$$+ \overset{d\to u\to K^{+-}}{0.007} + \overset{d\to d\to K^{+-}}{0.028} + \overset{d\to s\to K^{+-}}{0.005} + \overset{d\to c\to K^{+-}}{0.002}$$
$$+ \overset{u(g)\to\pi}{0.083} + \overset{u(g)\to K^+K^-}{0.023} + \overset{u(g)\to K^0\bar{K}^0}{0.005} \qquad (34)$$
$$= 1.0\,. \qquad (35)$$

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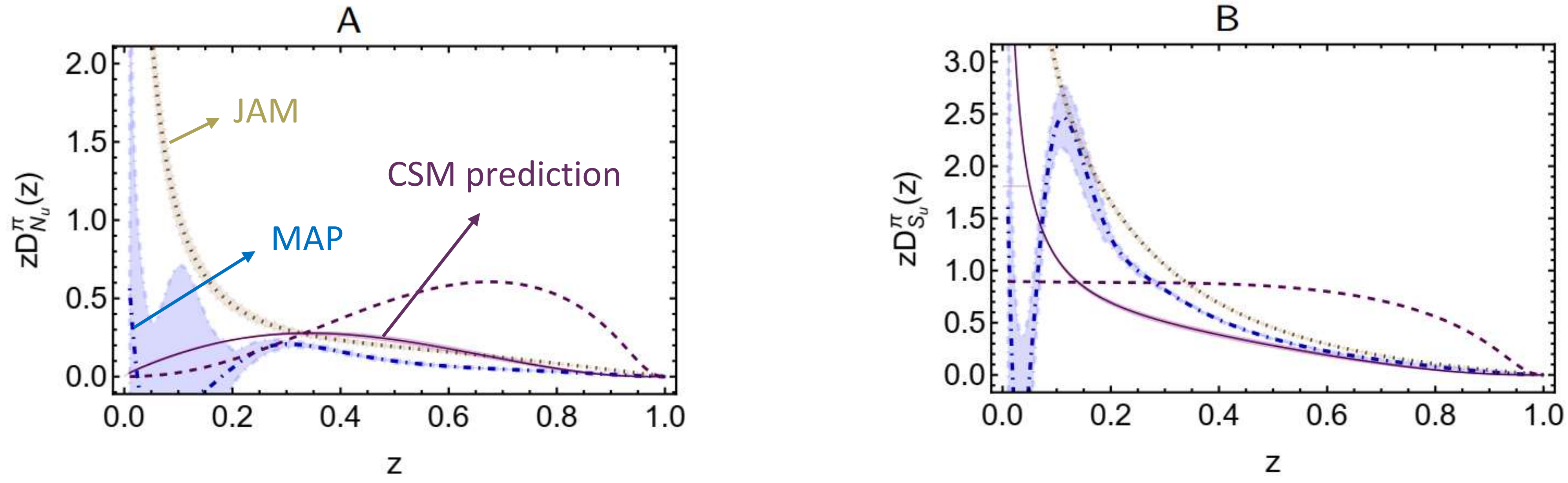
Pion FF Results



- A, B $u \to \pi^{+,0}$ (favoured) nonsinglet and singlet.

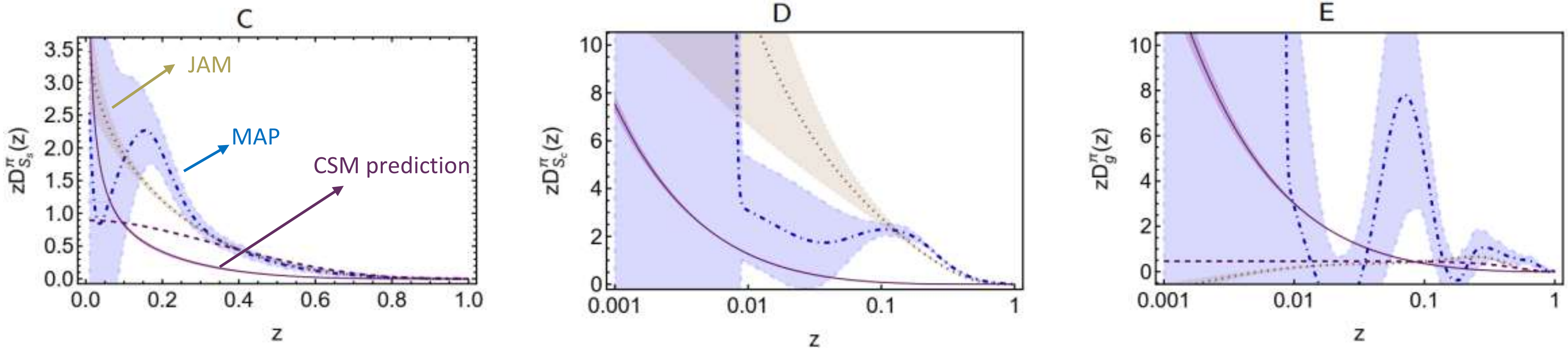
$$zD_N = \frac{3}{2} z[D_q^{\pi^+}(z) - D_{\bar{q}}^{\pi^+}(z)]$$

