BARYONS 2025

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Resolving the flavor-dependency puzzle with quark three-body forces

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Intro & Motivation

Known discrepancies of quark models

- **Configuration dependent:** single parameter set cannot describe mesons and baryons simultaneously.
  If meson data fit → baryon spectrum goes off (~100 MeV error)

- **Large spin splittings:** $\Delta - N$ split, $\Sigma^* - \Sigma, \Sigma - \Lambda$ split, etc… → naive usual hyperfine term

- **Flavor dependencies (for baryons):** light overestimated, heavy less over- or underestimated

[S. Pepin and Fl. Stancu, PRD (2002), W. Park et. al., PRD (2015), M. Karliner and J. L. Rosner, PRD (2014), C. Semay and B. Silvestre-Brac, Nucl. Phys. A (1999)]

Intro & Motivation

Present remedies

- **Fit (baryons + mesons) together:** flavor dependency remains

- **Use separate parameter sets:** one for mesons, one for baryons

- **Use χ^2 (including uncertainties) as the test statistic rather than RMSE:**

$$\text{RMSE} = \sum_i \left(m_i^{\text{thr}} - m_i^{\text{exp}}\right)^2 \rightarrow \chi^2 = \sum_i \frac{\left(m_i^{\text{thr}} - m_i^{\text{exp}}\right)^2}{\sigma_i^2}$$

to avoid overestimation of nucleon series

Intro & Motivation

Since QCD allows 3Q forces, a two-body potential for quarkonium cannot describe baryons well.

~~Present remedies~~ We add explicit 3Q forces for baryons

- ~~**Fit (baryons + mesons) together:** flavor dependency remains~~

  Fit one parameter set to meson-data only → predict the baryons (no refit)

- ~~**Use separate parameter sets:** one for mesons, one for baryons~~

  A unified single parameter set

- ~~**Use χ^2 (including uncertainties) as the test statistic rather than RMSE:**~~

$$\text{RMSE} = \sum_i \left(m_i^{\text{thr}} - m_i^{\text{exp}}\right)^2 \quad \longleftarrow \quad \chi^2 = \sum_i \frac{\left(m_i^{\text{thr}} - m_i^{\text{exp}}\right)^2}{\sigma_i^2}$$

Use RMSE: uncertainty of nucleon is too small, it overfits to nucleon

Model description (two-body potential) One-gluon exchange (OGE) model

- **Static potential (color-type)**

$$V_{ij}^{C}(r) = -\frac{3}{4}F_i^c F_j^c\left(-\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a_0^2} - D\right)$$

For (anti)fundamental quarks
$F_i^c = \lambda_i^c/2,\ F_i^c = -(F_i^c)^*\ (c = 1, \cdots, 8)$
Coulomb term: short range lattice data fit
Linear term: Wilson loop area law
Constant term: related to the vacuum energy
(depends on the renorm. scheme in lattice QCD)

- **Spin-dependent potential (color-spin type)**

$$V_{ij}^{CS}(r) = \frac{\hbar^2\kappa'_{ij}}{m_i m_j c^4 r_{0ij}}\frac{e^{-(r_{ij}/r_{0ij})^2}}{r_{ij}}F_i^c F_j^c\ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$$

Y. Koma and M. Koma, Nucl. Phys. B (2007),
S. Noh et. al., PLB (2025)

Here the parameters r_{0ij}, κ'_{ij} are defined as
$r_{0ij} = (\alpha + \beta m_{ij})^{-1}, \kappa'_{ij} = \kappa_0(1 + \gamma m_{ij}), m_{ij} = \frac{m_i m_j}{m_i + m_j}$

in heavy quark limit, $V^{CS}\sim$ a contact interaction
(as in the perturbative QCD)

$$V_{ij}^{CS}(r) \xrightarrow{m_{i,j}\to\infty} \propto \delta^{(3)}(\mathbf{r})$$

$$\frac{\kappa'_{ij}}{r_{0ij}}\frac{e^{-(r/r_{0ij})^2}}{r} \xrightarrow{m_{i,j}\to\infty} \frac{2\pi\kappa_0}{\alpha}\delta^{(3)}(\mathbf{r}) \quad \text{if } \gamma = \beta/\alpha$$

Model description

Here we don't consider

- Coupled channel effects (unquenching):
  $\bar{q}q(^3P_0)$ pair-creation vertex showed large but nearly multiplet-independent downward shifts for the $1S, 1P, 2S$ levels—such corrections can be absorbed into quenched fitting parameters [Barnes–Swanson PRC (2007) for mesons; Tecocoatzi et al. EPJA (2007) for baryons]
- Goldstone boson exchange (chiral quark model):
  For example, pion-exchange effect becomes significant $r \sim 1$ fm scale, improving excited spectroscopy. We consider the ground-state baryons. [L. Ya. Glozman and D. O. Riska, Phys. Rep. (1996)]
- Other Fermi-Breit terms (Tensor, LS, Darwin):
  Less significant than the hyperfine color-spin term for ground-state baryons.

Mesons

We use a simple non-relativistic quark model Hamiltonian to describe meson data:

$$H = \sum_i \left(m_i + \frac{\mathbf{p}_i^2}{2m_i} \right) - \frac{3}{4} \sum_{i<j} F_i^c F_j^c \left(V_{ij}^C + V_{ij}^{CS} \right)$$

Gaussian spatial basis function (Gaussian expansion method; GEM) is used:

$$\phi_{nl}(\mathbf{r}) = N_{nl}\, r^l\, e^{-\nu_n r^2} Y_{lm}(\hat{\mathbf{r}})$$

30 Gaussians used to describe spatial wavefunctions.

Mesons

Fit the model parameters to meson data (RMSE minimized)

Meson ($\mathcal{J}^{P(C)}$)	$\eta_c\,(0^{-+})$	$J/\psi\,(1^{--})$	$D\,(0^{-})$	$D^*\,(1^{-})$	$\pi\,(0^{-+})$	$\rho\,(1^{--})$	$K\,(0^{-})$	$K^*\,(1^{-})$
Experimental Value (MeV)	2983.6	3096.9	1864.8	2010.3	139.57	775.11	493.68	891.66
30 Gaussians ($\sigma =$ 1.36 MeV)	2984.0	3095.8	1863.5	2013.3	139.58	774.17	493.77	891.45
Deviation ($M - M_{\text{exp}}$ in MeV)	+0.4	−1.1	−1.3	+3.0	+0.01	−0.94	+0.09	−0.21

Parameters fit to ground and spin excited data (RMSE = 1.36 MeV):

$$\kappa = 85.2189 \text{ MeV fm} \qquad a_0 = 0.0298683 \text{ (fm/MeV)}^{1/2} \qquad D = 1082.19 \text{ MeV} \qquad \kappa_0 = 207.473 \text{ MeV}$$

$$\alpha = 0.911659 \text{ fm}^{-1} \qquad \beta = 0.00090649 \text{ (MeV fm)}^{-1} \qquad \gamma = 0.00120125 \text{ MeV}^{-1}$$

$$m_{ud} = 316.551 \text{ MeV} \qquad m_s = 595.328 \text{ MeV} \qquad m_c = 1882.96 \text{ MeV}$$

(within the lattice–informed range)

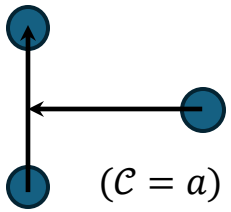
Baryons (two-body Hamiltonian)

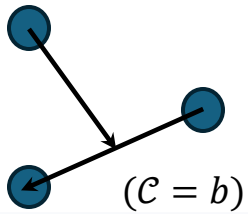
Using the meson fit parameters, predict baryons.
Describe the baryon eigenfunction as (S-, P-, D-waves)

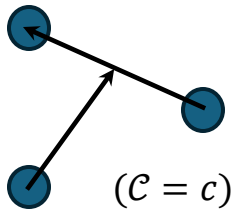
$$\Psi^{\text{baryon}}_{\mathcal{J}^P M} = \sum_{\mathcal{C}=a,b,c} \sum_{\{l,L,J,S,n,N\}} \mathcal{B}_{\{l,L,J,S,n,N\}} \mathcal{S}^{\text{baryon}} \times \left[[\phi_{n_\mathcal{C} l_\mathcal{C}}(\mathbf{r}_\mathcal{C}) \phi_{N_\mathcal{C} L_\mathcal{C}}(\mathbf{R}_\mathcal{C})]^J_P \otimes S \right]^{\mathcal{J}} \otimes F \otimes \mathcal{P}_1 C$$

S: spin part, F: flavor part, $\mathcal{P}_1 C$: singlet projected color part, $\mathcal{S}$: proper symmetrizer
$\mathcal{C}$: Jacobi coordinates ($\mathcal{C} = a, b, c$)

$$\mathbf{r}_a = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}, \qquad \mathbf{R}_a = \sqrt{\frac{2}{3}} \left(\frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3 \right) \quad (\mathcal{C} = a)$$







Baryons (two-body Hamiltonian)

Use infinitesimally shifted Gaussian (ISG) identity [Hiyama et. al, PPNP (2001)]

$$r^l e^{-\nu_n r^2} Y_{lm}(\hat{\mathbf{r}}) = \lim_{\epsilon \to 0} \frac{1}{(\nu_n \epsilon)^l} \sum_{k=1}^{k_{\max}} C_{lm,k} e^{-\nu_n (\mathbf{r} - \epsilon \mathbf{D}_{lm,k})^2}$$

