

Estimation of potential size based on momentum distribution of constituent particle

Eisuke Kawamura^A, Kotaro Murakami^{A,B}, Daisuke Jido^A

Institute of Science Tokyo^A, RIKEN iTHEMS^B

November 13, 2025

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$ 4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

6. Back up slide

1. Introduction

1.1 Motivation

1. J-PARC experiment for $\bar{K}NNN$ nuclei creation

2. Interest for size of hadronic compound system

- A hint as to what kind of structure they are (their component).
 - Compact system \Rightarrow Constituent quarks
 - Broad system \Rightarrow Constituent hadrons
- Can we obtain the system size experimentally?

3. Definition: System size = Interaction size = potential size

- When it is quark system, Interaction size = Confinement size
- When it is hadronic system, Interaction size = Meson exchange size
- Emphasize: system size \neq spread of wave function size
 - Spread of the WF depends on the binding energy.

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

6. Back up slide

2. Model

2.1 Spectator Nucleon

2.2 Definition

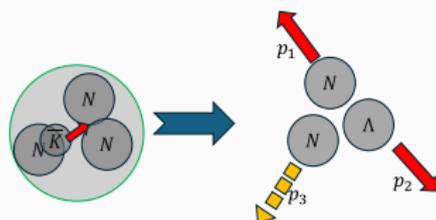
1. $\bar{K}NNN \rightarrow \Lambda + N + N$ decay process

- $\bar{K}NN \rightarrow \Lambda + N$

- Remaining N acts as a spectator

2. Spectator's momentum is conserved during the decay

3. We can obtain the information of momentum distribution inside the nucleus [1]



In this work

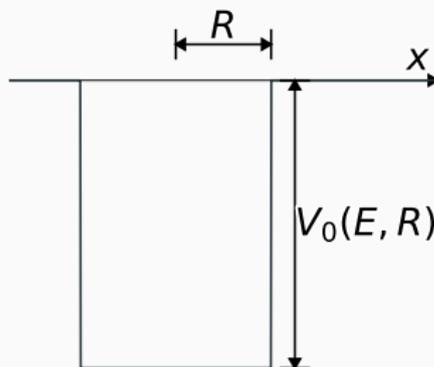
We confirm whether we can estimate the system size from momentum distribution of constituent particle.

We will demonstrate with a simple model.

[1]: P. Kienle, Y. Akaishi, and T. Yamazaki. Phys. Lett., B632:187-191, 2006

Simple model

- 3d-Square-Well potential
- 1s orbital wave function
- Analytic solution



We regard...

- Potential radius R is the system size.
- Potential depth V_0 depends on R and binding energy E .

Goal

- Estimation of R from p and E
 - It would be good if R has less dependence on E than p .
 - E is difficult to be measure accurately.

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

6. Back up slide

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

3.1 Simple estimation

3.2 Plot of $r_{\langle p^2 \rangle} - \sqrt{\langle r^2 \rangle}$

3.3 Deeply vs Shallowly

Heisenberg uncertainty principle (HUP)

$$\frac{\sqrt{\langle r^2 \rangle} \sqrt{\langle p^2 \rangle}}{\hbar} \geq \frac{3}{2} \quad (2)$$

Definition of $r_{\langle p^2 \rangle}$ (let (2)'s " \geq " as "=", estimate $\sqrt{\langle r^2 \rangle}$ from $\sqrt{\langle p^2 \rangle}$)

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}} \quad (3)$$

Simple estimation

All of wave functions satisfies the following;

$$r_{\langle p^2 \rangle} \stackrel{?}{\simeq} \sqrt{\langle r^2 \rangle} \quad (4)$$

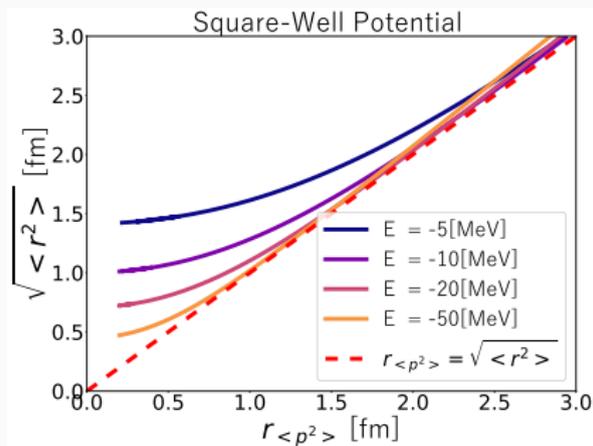
Is it true?