$$zD_S = \frac{3}{2} z[D_q^{\pi^+}(z) + D_{\bar{q}}^{\pi^+}(z)]$$

  - Agreement only on $z > 0.5$, *i.e.*, valence quark domain.
  - JAM nonsinglet FF result (zD_N) exhibits unexpected & unphysical divergence on $z \simeq 0$.

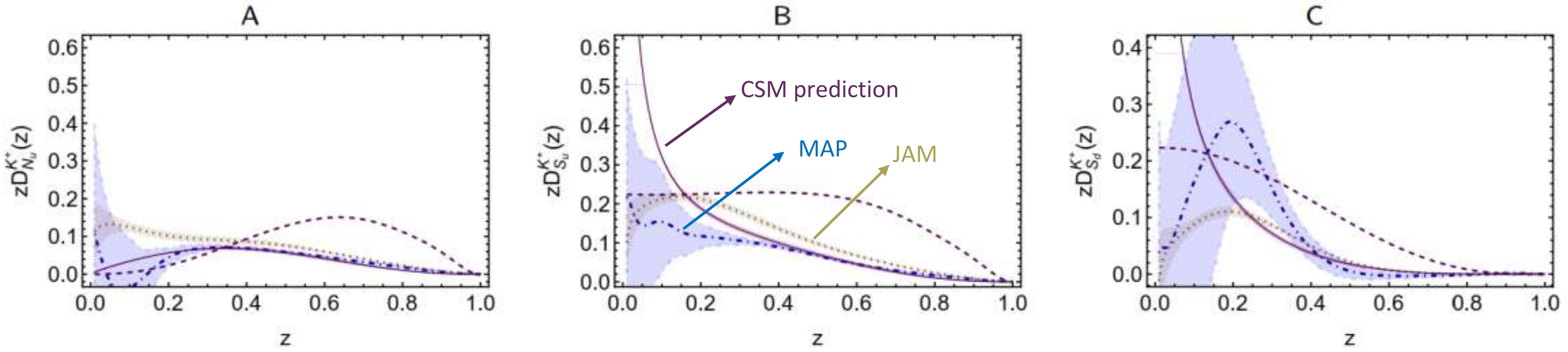
    This is the domain of glue and sea dominance; so, $zD_N = \frac{3}{2} z[D_q^{\pi^+}(z) - D_{\bar{q}}^{\pi^+}(z)]$ should vanish.
  - MAPFF fits highlight that FFs are practically unconstrained on $z \lesssim 0.2$.

Pion FF Results



- C $s \to \pi$ (unfavoured).
  - One might say there is qualitative agreement on the far valence domain, but only in the sense that this FF is small.
  - Otherwise, any agreement is only the result of an accidental curve crossing.
- D, E $\mathrm{c}, \mathrm{g} \to \pi$ (unfavoured).
  - There is ***no*** agreement on these FFs, which are very poorly constrained by data.
  - MAPFF: practically unconstrained on $z \lesssim 0.2$.
  - JAM: negative contribution on z<0.01, which is unphysical.

Kaon FF Results



- A, B $u \to K$ (favoured) nonsinglet and singlet.

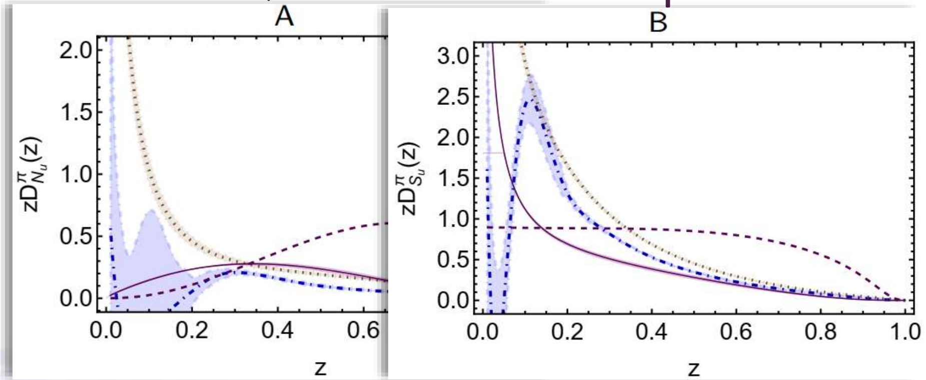
$$zD_N = z[D_q^{\pi^+}(z) - D_{\bar{q}}^{\pi^+}(z)]$$