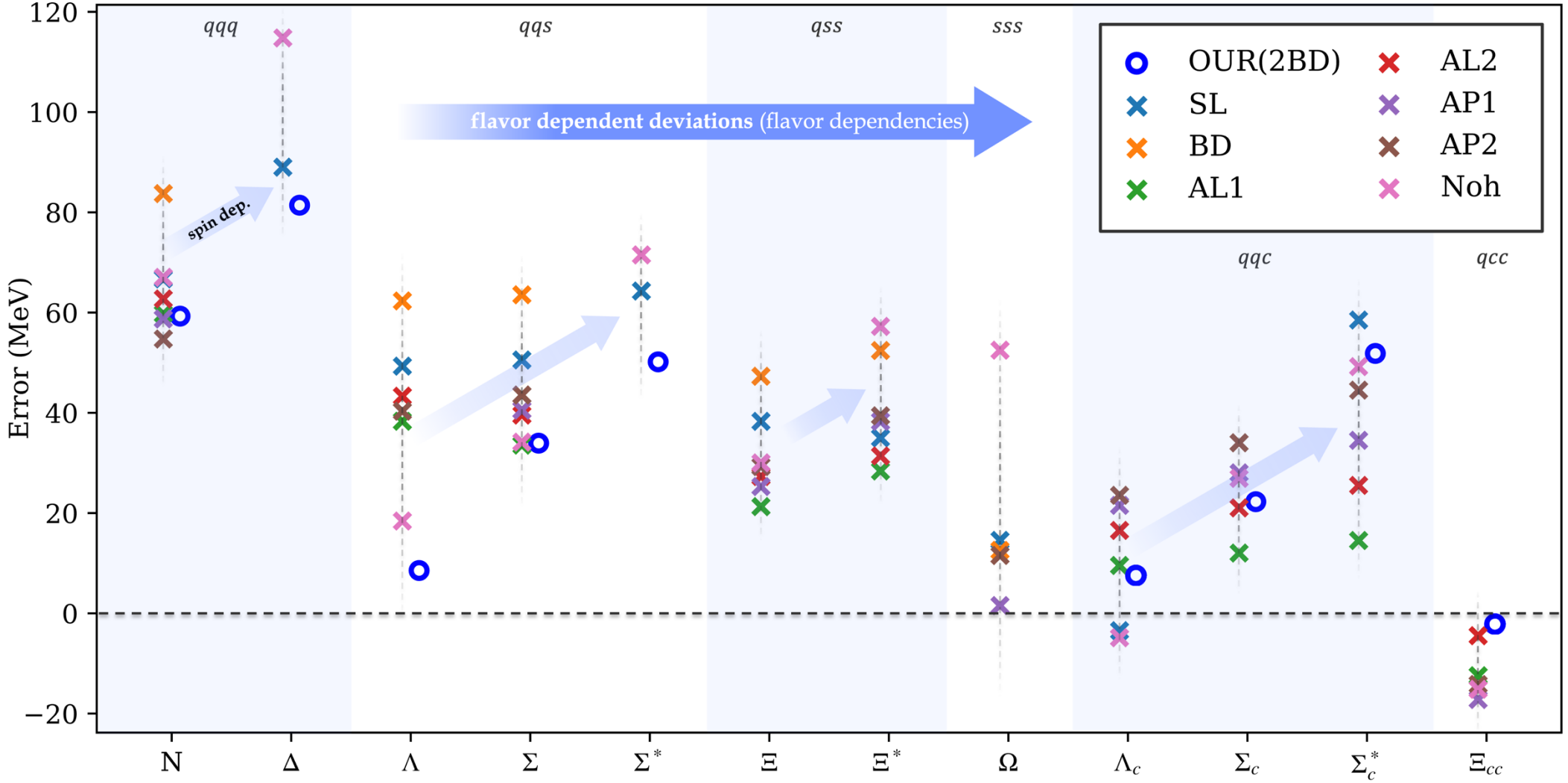
Solving the Schrödinger equation (generalized EVP) $H\Psi = EO\Psi$
Ground state baryons:

Baryon $(\mathcal{J}^P)$	$p\,(1/2^+)$	$\Delta\,(3/2^+)$	$\Sigma\,(1/2^+)$	$\Lambda\,(1/2^+)$	$\Sigma_c\,(1/2^+)$	$\Lambda_c\,(1/2^+)$	$\Xi_{cc}\,(?^?)$[5]	$\Sigma^*\,(3/2^+)$	$\Sigma_c^*\,(3/2^+)$
Experimental Mass (MeV)	938.27	1232.00	1197.45	1115.68	2453.97	2286.46	3621.46	1383.70	2518.48
$M^{\text{GEM}}_{S,P,D}$ (MeV, $\sigma = 46.28$)	997.56	1313.42	1231.40	1124.22	2476.24	2294.00	3619.29	1433.87	2570.33
Deviation $(M^{\text{GEM}}_{S,P,D} - M_{\text{exp}})$	+59.29	+81.42	+33.95	+8.54	+22.27	+7.54	−2.17	+50.17	+51.85

→ a significant discrepancy with exp. data.

Error plot of various quark models

GEM or solving Faddeev eq.



SL [D. P. Stanley and D. Robsen, PRL (1980)]

BD [R. K. Bhaduri et. al., Nuovo Cim. A (1981)]

AL1

AL2

AP1 [C. Semay and B. Silvestre-Brac, Z. Phys. C (1994),

AP2 B. Silvestre-Brac, Few-body Syst. (1996)]

Noh [S. Noh et. al., PLB (2024)]

Pairwise combination of (meson-fit) two-body potential cannot reproduce baryons.

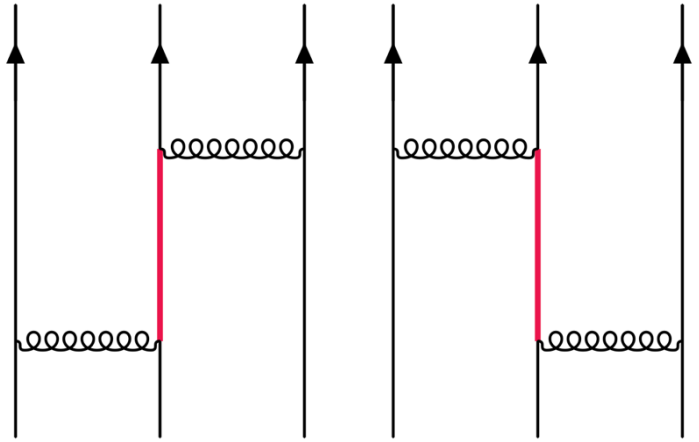
Quark three-body interaction (color, spin part)

[J. Fujita and H. Miyazawa, PTP (1957)]

Two pion exchange with an intermediate $\Delta \rightarrow$ 3N force
Two gluon exchange (TGE) with an intermediate $q^* \rightarrow$ 3Q force ?

Fujita-Miyazawa mechanism

Consider 2nd quark first



$$V_{23}S^{(2)}(p_2)V_{12} \;+\; V_{12}S^{(2)}(p_2)V_{23}$$

For the intermediate (color-spin excited) quark propagator, $S(p_2) \sim 1/M_2$ (static approximation) and use simply $M_i = km_i$

2 types in one-gluon exchange (no spatial dep.)

$$V_{ij}^{C} \sim \lambda_i \cdot \lambda_j \qquad V_{ij}^{CS} \sim \lambda_i \cdot \lambda_j \; \sigma_i \cdot \sigma_j$$

⬇ time-ordered combination

3 types in two-gluon exchange (TGE)

$$\boldsymbol{V^{C} \leftrightarrow V^{C}}: L_{(2)}^{C-C} = V_{23}^{C}S^{(2)}(p_2)\,V_{12}^{C} + V_{12}^{C}S^{(2)}(p_2)V_{23}^{C} \;\rightarrow\; L^{C-C}$$

$$\boldsymbol{V^{CS} \leftrightarrow V^{CS}}: L_{(2)}^{S-S} = V_{23}^{CS}S^{(2)}(p_2)V_{12}^{CS} + V_{12}^{CS}S^{(2)}(p_2)V_{23}^{CS} \;\rightarrow\; L^{S-S}$$

$$\boldsymbol{V^{C} \leftrightarrow V^{CS}}: L_{(2)}^{C-S} = V_{23}^{C}S^{(2)}(p_2)V_{12}^{CS} + V_{12}^{CS}S^{(2)}(p_2)V_{23}^{C} \;\rightarrow\; L^{C-S}$$

Quark three-body interaction (color, spin part)

Collecting all the symmetric combinations [$M_i = km_i$ $(k > 1)$]

$$L_{123}^{C-C} = \frac{4}{3} \sum_{\mathrm{cyc}(l,i,j)} \frac{\lambda_i^c \lambda_j^c}{M_l} + 2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c \sum_{l=1}^{3} \frac{\mathrm{sgn}(q_l)}{M_l},$$

$$L_{123}^{S-S} = \sum_{\mathrm{cyc}(l,i,j)} \frac{4\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\, \lambda_i^c \lambda_j^c}{3\, m_l M_l (m_1 m_2 m_3)} + \frac{2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c}{m_1 m_2 m_3} \sum_{\mathrm{cyc}(l,i,j)} \frac{\mathrm{sgn}(q_l)}{m_l M_l}\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - \frac{2\, f^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c\, \epsilon^{ijk} \sigma_1^i \sigma_2^j \sigma_3^k}{m_1 m_2 m_3} \sum_{l=1}^{3} \frac{1}{m_l M_l}$$

$$L_{123}^{C-S} = \frac{4}{3} \sum_{\mathrm{cyc}(l,i,j)} \frac{\lambda_i^c \lambda_j^c}{M_l} \left(\frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_i}{m_l m_i} + \frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_j}{m_l m_j} \right) + 2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c \sum_{\mathrm{cyc}(l,i,j)} \frac{\mathrm{sgn}(q_l)}{m_l M_l} \left(\frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_i}{m_i} + \frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_j}{m_j} \right).$$

For baryons, $\mathrm{sgn}(q_i) = 1$, $d^{abc}\lambda_1^a\lambda_2^b\lambda_3^c = 80/9$, $f^{abc}\lambda_1^a\lambda_2^b\lambda_3^c = 0$.

Treat $L^{3Q} = \sum_{i<j<k} \left(A L_{ijk}^{C-C} + B L_{ijk}^{S-S} + C L_{ijk}^{C-S} \right) \times \delta^{(3)}(\mathbf{r})\delta^{(3)}(\mathbf{R})$ as a perturbation.