Simple estimation

All of wave functions satisfies the following;

$$r_{\langle p^2 \rangle} \stackrel{?}{\simeq} \sqrt{\langle r^2 \rangle} \quad (5)$$

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}} \quad (6)$$



- $r_{\langle p^2 \rangle} - \sqrt{\langle r^2 \rangle}$ graph
- Analytic solution
- $m = 1000$ MeV

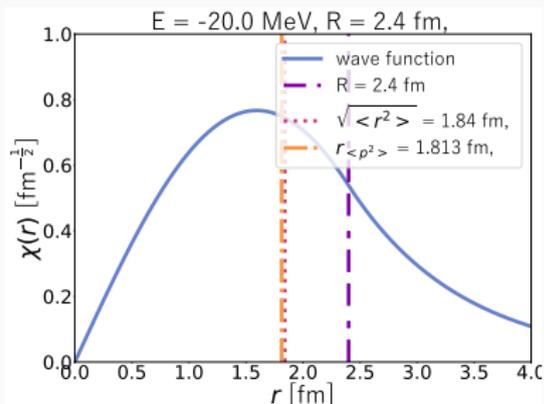
Large $r_{\langle p^2 \rangle} \rightarrow r_{\langle p^2 \rangle} \simeq \sqrt{\langle r^2 \rangle}$

Small $r_{\langle p^2 \rangle} \rightarrow r_{\langle p^2 \rangle} \neq \sqrt{\langle r^2 \rangle}$

Simple estimation is not true!

Deeply bound state

WF is inside the potential.



Trivial relation

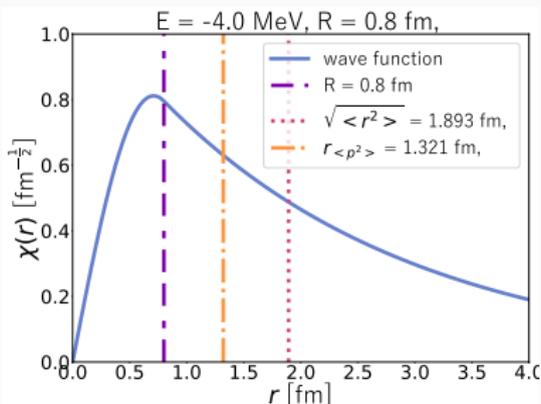
$$r_{\langle p^2 \rangle} \sim \sqrt{\langle r^2 \rangle} \sim R \quad (7)$$

We can estimate R via $\sqrt{\langle r^2 \rangle}$.

Let's consider shallowly bound state!

Shallowly bound state

WF spreads outside the potential.



Non-trivial relation

$$r_{\langle p^2 \rangle} \not\sim \sqrt{\langle r^2 \rangle} \not\sim R \quad (8)$$

We cannot estimate R via $\sqrt{\langle r^2 \rangle}$.

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

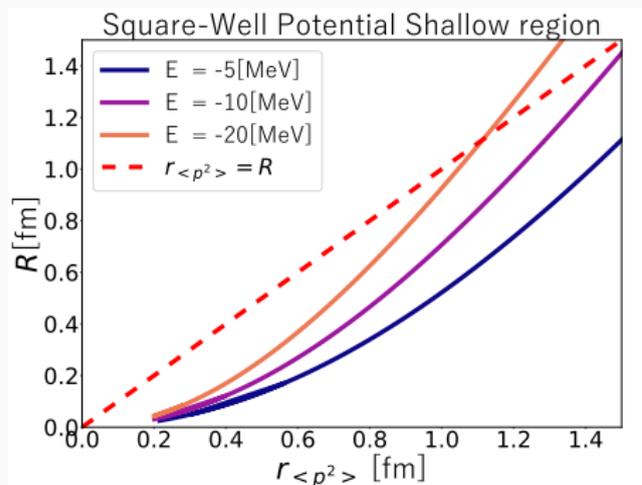
6. Back up slide

4. $\sqrt{\langle p^2 \rangle}$ vs R

4.1 Shallowly bound state

4.2 Numerical example

4.3 Deuteron example



- $r_{\langle p^2 \rangle} - R$ graph
- Analytic solution
- $m = 1000$ MeV
- Shallowly region

Shallowly bound state

$$R < r_{\langle p^2 \rangle} \quad (9)$$

$\therefore r_{\langle p^2 \rangle}$ gives an upper bound on R .

$\therefore r_{\langle p^2 \rangle}$ is small \Rightarrow Compact!