$$zD_S = z[D_q^{\pi^+}(z) + D_{\bar{q}}^{\pi^+}(z)]$$

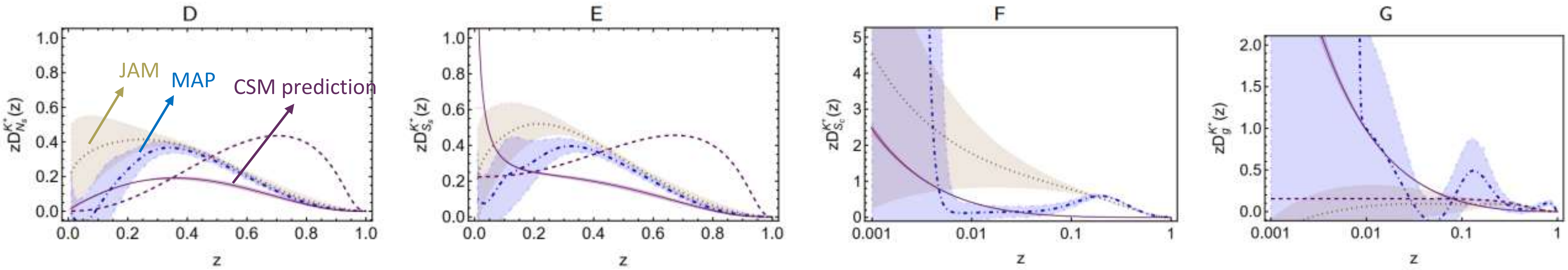
  - Agreement on $z > 0.4$, *i.e.*, valence quark domain.
  - JAM nonsinglet FF result (zD_N) is finite and nonzero on $z \simeq 0$, again, unexpected.

    Singlet FF result (zD_S) is also finite and nonzero, in contradiction of its $u \to \pi$ result and our prediction.

- C d$\to K$ (favoured) singlet.
  - qualitative agreement on the far valence domain.
  - JAM and MAPFF fits produce nonzero finite values on $z \simeq 0$.



Kaon FF Results



- D, E s→ K (favoured) nonsinglet and singlet.
  - Agreement is seen on $z \gtrsim 0.7$; but nothing beyond that.
  - Both JAM and MAPFF produce nonzero finite values on $z \simeq 0$.

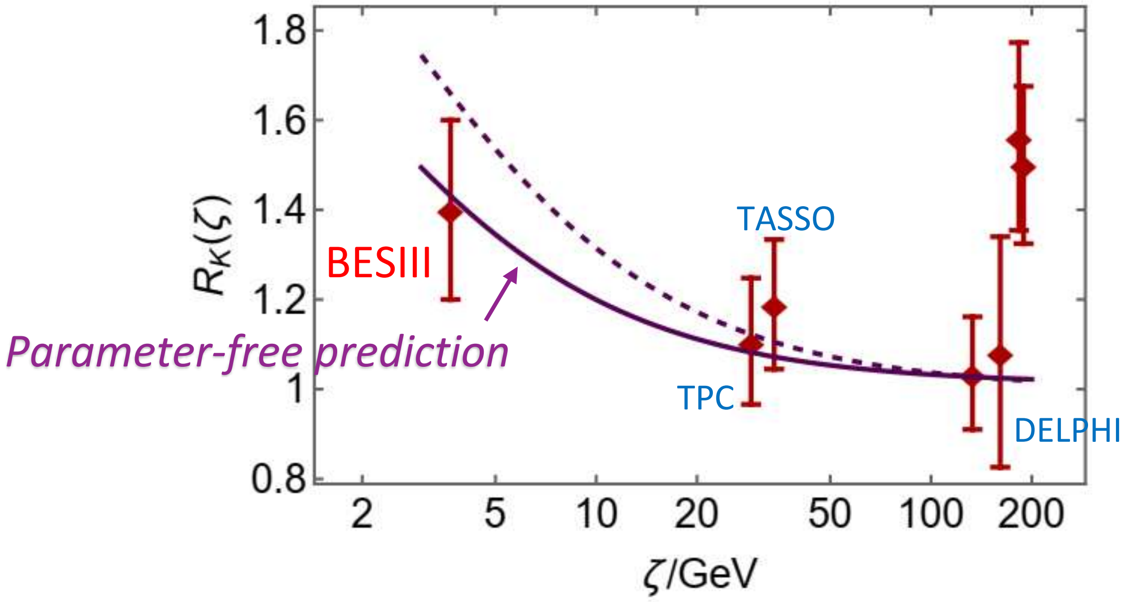
$$zD_N = z[D_q^{\pi^+}(z) - D_{\bar{q}}^{\pi^+}(z)]$$

$$zD_S = z[D_q^{\pi^+}(z) + D_{\bar{q}}^{\pi^+}(z)]$$

- F, G c, g → K (unfavoured).
  - Again, there is *no* agreement on these FFs, which are very poorly constrained by data.
  - JAM and MAP: c quark FF is divergent, inconsistent with the their other FFs.