Unknown fitting parameters

contact interaction

Quark three-body interaction (color, spin part)

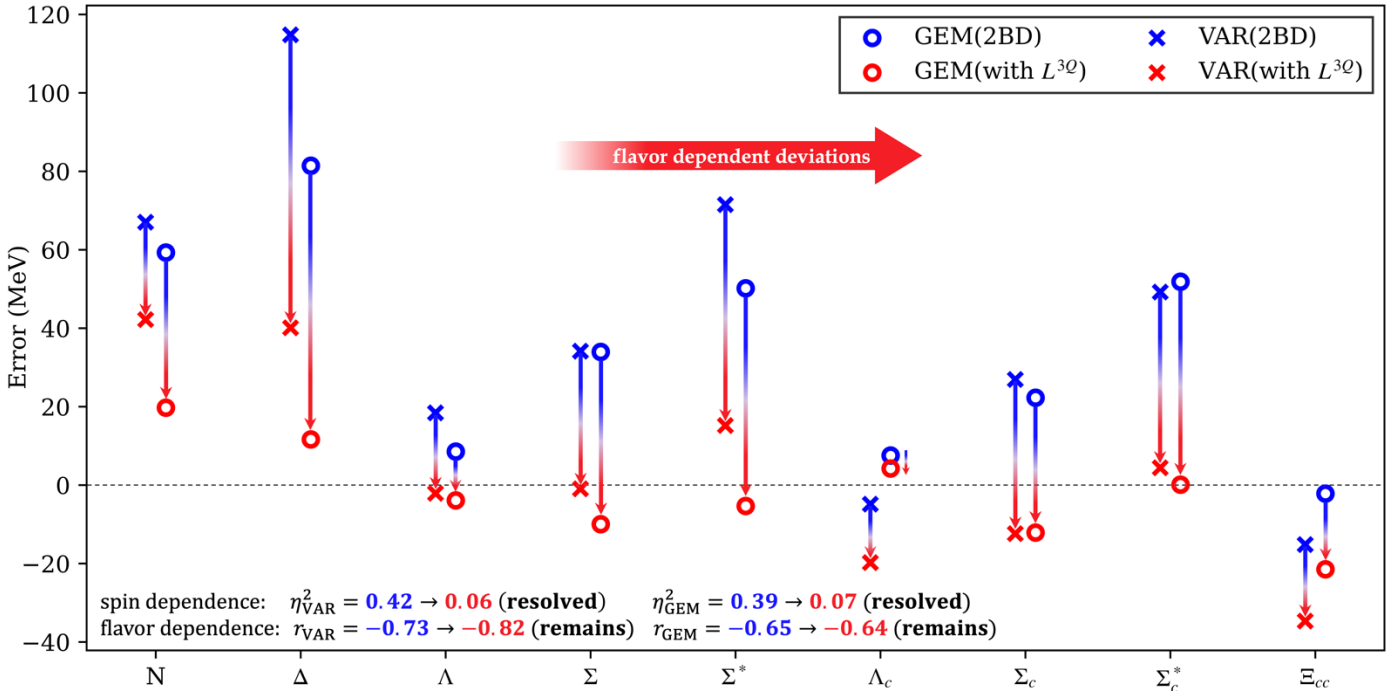
Treat $L^{3Q} = \sum_{i<j<k} \left(A L^{C-C}_{ijk} + B L^{S-S}_{ijk} + C L^{C-S}_{ijk} \right) \times \delta^{(3)}(\mathbf{r}) \delta^{(3)}(\mathbf{R})$ as a perturbation.

Unknown fitting parameters

Fit to the ground state baryon data,

Baryon $(M_{\rm exp}, \mathcal{J}^P)$	without L^{3Q} $(\sigma = 46.28)$	with L^{3Q} $(\sigma = 12.71)$
$p(938.27, 1/2^+)$	997.56(+59.29)	958.00(+19.73)
$\Delta(1232, 3/2^+)$	1313.42(+81.42)	1243.64(+11.64)
$\Sigma(1197.45, 1/2^+)$	1231.40(+33.95)	1187.48(−9.97)
$\Lambda(1115.68, 1/2^+)$	1124.22(+8.54)	1111.78(−3.90)
$\Sigma_c(2453.97, 1/2^+)$	2476.24(+22.27)	2441.86(−12.11)
$\Lambda_c(2286.46, 1/2^+)$	2294.00(+7.54)	2290.70(+4.23)
$\Xi_{cc}(3621.46, ?^?)$[a]	3619.29(−2.17)	3599.99(−21.47)
$\Sigma^*(1383.7, 3/2^+)$	1433.87(+50.17)	1378.37(−5.33)
$\Sigma_c^*(2518.48, 3/2^+)$	2570.33(+51.85)	2518.59(+0.11)

[a] We assumed $\mathcal{J}^P = 1/2^+$.



Quark three-body interaction (color, spin part)

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Unknown fitting parameters

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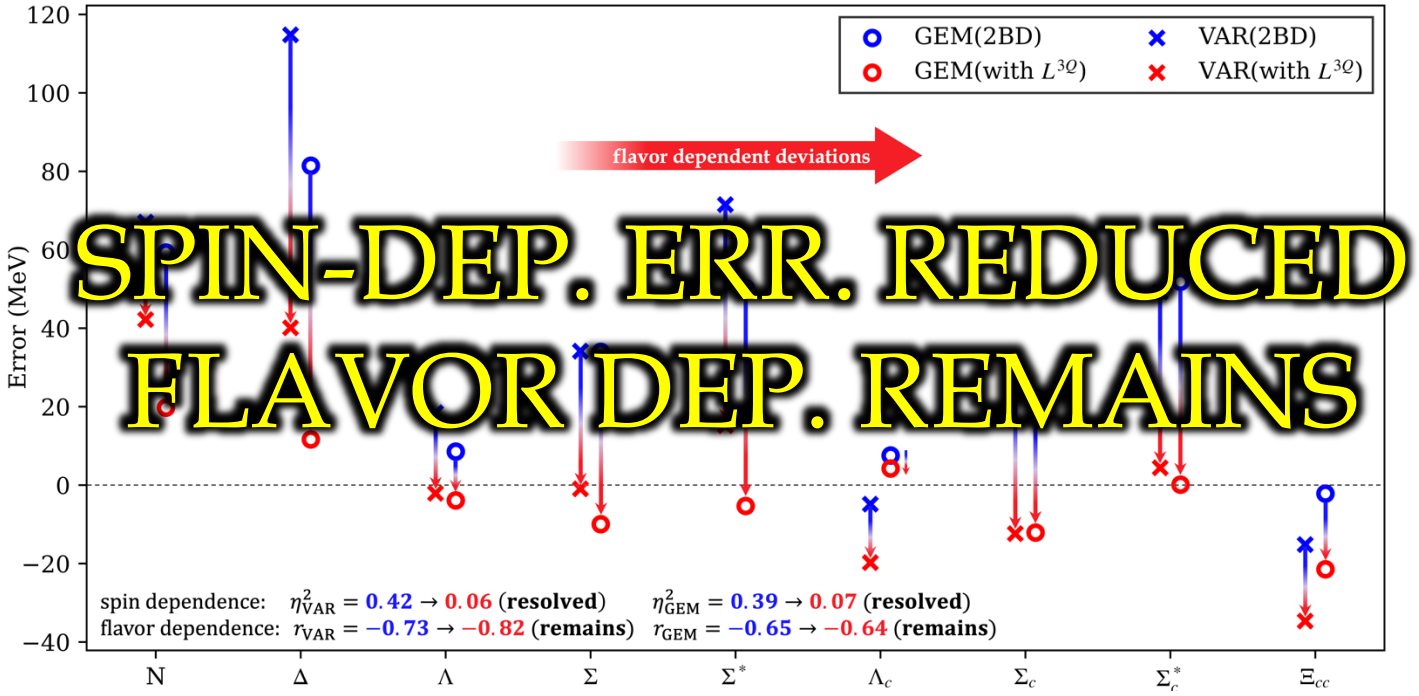

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FLAVOR DEP. REMAINS

Quark three-body interaction (short-range corrections)

Now focus on the spatial part.

Contact interaction does not capture nonperturbative effects of QCD.

→ How can we refine the spatial part of the three-body force?

$$L^{3Q} = \sum_{i<j<k} \left(A L_{ijk}^{C-C} + B L_{ijk}^{S-S} + C L_{ijk}^{C-S} \right) \times [\textbf{Spatial part?}]$$

Unknown fitting parameters

In the OGE-derived V^{CS}, **heavier quarks interact more closely (short-range nature):**

$$V_{ij}^{CS}(r) = \frac{\hbar^2 \kappa'_{ij}}{m_i m_j c^4 r_{0ij}} \frac{e^{-(r_{ij}/r_{0ij})^2}}{r_{ij}} F_i^c F_j^c \, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \xrightarrow{m_{i,j}\to\infty} \propto \delta^{(3)}(\mathbf{r})$$

Newly introduced three-body force must likewise respect the short-range effect.*

*Asymptotic freedom causes $\alpha_s(Q^2)$ to decrease with increasing Q (i.e., shorter distances),
so that TGE naturally generates local three-body operators whose strength is confined to a short range.

Quark three-body interaction (short-range corrections)

Model the spatial operator as the minimal combination of mr **for heavier quarks to interact more closely**.