- $\sqrt{\langle p^2 \rangle} = 300$ MeV
- $r_{\langle p^2 \rangle} = 1$ fm
- $E \sim -10$ MeV, $m = 1$ GeV



- Shallowly bound state
- $R \sim 0.8$ fm

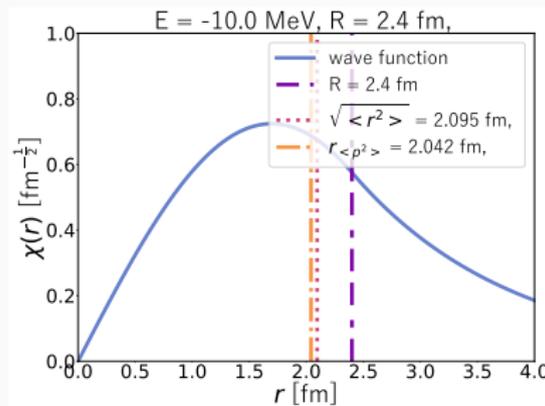
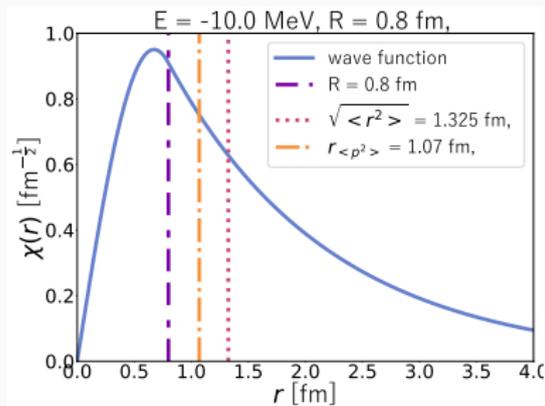
→ Compact system

- $\sqrt{\langle p^2 \rangle} = 150$ MeV
- $r_{\langle p^2 \rangle} = 2$ fm
- $E \sim -10$ MeV, $m = 1$ GeV



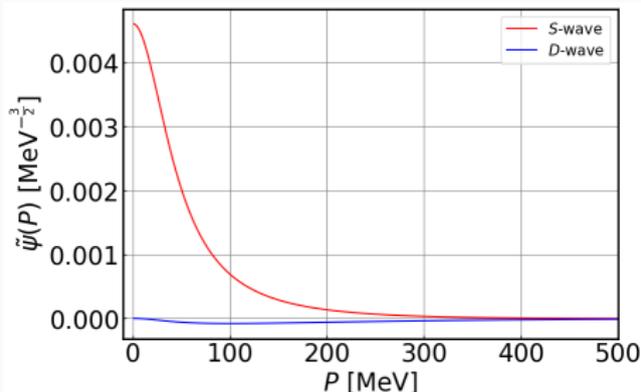
- Deeply bound state
- $R \sim 2.4$ fm

→ Hadronic system



Deuteron (CD-Bonn potential)

- $E = -2.2\text{MeV} / 2$ particles
- $\sqrt{\langle p^2 \rangle} = 121 \text{ MeV}$ [2]
 - $r_{\langle p^2 \rangle} = 2.4 \text{ fm}$
- $m = 939 \text{ MeV}$



Apply to the square-well potential

$$\Rightarrow R = 1.44\text{fm} \simeq \frac{200\text{MeV} \cdot \text{fm}}{140\text{MeV}} = \frac{\hbar}{m_{\pi}} \quad (10)$$

This is pion exchange range.

This example supports the validity of the theory.

[2] R. Machleidt, PRC, 2001

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

6. Back up slide

5. Summary

5.1 Definition, Purpose,
Conclusion

5.2 Future work

Definition

- Potential radius regarded as system size

Purpose

- Can we estimate potential radius from momentum distribution?

Conclusion

- If we can know $r_{\langle p^2 \rangle}$, we can know whether the system is compact.
 - $r_{\langle p^2 \rangle}$ is small \Rightarrow The system is compact.
- To determine the system size, observe the momentum distribution.

- More information obtained from the momentum distribution
- Potential shape dependence
 - Consider repulsion as well.
- Compare to already-measured data
 - Hyperon
 - Halo nucleon

Thank you for your attention.