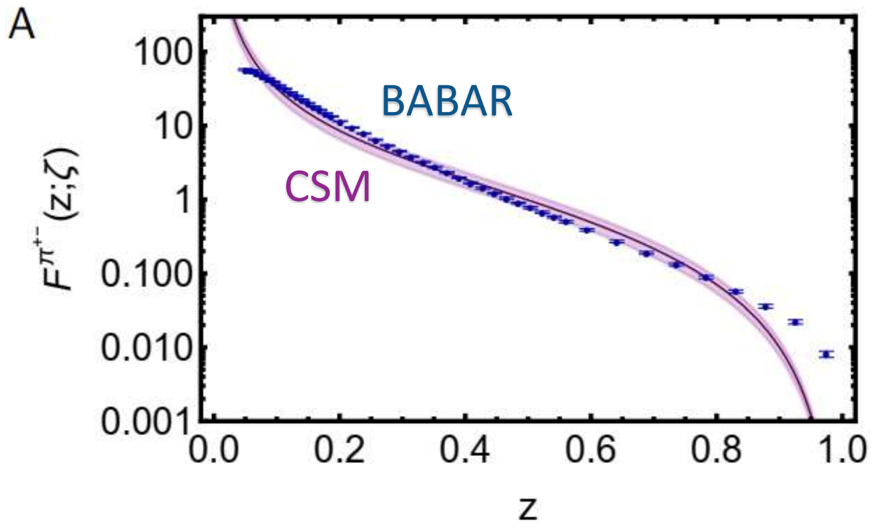
Hadron Jet Multiplicities

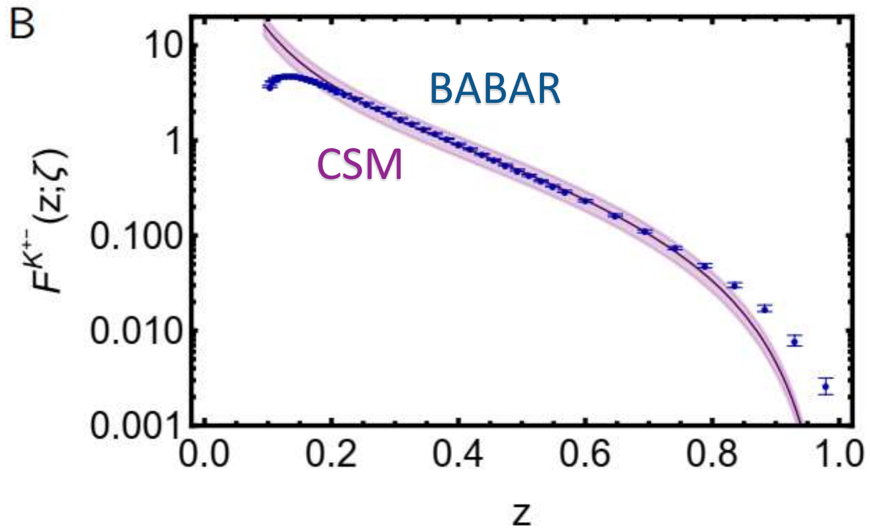
- Relative multiplicity of charged and neutral kaons in $e^+ + e^- \to h + X$ $\quad R_K(\zeta) = \dfrac{M^{K^+}(\zeta) + M^{K^-}(\zeta)}{M^{K^0}(\zeta) + M^{\bar{K}^0}(\zeta)}$
- Data exist, most recently from BESIII, last year



- ✓ Predictions for hadron jet multiplicities reveal SU(3)-flavour symmetry breaking in $R_K(\zeta)$.
- ✓ Breaking significant at reaction energy scales $\zeta \approx 3m_p$, but decreases in size with increasing reaction energy.

- z-dependent multiplicities





Summary

- A unified treatment of the pion, kaon DFs and FFs was accomplished.
- Give insights into the link between two important phenomena in QCD: EHM and confinement.
- Continuum predictions provide coherent picture of fragmentation across all parton types $\Rightarrow$ Through comparison with fits:
  - largely model dependent, mutually inconsistent
  - QCD constrain is not satisfied
  - momentum sum rule is not enforced, some are larger, some are smaller
  - unconstrained on small z domain
  - unexpected & unphysical behaviour on $z \simeq 0$
- These must be improved if anything objective is to be learnt from modern and anticipated facilities.

Thank you!