Yukawa-type $$L_{\mathrm{Y}}^{3Q} = \sum_{i<j<k} \left(A L_{ijk}^{C-C} + B L_{ijk}^{S-S} + C L_{ijk}^{C-S} \right) \times \exp\left[-\frac{c}{\hbar} (m_{ij} r_{ij} + m_{jk} r_{jk} + m_{ki} r_{ki}) \right]$$

Unknown fitting parameters

$$m_{ij} = \frac{2 m_i m_j}{m_i + m_j}$$

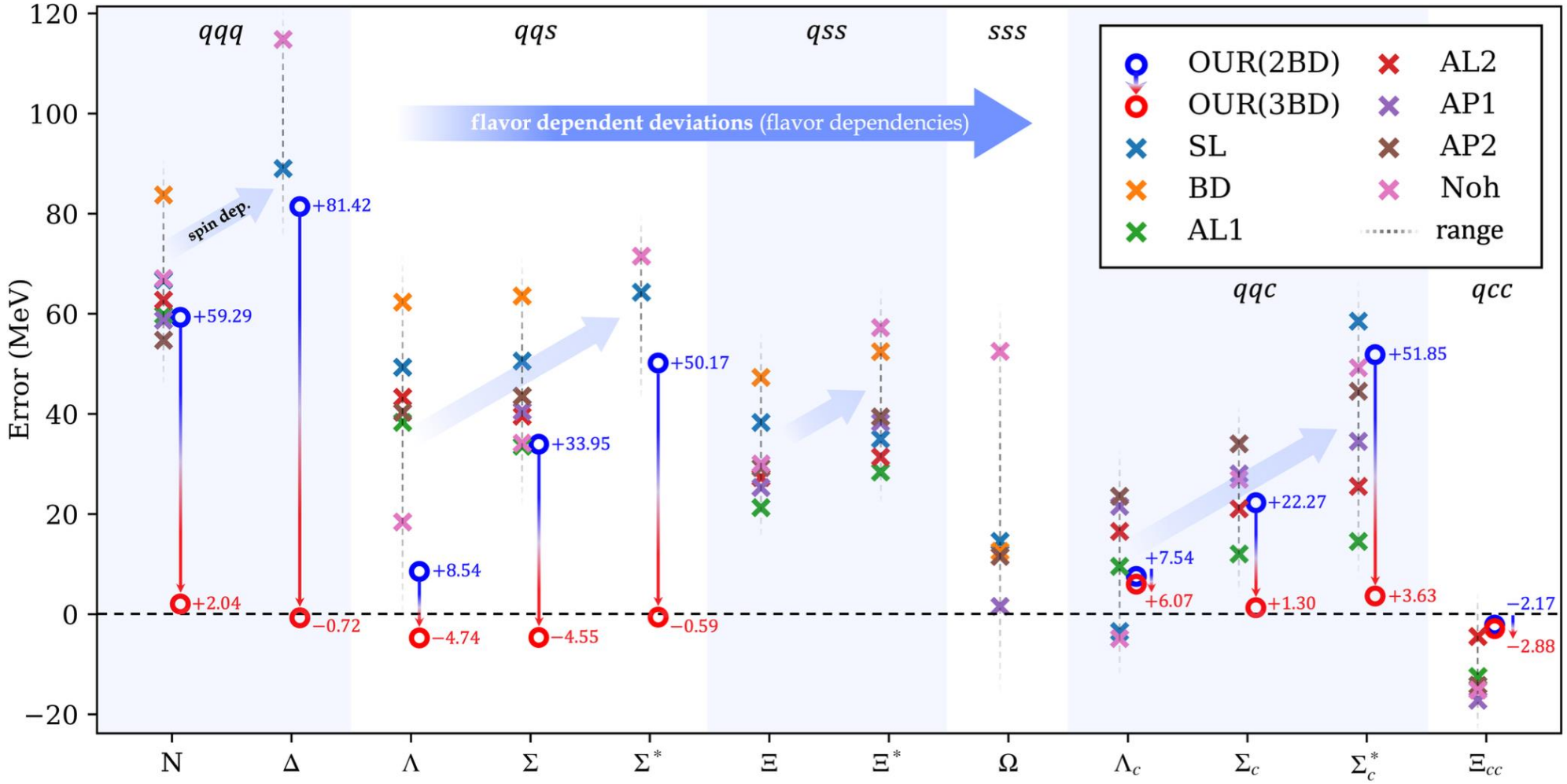
Gaussian-type $$L_{\mathrm{G}}^{3Q} = \sum_{i<j<k} \left(A L_{ijk}^{C-C} + B L_{ijk}^{S-S} + C L_{ijk}^{C-S} \right) \times \exp\left[-\frac{c^2}{8\hbar^2} (m_{ij}^2 r_{ij}^2 + m_{jk}^2 r_{jk}^2 + m_{ki}^2 r_{ki}^2) \right]$$

Unknown fitting parameters

Three-body spatial profile		(a) $\delta(\mathbf{r})$ approximation		(b) Mass-dep. Yukawa		(c) Mass-dep. Gaussian	
Observed mass (M_{exp}, J^P)	Two-body mass $(\sigma = 46.28)$	with L^{3Q} $(\sigma = 12.71)$	M^{3Q} (MeV)	with L_{Y}^{3Q} $(\sigma = 3.67)$	M^{3Q} (MeV)	with L_{G}^{3Q} $(\sigma = 6.41)$[6]	M^{3Q} (MeV)
$p(938.27, 1/2^+)$	997.56(+59.29)	958.00(+19.73)	−39.56	940.31(+2.04)	−57.25	947.08(+8.81)	−50.48
$\Delta(1232, 3/2^+)$	1313.42(+81.42)	1243.64(+11.64)	−69.78	1231.28(−0.72)	−82.14	1232.82(+0.82)	−80.60
$\Sigma(1197.45, 1/2^+)$	1231.40(+33.95)	1187.48(−9.97)	−43.92	1192.9(−4.55)	−38.50	1189.16(−8.29)	−42.24
$\Lambda(1115.68, 1/2^+)$	1124.22(+8.54)	1111.78(−3.90)	−12.44	1110.94(−4.74)	−13.28	1110.88(−4.8)	−13.34
$\Sigma_c(2453.97, 1/2^+)$	2476.24(+22.27)	2441.86(−12.11)	−34.38	2455.27(+1.30)	−20.97	2447.52(−6.45)	−28.72
$\Lambda_c(2286.46, 1/2^+)$	2294.00(+7.54)	2290.70(+4.23)	−3.31	2292.53(+6.07)	−1.47	2292.70(+6.24)	−1.30
$\Xi_{cc}(3621.46, ?^?)$[7]	3619.29(−2.17)	3599.99(−21.47)	−19.30	3618.58(−2.88)	−0.71	3619.27(−2.19)	−0.02
$\Sigma^*(1383.7, 3/2^+)$	1433.87(+50.17)	1378.37(−5.33)	−55.50	1383.11(−0.59)	−50.76	1381.71(−1.99)	−52.16
$\Sigma_c^*(2518.48, 3/2^+)$	2570.33(+51.85)	2518.59(+0.11)	−51.74	2522.11(+3.63)	−48.22	2526.82(+8.27)	−43.58

$$L^{3Q} = \sum_{i<j<k} \left(AL_{ijk}^{C-C} + BL_{ijk}^{S-S} + CL_{ijk}^{C-S} \right) \exp\left[-\frac{c}{\hbar}(m_{ij}r_{ij} + m_{jk}r_{jk} + m_{ki}r_{ki}) \right]$$

$$A = -23000.3\ \mathrm{MeV}^2, \qquad B = 5.94444 \times 10^{13}\ \mathrm{MeV}^6, \qquad C = -8.78859 \times 10^{8}\ \mathrm{MeV}^4.$$

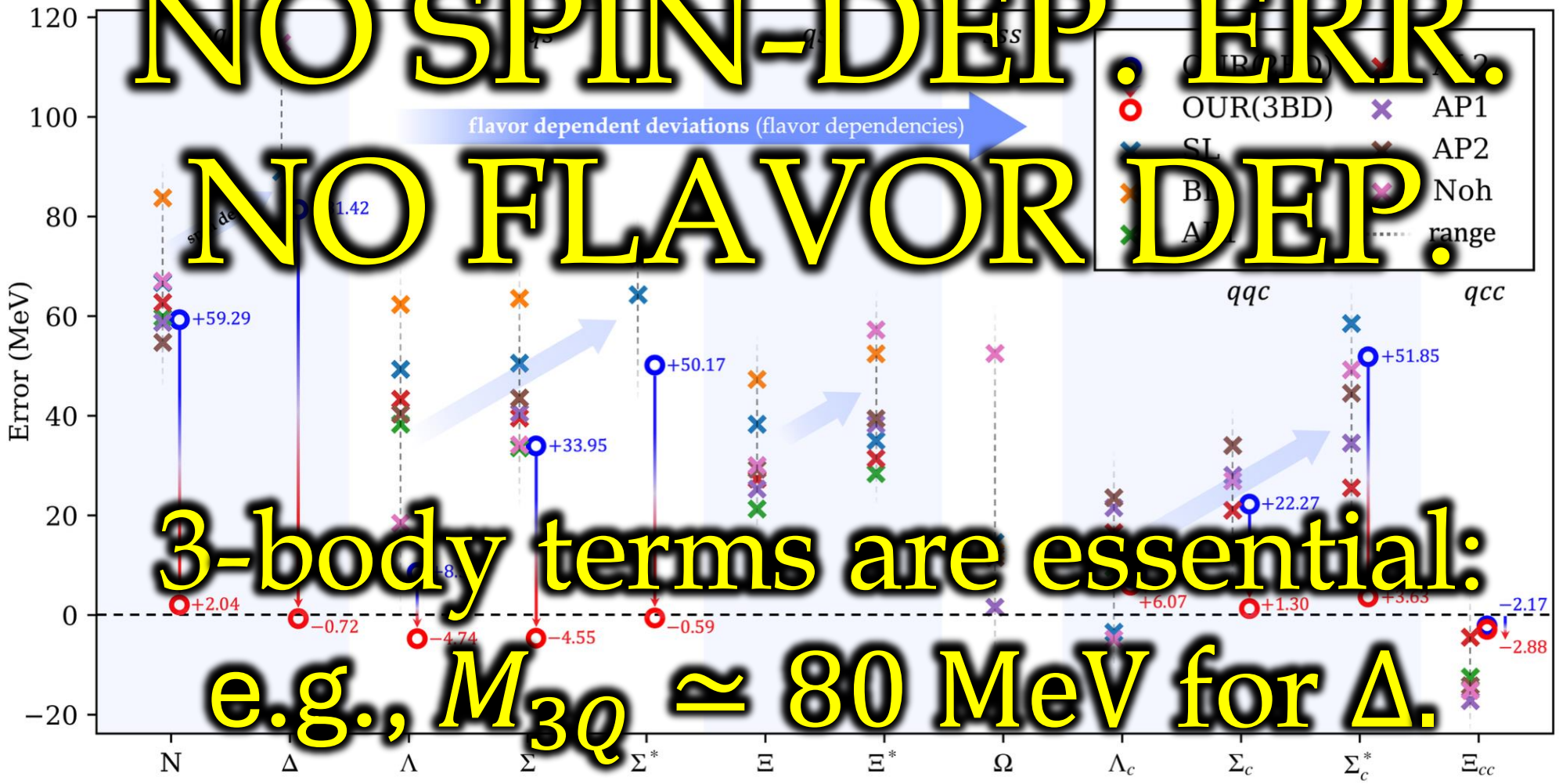


Three-body force well reproduce spin splittings (e.g. $N - \Delta$) and resolves flavor dependency!

$$L^{3Q} = \sum_{i<j<k} \left(AL_{ijk}^{C-C} + BL_{ijk}^{S-S} + CL_{ijk}^{C-S} \right) \exp\left[-\frac{c}{\hbar}(m_{ij}r_{ij} + m_{jk}r_{jk} + m_{ki}r_{ki}) \right]$$

$$A = -23000.3\ \mathrm{MeV}^2, \qquad B = 5.94444 \times 10^{13}\ \mathrm{MeV}^6, \qquad C = -8.78859 \times 10^8\ \mathrm{MeV}^4.$$

NO SPIN-DEP. ERR.
NO FLAVOR DEP.



3-body terms are essential:
e.g., $M_{3Q} \simeq 80$ MeV for Δ.

Three-body force well reproduce spin splittings (e.g. $N - \Delta$) and resolves flavor dependency!

Quark three-body interaction (generalization)

TGE color-spin term — heavy-pair localized

$$L_f^{3Q} = \sum_{i<j<k} \left(AL_{ijk}^{C-C} + BL_{ijk}^{S-S} + CL_{ijk}^{C-S} \right) \times f(m_{ij}r_{ij}, m_{jk}r_{jk}, n_{ki}r_{ki})$$

Arbitrary function f that satisfies:

$f(m_{ij}r_{ij})$ is decreasing, $f(\infty) = 0$

Summary

1. It is a function of the dimensionless mass-scaled distance $x = m_{ij}r_{ij}$.
2. It is strictly monotonically decreasing in x.
3. It decays to zero at long range, i.e., $\lim_{x\to\infty} f(x) = 0$.
4. (When treated as a perturbation; Kato-Rellich theorem) spatial potential may be at most as singular as $f(r) = \mathcal{O}(r^{-a})$ as $r \to 0$, with $a < 2$, so that the divergence never reaches the critical $1/r^2$ behavior [213, 214].

can resolve the residual flavor dependent errors. [proof skipped]

Flavor dependency in the quark model is attributed to the three-body (TGE) force!

in communication with: *Three-body forces in the quark model*, Prog. Part. Nucl. Phys. **(2026)**

Summary

Motivation:
Meson-fit 2-body quark model cannot reproduce baryon spectra (spin splitting & flavor dependency).

Method:
Fujita-Miyazawa mechanism for two-gluon exchange (TGE) constructs 3-body color-spin part.
Non-perturbative short-range effect is encoded into the spatial part.

Result:
Both spin splitting and flavor dependency are resolved with 3-body forces.

Supplementary materials

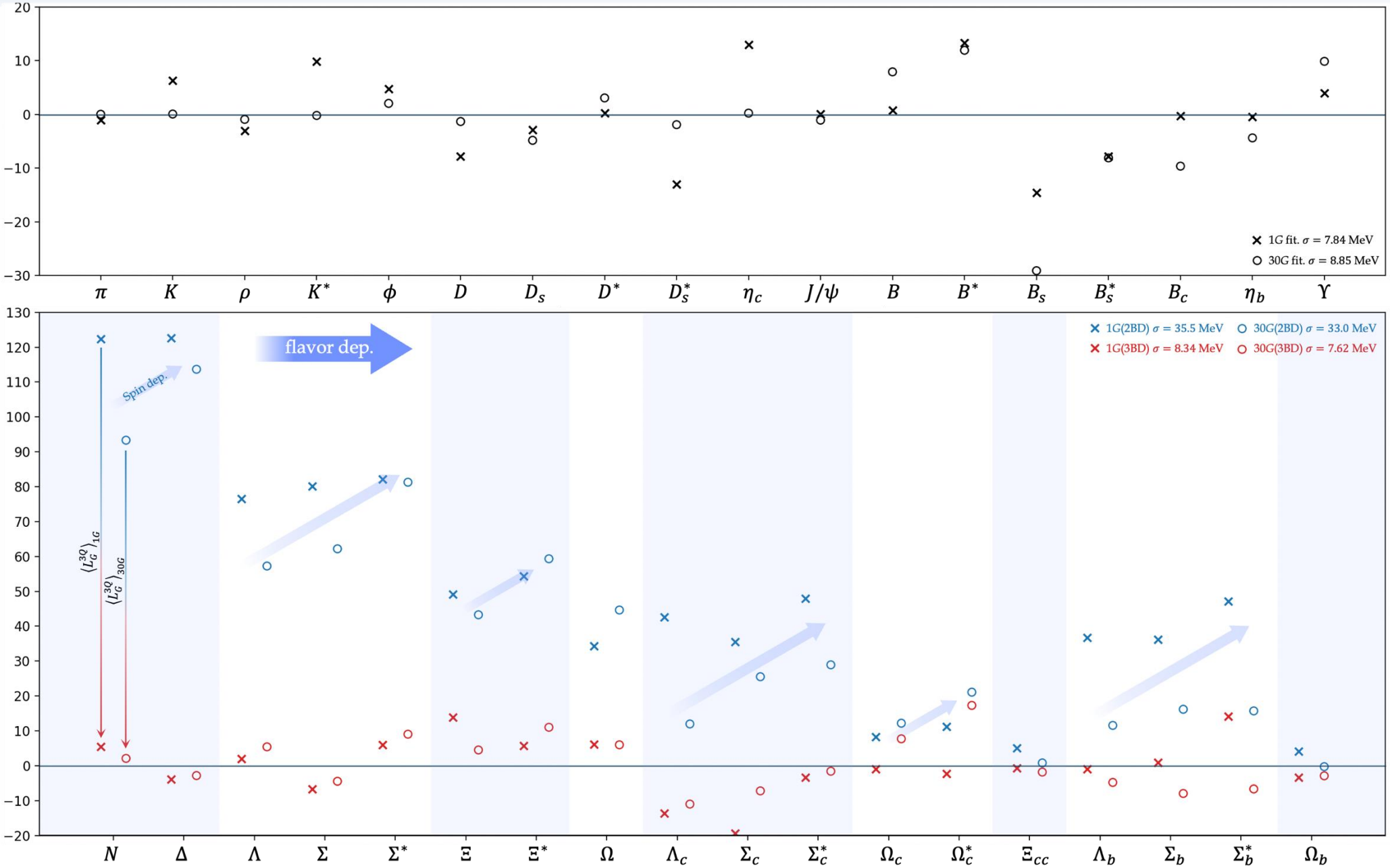


Table 2.1: Lattice and quark model determinations of Cornell-type static potential parameters. Here, σ denotes the coefficient of the linear term, κ denotes the coefficient of the Coulomb term, and $-D$ represents the vacuum constant.

Reference	N_f/Setup	Scale def./value	σ (GeV^2)	κ	D (GeV)	Remarks
Ref. [54]	0 (quenched)	$r_0 = 0.5$ fm	≈ 0.21	≈ 0.295	≈ 0.686	Derived by imposing $V(r_0) = 0$, so $D = \sigma r_0 - \kappa/r_0$.
Ref. [52]	3 (KS fermions)	$r_1^2 F(r_1) = 1.0$	≈ 0.18	≈ 0.36	—	Constant term depends on the renormalization choice.
Ref. [70]	0 (quenched)	r_0 from $Q\bar{Q}$ potential	≈ 0.21	≈ 0.14	≈ 0.8–1.0	Constant term for the three-quark (Y-shaped) system.
Ref. [69]	2+1 (DWF)	$a \approx 0.08$ fm	≈ 0.221	≈ 0.471	≈ 0.689	Constant $-D$ fitted directly.
Ref. [27]	Quark model	Meson-fit	0.112	0.456	1.084	Phenomenological fit. (VAR)
Present work	Quark model	Meson-fit	0.112	0.432	1.082	Phenomenological fit. (GEM)

[52] C. Bernard, T. Burch, K. Orginos, and D. Toussaint, T. A. DeGrand, C. DeTar, S. Gottlieb, U. M. Heller, J. E. Hetrick, and B. Sugar, *Static quark potential in three flavor QCD*, Phys. Rev. D **62**, 034503 (2000).

[54] G. S. Bali, *QCD forces and heavy quark bound states*, Phys. Rep. **343**, 1 (2001).

[69] T. Kawanai and S. Sasaki, *Potential description of charmonium and charmed-strange mesons from lattice QCD*, Phys. Rev. D **92**, 094503 (2015).

[70] T. T. Takahashi, H. Suganuma, Y. Nemoto, and H. Matsufuru, *Detailed analysis of the three-quark potential in SU(3) lattice QCD*, Phys. Rev. D **65**, 114509 (2002).

[27] S. Noh, A. Park, H. Yun, S. Cho, and S. H. Lee, *The inevitable quark three-body Force and its implications for exotic states*, Phys. Lett. B **862**, 139278 (2025).

Table 2.2: Recent values of mass shifts from coupled-channel effects. Negative values indicate downward shifts. Each channel includes its spin-excited counterparts; for example, $K\bar{K}$ represents $K\bar{K}$, $\bar{K}K^*$, $K^*\bar{K}$, and $\bar{K}^*K^*$. We do not explicitly list all the channels for mesons.

Meson (type)	**Shifts** (state or $n^{2S+1}L_{\mathcal{J}}$; MeV)	**Model**	**Coupled channels**
Light mesons $(\pi, \rho, \omega, \eta)$	$\approx -37(\pi)$, $-100(\rho)$, $-66(\omega)$, $-65(\eta)$	Chiral quark model	$\pi\pi$, $\pi\rho$, $\rho\rho$, $K\bar{K}$, $\cdots$ [110]
$D(c\bar{q})$ and excitations	$-360(D)$, $-434(D^*)$, $-395(2^1S_0)$, $-393(2^3S_1)$	Non-rel. quark model	$D\pi$, $D\eta$, $D\rho$, $D\omega$ [116]
$D_s(c\bar{s})$ and excitations	$-304(1^1S_0)$, $-359(1^3S_1)$, $-373(2^1S_0)$, $-334(2^3S_1)$	Non-rel. quark model	DK, $D_s\eta$, $D_s\phi$, $\cdots$ [117]
Charmonium $(c\bar{c})$ series	$-57.8(\eta_c(1S))$, $-100.3(\eta_c(2S))$, $-66.8(J/\psi(1S))$, $-111.6(\psi(2S))$, $-86(\chi_{c0}(1P))$, $-91(\chi_{c1}(1P))$, $-125(\chi_{c0}(2P))$, $-135(\chi_{c1}(2P))$	Chiral quark model	$^3P_0$ including $D\bar{D}$, $D_s\bar{D}_s$, $\cdots$ [118]
Bottomnium $(b\bar{b})$ series	$-52(\eta_b)$, $-49(\Upsilon)$, $-53(\eta_b')$, $-53(\Upsilon')$, $-65(h_b)$	Rel. quark model	BB, $B_{s,c}B_{s,c}$, $(\eta_b$ or $\Upsilon)^2$ [113]

Baryon	**Shift**	**Model**	**Coupled channels** (their contributions) [115]
N	-377	Non-rel. quark model	$N\pi(-178)$, $\Sigma K(-2)$, $\Delta\pi(-158)$, $N\eta(-8)$, $N\eta'(-5)$, $\Sigma^*K(-11)$, $\Lambda K(-15)$
Δ	-410	Non-rel. quark model	$N\pi(-120)$, $\Sigma K(-16)$, $\Delta\pi(-186)$, $\Delta\eta(-46)$, $\Delta\eta'(-25)$, $\Sigma^*K(-17)$
Λ	-359	Non-rel. quark model	$N\bar{K}(-123)$, $\Sigma\pi(-78)$, $\Xi K(-3)$, $\Lambda\eta(-4)$, $\Lambda\eta'(-5)$, $\Sigma^*\pi(-124)$, $\Xi^*K(-22)$
Σ	-345	Non-rel. quark model	$N\bar{K}(-6)$, $\Sigma\pi(-79)$, $\Lambda\pi(-34)$, $\Sigma\eta(-22)$, $\Sigma\eta'(-5)$, $\Xi K(-29)$, $\Delta\bar{K}(-113)$, $\Sigma^*\pi(-30)$, $\Sigma^*\eta(-20)$, $\Sigma^*\eta'(0)$, $\Xi^*K(-8)$
Σ^*	-394	Non-rel. quark model	$N\bar{K}(-33)$, $\Sigma\pi(-33)$, $\Lambda\pi(-60)$, $\Sigma\eta(-15)$, $\Xi K(-6)$, $\Delta\bar{K}(-91)$, $\Sigma^*\pi(-101)$, $\Sigma^*\eta(-8)$, $\Sigma^*\eta'(-24)$, $\Xi^*K(-23)$
Ξ	-328	Non-rel. quark model	$\Sigma\bar{K}(-163)$, $\Lambda\bar{K}(-7)$, $\Xi\pi(-8)$, $\Xi\eta'(-16)$, $\Xi\eta(-4)$, $\Sigma^*\bar{K}(-42)$, $\Xi^*\pi(-44)$, $\Xi^*\eta(-20)$,
Ξ^*	-376	Non-rel. quark model	$\Sigma\bar{K}(-38)$, $\Lambda\bar{K}(-47)$, $\Xi\pi(-54)$, $\Xi\eta'(-16)$, $\Xi\eta(0)$, $\Sigma^*\bar{K}(-138)$, $\Xi^*\pi(-39)$, $\Xi^*\eta(-2)$, $\Xi^*\eta'(-24)$, $\Omega K(-18)$
Ω	-374	Non-rel. quark model	$\Xi\bar{K}(-163)$, $\Xi^*\bar{K}(-138)$, $\Omega\eta(-19)$, $\Omega\eta'(-54)$

Baryon (type)	**Quenched** (MeV)	**Shift** (MeV)	**Model and method**
$\Lambda_c(2P, 3/2^-)$	$3000 \sim 3012$	$-67 \sim -72$	Non-rel. quark model and $^3P_0$ (D^*N continuum) [119]
		-92	Non-rel. quark model, $^3P_0$ (D^*N continuum), $D^*N - D^*N$ interaction (unitarized) [120]
$\Lambda_c(2P, 1/2^-)$	$2989 \sim 2996$	$-15 \sim -56$	Non-rel. quark model and $^3P_0$ (D^*N continuum) [119]
		-53	Non-rel. quark model, $^3P_0$ (D^*N continuum), $D^*N - D^*N$ interaction (unitarized) [120]
$\Omega_c(1P, 1/2^-)$	–	-97	Non-rel. quark model, $^3P_0$; coupled to $\Xi_c\bar{K}$ and $\Xi_c'\bar{K}$ (S-wave); predicts $M_{\rm phy} = 2945$ MeV [121]

(Two-body) potential

$$V_{ij}^{C}(r) = -\frac{3}{4}F_i^C F_j^C\left(-\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a_0^2} - D\right)$$

$$V_{ij}^{CS}(r) = \frac{\hbar^2 \kappa'_{ij}}{m_i m_j c^4 r_{0ij}} \frac{e^{-(r_{ij}/r_{0ij})^2}}{r_{ij}} F_i^C F_j^C\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

params	κ	$1/a_0^2$ (MeV^2)	D (GeV)
lattice	$0.3 - 0.5$	$(350 - 450\ \text{MeV})^2$	$0.6 - 0.8$ (depends on renorm. scheme)
quenched quark model			$0.9 - 1.1$ (depends on fitting data)

The nearly uniform loop-induced mass shifts are **effectively absorbed into the D or constituent quark mass m_i.** [Barnes–Swanson PRC (2007) for mesons; Tecocoatzi et al. EPJA (2007) for baryons].

Baryon basis functions

e.g., for proton

$$\left[\left(\mathcal{S}_{123}\left[\phi_{n(l=0,2,4,\cdots)}(\mathbf{r}_a)\phi_{NL}(\mathbf{R}_a)\right]^{J=0}\right)\otimes\frac{1}{\sqrt{2}}\left(\begin{array}{|c|c|}\hline u^1&u^2\\\hline d^3\\\cline{1-1}\end{array}_F\begin{array}{|c|c|}\hline \uparrow^1&\uparrow^2\\\hline \downarrow^3\\\cline{1-1}\end{array}_S+\begin{array}{|c|c|}\hline u^1&u^3\\\hline d^2\\\cline{1-1}\end{array}_F\begin{array}{|c|c|}\hline \uparrow^1&\uparrow^3\\\hline \downarrow^2\\\cline{1-1}\end{array}_S\right)\right]^{1/2+}$$

$$\left[\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}_{J=0}\otimes\frac{1}{\sqrt{2}}\left(\begin{array}{|c|c|}\hline u^1&u^2\\\hline d^3\\\cline{1-1}\end{array}_F\begin{array}{|c|c|}\hline \uparrow^1&\uparrow^3\\\hline \downarrow^2\\\cline{1-1}\end{array}_S-\begin{array}{|c|c|}\hline u^1&u^3\\\hline d^2\\\cline{1-1}\end{array}_F\begin{array}{|c|c|}\hline \uparrow^1&\uparrow^2\\\hline \downarrow^3\\\cline{1-1}\end{array}_S\right)\right]^{1/2+}$$

$$\left[\frac{1}{\sqrt{2}}\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}_{J=2}\begin{array}{|c|c|}\hline u^1&u^2\\\hline d^3\\\cline{1-1}\end{array}_F+\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}_{J=2}\begin{array}{|c|c|}\hline u^1&u^3\\\hline d^2\\\cline{1-1}\end{array}_F\right)\otimes\begin{array}{|c|c|c|}\hline&&\\\hline\end{array}_{S=3/2}\right]^{1/2+}$$

J: total orbital angular momentum, $\{J^P = 1/2^+\} = \{J\}\otimes\{S\}$

BARYONS 2025

Spin-dependent three-body piece are not calculated on lattice yet.

Lattice QCD

Three-body fit (static piece) [T. T. Takahashi et. al, PRL (2001)]

$$V_{3Q} = -A_{3Q}\sum_{i<j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|} + \sigma_{3Q}L_{\min} + C_{3Q}$$

2-body Coulomb

Y-shaped flux tube

2-body constant

Breit-Fermi reduction

[Y shaped] ≃ [Δ shaped] approximation

mass dependent

Spin dependent term [Koma & Koma, Nucl. Phys. B (2007)]

$$\frac{\mathbf{S}_1\cdot\mathbf{S}_2}{3m_1m_2}\left(c_F^{(1)}c_F^{(2)}V_4(r) - 48\pi\alpha_s d_v C_F\delta^3(\mathbf{r})\right) + \text{[spin-orbit, tensor]}$$

fit to $V_4 \sim -C_1\frac{e^{-mr}}{r} + \frac{C_2}{r}$

Quark model

$$V_{3Q} = \sum_{i<j}\left(V_{ij}^{C} + V_{ij}^{CS}\right) + L_{\Upsilon}^{3Q}\,[\text{TGE part; } \mathcal{O}(\alpha_s^2)] + \text{[spin-orbit, tensor part]}$$

Time ordered perturbation (color-spin part)
Shirt-range spatial profile

not significant for ground states

Breit-Fermi reduction

BARYONS 2025

Baryon spectrum
(detailed)

We record only those eigenvalues that can be identified with the PDG-listed $1/2^+$, $3/2^+$ states. When the three-body interaction is included, higher orbital and radial excitations are described less accurately than expected. This is likely due to a slight bias introduced by calibrating the parameters to the ground states.

(a) Mass without short-range spatial dependence and optimized Gaussian parameters

Baryons ($M_{\rm exp}$, $\mathcal{J}^P$)	Mass (L^{3q}) (MeV)	ν values (fm^{-2})	Number of bases	Mass without L^{3Q} (MeV)	L^{3Q} (MeV)	AL^{C-C} (MeV)	BL^{S-S} (MeV)	CL^{C-S} (MeV)
$p(938.27, 1/2^+)$	958.00(+19.73)	$1.351 \times 1.25^{0\sim11}$	1440	997.56(+59.29)	−39.563	−54.491	−17.767	+32.694
$\Delta(1232, 3/2^+)$	1243.64(+11.64)	$0.85 \times 1.25^{0\sim11}$	2016	1313.42(+81.42)	−69.783	−54.491	+18.200	−33.492
$\Sigma(1197.45, 1/2^+)$	1187.48(−9.97)	$1.5 \times 1.25^{0\sim5}$	720	1231.40(+33.95)	−43.915	−45.985	−11.951	+14.021
$\Sigma_c(2453.97, 1/2^+)$	2441.86(−12.11)	$1.7 \times 1.25^{0\sim5}$	720	2476.24(+22.27)	−34.374	−39.381	−3.1448	+8.1513
$\Lambda(1115.68, 1/2^+)$	1111.78(−3.90)	$1.85 \times 1.275^{0\sim7}$	1088	1124.22(+8.54)	−12.443	−45.985	−3.7981	+37.341
$\Lambda_c(2286.46, 1/2^+)$	2290.70(+4.23)	$2.1 \times 1.25^{0\sim5}$	612	2294.00(+7.54)	−3.3041	−39.381	−0.36045	+36.437
$\Xi_{cc}(3621.55, ?^?)$[a]	3599.99(−21.47)	$1.587 \times 1.25^{0\sim8}$	1620	3619.29(−2.17)	−19.299	−24.271	+0.14172	+4.8300
$\Sigma^*(1383.7, 3/2^+)$	1378.37(−5.33)	$1.344 \times 1.25^{0\sim5}$	1224	1433.87(+50.17)	−55.502	−45.985	+5.4351	−14.951
$\Sigma_c^*(2518.48, 3/2^+)$	2518.59(+0.11)	$1.538 \times 1.3^{0\sim5}$	1224	2570.33(+51.85)	−51.744	−39.381	+2.2656	−14.628
$N(1880, 1/2^+)$	1847.27(−32.73)	$1.351 \times 1.25^{0\sim11}$	1440	1883.60(+3.60)	−36.333	−54.491	−21.611	+39.769
$N(2300, 1/2^+)$	2335.48(+35.48)	$1.7 \times 1.25^{0\sim7}$	640	2375.60(+75.60)	−40.116	−54.491	−17.109	+31.484
$\Delta(1600, 3/2^+)$	1644.14(+44.14)	$0.85 \times 1.25^{0\sim11}$	2016	1710.61(+110.61)	−66.474	−54.491	+14.262	−26.246
$\Delta(1920, 3/2^+)$	1929.91(+9.91)	$0.9 \times 1.25^{0\sim9}$	1400	2002.57(+82.57)	−72.664	−54.491	+21.629	−39.802
$\Sigma(1660?, 1/2^+)$	1722.02(+62.02)	$1.2 \times 1.25^{0\sim5}$	720	1768.70(+108.70)	−46.680	−45.985	−11.714	+11.019
$\Lambda(1600, 1/2^+)$	1643.64(+43.64)	$2.0 \times 1.25^{0\sim5}$	612	1664.23(+64.23)	−20.590	−45.985	−1.9129	+27.308
$\Lambda(1713, 1/2^+)$	1772.58(+59.58)	$1.85 \times 1.275^{0\sim7}$	1088	1782.48(+69.48)	−9.8996	−45.985	−3.1851	+39.271
$\Sigma^*(2080, 3/2^+)$	2082.45(+2.45)	$1.344 \times 1.25^{0\sim5}$	1224	2143.74(+63.74)	−61.285	−45.985	+8.7382	−24.038

(b) Mass with Yukawa-type spatial profile of Eq. (27) and numerically stable expectation values

Baryons ($M_{\rm exp}$, $\mathcal{J}^P$)	Mass (MeV)	$L_{\rm Y}^{3Q}$ (MeV)	$Af_{\rm Y}L^{C-C}$ (MeV)	$Bf_{\rm Y}L^{S-S}$ (MeV)	$Cf_{\rm Y}L^{C-C}$ (MeV)	Kinetic[b] (MeV)	V^C (MeV)	V^{CS} (MeV)	r_{12} (fm)	$r_{23} = r_{31}$ (fm)
$p(938.27, 1/2^+)$	940.31(+2.04)	−57.252	−97.752	−20.632	+61.132	878.90	−640.06	−191.23	0.727	0.727
$\Delta(1232, 3/2^+)$	1231.28(−0.72)	−82.139	−57.665	+12.468	−36.942	652.55	−427.12	141.37	0.838	0.838
$\Sigma(1197.45, 1/2^+)$	1192.90(−4.55)	−38.503	−45.274	−7.617	+14.388	761.77	−654.12	−104.68	0.742	0.696
$\Sigma_c(2453.97, 1/2^+)$	2455.27(+1.30)	−20.973	−25.089	−1.297	+5.413	612.39	−662.80	11.661	0.737	0.687
$\Lambda(1115.68, 1/2^+)$	1110.94(−4.74)	−13.278	−64.107	−3.428	+54.257	801.27	−689.43	−216.07	0.627	0.656
$\Lambda_c(2286.46, 1/2^+)$	2292.53(+6.07)	−1.469	−35.361	−0.210	+34.102	786.48	−789.62	−218.93	0.634	0.638
$\Xi_{cc}(3621.55, ?^?)$	3618.58(−2.88)	−0.714	−0.904	+0.00342	+0.187	637.92	−1037.3	−58.389	0.438	0.590
$\Sigma^*(1383.7, 3/2^+)$	1383.11(−0.59)	−50.758	−40.208	+3.076	−13.626	639.48	−549.79	115.75	0.791	0.759
$\Sigma_c^*(2518.48, 3/2^+)$	2522.11(+3.63)	−48.219	−35.720	+1.330	−12.489	628.23	−652.67	78.736	0.748	0.709
$N(1880, 1/2^+)$	1873.69(−6.31)	−9.906	−19.972	−5.127	+15.193	1089.3	−23.691	−131.58	1.061	1.061

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S-wave estimation

Meson:
RMSE = 7.7 MeV

Baryon:
RMSE = 8.3 MeV

Tetraquark

Three-body contribution to tetraquarks

(a) Mesons with 30 Gaussians: $\sigma = 7.66$ MeV.

Meson	π	ρ	η_c	ϕ	η_b	J/ψ	K	Υ	D
Observed mass (MeV)	139.57	775.11	2984.10	1019.46	9398.70	3096.90	497.61	9460.40	1864.84
Estimate (MeV)	138.46	772.07	2997.10	1024.22	9398.26	3096.94	503.90	9464.38	1857.05
Deviation (MeV)	−1.11	−3.04	+13.00	+4.76	−0.44	+0.04	+6.29	+3.98	−7.79
Meson	K^*	D_s	D^*	B	D_s^*	B_s	B^*	B_c	B_s^*
Observed mass (MeV)	895.81	1968.30	2010.26	5279.72	2112.10	5366.77	5325.20	6274.47	5415.40
Estimate (MeV)	905.68	1965.42	2010.52	5280.43	2099.11	5352.17	5338.54	6274.14	5407.91
Deviation (MeV)	+9.87	−2.88	+0.26	+0.71	−12.99	−14.60	+13.34	−0.33	−7.49

(b) Baryons using meson fit parameters $\sigma = 8.34$ MeV.

Baryon	N	Λ	Σ	Δ	Ξ	Σ^*	Ξ^*	Ω	Λ_c
Observed Mass (MeV)	938.27	1115.68	1197.45	1232.00	1321.71	1383.70	1535.00	1672.45	2286.46
Estimate (MeV)	943.74	1117.63	1190.80	1228.13	1335.53	1389.74	1540.69	1678.64	2272.79
Deviation (MeV)	+5.47	+1.94	−6.65	−3.87	+13.82	+6.04	+5.69	+6.19	−13.67
Baryon	Σ_c	Σ_c^*	Ω_c	Ω_c^*	Ξ_{cc}	Λ_b	Σ_b	Σ_b^*	Ω_b
Observed Mass (MeV)	2453.97	2518.48	2695.20	2765.90	3621.20	5619.60	5815.64	5834.74	6046.10
Estimate (MeV)	2434.58	2515.18	2694.20	2763.62	3620.46	5618.64	5816.58	5848.85	6042.71
Deviation (MeV)	−19.39	−3.30	−1.00	−2.28	−0.74	−0.96	+0.94	+14.11	−3.39

(c) Single S-wave estimation of quark three-body contribution for tetraquarks.

Tetra quarks	Variational params $a_1,\ a_2,\ a_3$ (fm^{-2})	$\sum A f_G L^{C-C}$ (MeV)	$\sum B f_G L^{S-S}$ (MeV)	$\sum C f_G L^{C-S}$ (MeV)	Three-body sum (MeV)
T_{cc}	2.94, 6.49, 2.40	−2.40764	+0.42410	+9.00836	+7.02482 (repulsive)
T_{bb}	3.01, 16.28, 2.68	−1.72865	+0.12943	+7.08470	+5.48548 (repulsive)
$\chi_{c1}(3872)$	$a_3 < 0.01, r_{\bar{c}q-cq} > 8$ fm	$< 10^{-100}$	$< 10^{-100}$	$< 10^{-100}$	~ 0 (nearly molecular)

3Q force for dibaryons [A. Park, S. H. Lee]

[No spatial part considered]

$uuddss$ $(I = 0, S = 0)$: $F = \mathbf{1}, \mathbf{27}$ basis with three-body force (in MeV)

$$\sum AL_{ijk}^{C-C} = \begin{pmatrix} -22.993 & 0 \\ 0 & -22.993 \end{pmatrix}, \quad \sum BL_{ijk}^{S-S} = \begin{pmatrix} -97.585 & -6.111 \\ -6.111 & -133.458 \end{pmatrix}, \quad \sum CL_{ijk}^{C-S} = \begin{pmatrix} 64.813 & -5.316 \\ -5.316 & -38.747 \end{pmatrix}$$

$$L^{3Q} = \begin{pmatrix} -55.765 & -11.427 \\ -11.427 & -195.198 \end{pmatrix}$$

$uudsss$ $(I = 1/2, S = 2)$: $F = \mathbf{27}, \mathbf{8}$ basis with three-body force

$$\sum AL_{ijk}^{C-C} = \begin{pmatrix} -20.866 & 0 \\ 0 & -20.866 \end{pmatrix}, \quad \sum BL_{ijk}^{S-S} = \begin{pmatrix} -29.582 & 13.798 \\ 13.798 & -7.428 \end{pmatrix}, \quad \sum CL_{ijk}^{C-S} = \begin{pmatrix} -11.992 & -2.639 \\ -2.639 & 52.854 \end{pmatrix}$$

$$L^{3Q} = \begin{pmatrix} -62.441 & 11.159 \\ 11.159 & 24.561 \end{pmatrix}$$

In molecular scenario, actual 3-body term approaches to 0.

3Q force for the pentaquark $P_c(4380)$ [A. Park, S. H. Lee]

$uudc\bar{c}\,(I=1/2, S=3/2)$

$$\phi_1 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{3}{2}} (4\bar{5})_{S=0}],$$
$$\phi_2 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{3}{2}} (4\bar{5})_{S=1}],$$
$$\phi_3 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=1}],$$
$$\phi_4 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{1}} (4\bar{5})_{C=\mathbf{1}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=1}].$$

$uudc\bar{c}\,(I=1/2, S=3/2)$

$$\phi_1 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{3}{2}} (4\bar{5})_{S=1}],$$
$$\phi_2 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=0}],$$
$$\phi_3 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{8}} (4\bar{5})_{C=\mathbf{8}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=1}],$$
$$\phi_4 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{1}} (4\bar{5})_{C=\mathbf{1}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=0}],$$
$$\phi_5 = [(123)_{I=\frac{1}{2}} 4\bar{5}] \otimes [(123)_{C=\mathbf{1}} (4\bar{5})_{C=\mathbf{1}}] \otimes [(123)_{S=\frac{1}{2}} (4\bar{5})_{S=1}].$$

Table 6.1: Elements of $\sum L^X_{ijk}$ for $X = C-C$, $S-S$, $C-S$ (we list only $i \le j$). $C-C$ has an overall $1/m_u$, $S-S$ $1/m_u^5$, $C-S$ $1/m_u^3$. Here, for readability, we denote $\frac{\alpha}{\beta\sqrt{\gamma}}$ by $\alpha/\beta\sqrt{\gamma}$. The SU(3)$_F$ symmetry breaking parameter is $\delta = 1 - m_u/m_c$.

(a) $uudc\bar{c}\,(I=1/2, S=3/2)$

Type (X)	$C-C$ ($\times m_u^{-1}$)					$S-S$ ($\times m_u^{-5}$)					$C-S$ ($\times m_u^{-3}$)				
Component	1	δ	δ^2	δ^3	δ^4	1	δ	δ^2	δ^3	δ^4	1	δ	δ^2	δ^3	δ^4
$\langle\phi_1\vert\sum L^X\vert\phi_1\rangle$	0	+4/3	0	0	0	−8/3	−28	+36	−44/3	0	+112/3	−114	+114	−48	0
$\langle\phi_1\vert\sum L^X\vert\phi_2\rangle$	0	0	0	0	0	$+40\sqrt{5/3}$	$-40\sqrt{5/3}$	0	0	0	$+80\sqrt{5/3}$	$-40\sqrt{15}$	$+40\sqrt{5/3}$	0	0
$\langle\phi_1\vert\sum L^X\vert\phi_3\rangle$	0	0	0	0	0	$+80\sqrt{2}/3\sqrt{3}$	$-440\sqrt{2}/3\sqrt{3}$	$+80\sqrt{6}$	$-160\sqrt{2/3}$	$+40\sqrt{2/3}$	$-320\sqrt{2}/3\sqrt{3}$	$+160\sqrt{2/3}$	$-160\sqrt{2}/3\sqrt{3}$	0	0
$\langle\phi_1\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	$+16\sqrt{2}/3\sqrt{3}$	$+848\sqrt{2}/3\sqrt{3}$	$-192\sqrt{6}$	$+400\sqrt{2/3}$	$-112\sqrt{2/3}$	$-64\sqrt{2}/3\sqrt{3}$	$+32\sqrt{2/3}$	$-32\sqrt{2}/3\sqrt{3}$	0	0
$\langle\phi_2\vert\sum L^X\vert\phi_2\rangle$	0	+4/3	0	0	0	+40/3	−260/3	+332/3	−172/3	32/3	−400/9	+224/3	−512/9	+16	0
$\langle\phi_2\vert\sum L^X\vert\phi_3\rangle$	0	0	0	0	0	$+112\sqrt{10}/9$	$-184\sqrt{10}/9$	$+16\sqrt{10}$	$-32\sqrt{10}/3$	$+8\sqrt{10}/3$	$+64\sqrt{10}/3$	$-32\sqrt{10}$	$+32\sqrt{10}/3$	0	0
$\langle\phi_2\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	$+80\sqrt{10}/9$	$-80\sqrt{10}/9$	0	0	0	0	0	0	0	0
$\langle\phi_3\vert\sum L^X\vert\phi_3\rangle$	0	+4/3	0	0	0	−16/3	−188/3	−4/3	+12	−16/3	+32/9	+104/3	−392/9	+16	0
$\langle\phi_3\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	+160/9	−160/9	0	0	0	+320/3	0	0	0	0
$\langle\phi_4\vert\sum L^X\vert\phi_4\rangle$	0	+64/3	0	0	0	−128/3	−64/3	+128/3	−64/3	0	−1024/9	+128/3	−128/9	0	0

(b) $uudc\bar{c}\,(I=1/2, S=1/2)$

Type (X)	$C-C$ ($\times m_u^{-1}$)					$S-S$ ($\times m_u^{-5}$)					$C-S$ ($\times m_u^{-3}$)				
Component	1	δ	δ^2	δ^3	δ^4	1	δ	δ^2	δ^3	δ^4	1	δ	δ^2	δ^3	δ^4
$\langle\phi_1\vert\sum L^X\vert\phi_1\rangle$	0	+4/3	0	0	0	+64/3	−428/3	+620/3	−364/3	+80/3	−640/9	+344/3	−632/9	+16	0
$\langle\phi_1\vert\sum L^X\vert\phi_2\rangle$	0	0	0	0	0	$-160/3\sqrt{3}$	$+880/3\sqrt{3}$	$-160\sqrt{3}$	$+320/\sqrt{3}$	$-80/\sqrt{3}$	$+640/3\sqrt{3}$	$-320/\sqrt{3}$	$+320/3\sqrt{3}$	0	0
$\langle\phi_1\vert\sum L^X\vert\phi_3\rangle$	0	0	0	0	0	+224/9	−368/9	+32	−64/3	+16/3	+128/3	−64	+64/3	0	0
$\langle\phi_1\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	$-1760/3\sqrt{3}$	$+4064/3\sqrt{3}$	$-448\sqrt{3}$	$+736/\sqrt{3}$	$-160/\sqrt{3}$	$+128/3\sqrt{3}$	$-64/\sqrt{3}$	$+64/3\sqrt{3}$	0	0
$\langle\phi_1\vert\sum L^X\vert\phi_5\rangle$	0	0	0	0	0	+160/9	−160/9	0	0	0	0	0	0	0	0
$\langle\phi_2\vert\sum L^X\vert\phi_2\rangle$	0	+4/3	0	0	0	−256/3	+60	−52	+44/3	0	+176/3	−144	+144	−48	0
$\langle\phi_2\vert\sum L^X\vert\phi_3\rangle$	0	0	0	0	0	$-320/\sqrt{3}$	$+560/\sqrt{3}$	$-120\sqrt{3}$	$+40\sqrt{3}$	0	$+80/\sqrt{3}$	$-40\sqrt{3}$	$+40/\sqrt{3}$	0	0
$\langle\phi_2\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	0	0	0	0	0	+320/3	0	0	0	0
$\langle\phi_2\vert\sum L^X\vert\phi_5\rangle$	0	0	0	0	0	$+832/\sqrt{3}$	$-3136/3\sqrt{3}$	$+480\sqrt{3}$	$-896/\sqrt{3}$	$+224/\sqrt{3}$	$+128/3\sqrt{3}$	$-64/\sqrt{3}$	$64/3\sqrt{3}$	0	0
$\langle\phi_3\vert\sum L^X\vert\phi_3\rangle$	0	+4/3	0	0	0	−640/3	+724/3	−364/3	+20	+32/3	−208/9	+224/3	−512/9	+16	0
$\langle\phi_3\vert\sum L^X\vert\phi_4\rangle$	0	0	0	0	0	$-896/3\sqrt{3}$	$+2624/3\sqrt{3}$	$-352\sqrt{3}$	$+640/\sqrt{3}$	$-160/\sqrt{3}$	$+128/3\sqrt{3}$	$-64/\sqrt{3}$	$+64/3\sqrt{3}$	0	0
$\langle\phi_3\vert\sum L^X\vert\phi_5\rangle$	0	0	0	0	0	−320/9	+320/9	0	0	0	+320/3	0	0	0	0
$\langle\phi_4\vert\sum L^X\vert\phi_4\rangle$	0	+64/3	0	0	0	+128/3	−192	+128	−64/3	0	−256/3	0	0	0	0
$\langle\phi_4\vert\sum L^X\vert\phi_5\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\langle\phi_5\vert\sum L^X\vert\phi_5\rangle$	0	+64/3	0	0	0	−128/3	−64/3	+128/3	−64/3	0	−256/9	−256/3	+256/9	0	0