1. Introduction

2. Model

3. $\sqrt{\langle p^2 \rangle}$ vs $\sqrt{\langle r^2 \rangle}$

4. $\sqrt{\langle p^2 \rangle}$ vs R

5. Summary

6. Back up slide

6. Back up slide

6.1 Model ambiguity

6.2 Plot of approx

6.3 r_p solution

6.4 Analytic solutions

6.5 Deeply bound state

6.6 Shallowly bound state

6.7 Shape dependence of
potential

6.8 Coordinate space WF

6.9 Momentum space WF

Observed quantities: momentum of decay particle \rightarrow

Total BE and momentum distribution of constituent particle

Estimation target: potential radius R

- Ambiguity of p
 - Measurement error of p
 - Is momentum really conserved?
 - Does the nucleon really act as a spectator?
- Ambiguity of E
 - Measurement error of total binding energy
 - Error of “Total BE” \rightarrow “Nucleon-felt BE (E)”
 - (Binding energy of spectator nucleon E is not observable)
- Ambiguity of $V(r)$
 - Ambiguity of independent particle picture
 - Difference between model shape and actual shape of potential

It would be good if R has less dependence on E than p

Between the following two:

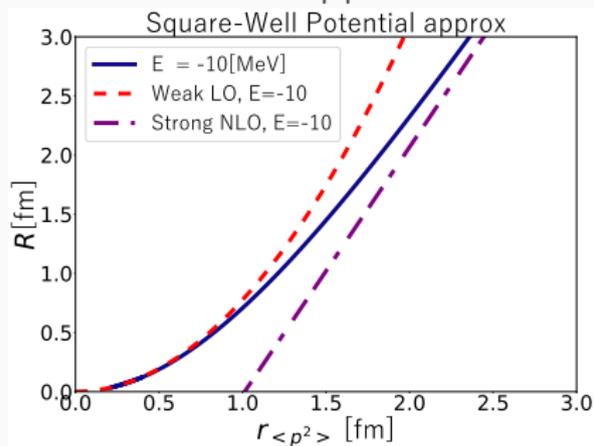
$$\text{Deeply : } R \simeq \frac{2\pi}{3} r_{\langle p^2 \rangle} - \frac{3}{2} \frac{1}{\rho} \quad (11a)$$

$$\rho \stackrel{\text{def}}{=} \sqrt{\frac{2m|E|}{\hbar^2}}$$

$$\text{Shallowly : } R \simeq \frac{\pi^2}{9} \rho r_{\langle p^2 \rangle}^2 \quad (11b)$$

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}}$$

We can obtain upper and lower bounds for R .

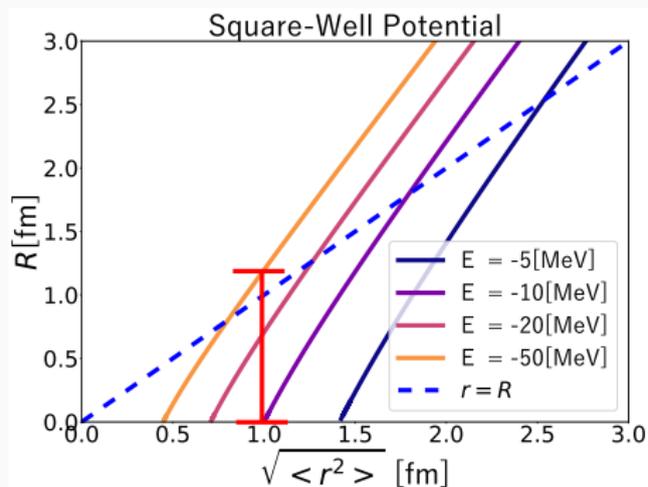
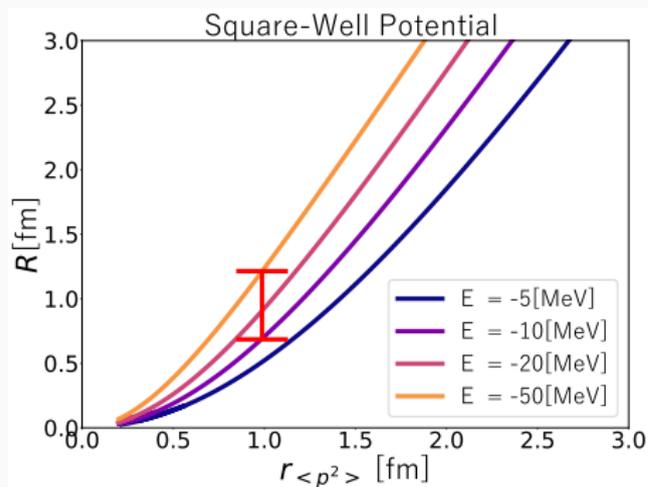


Deep or Shallow?

Roughly determined from

$E, r_{\langle p^2 \rangle}$

- $E = -10, r_{\langle p^2 \rangle} = 1.0 \Rightarrow$ Shallow



When we measure $r_{\langle p^2 \rangle}$, we can estimate R with less ambiguity than when we measure $\sqrt{\langle r^2 \rangle}$

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle r^2 \rangle}} \quad (12)$$

- $R - r_{\langle p^2 \rangle}$
- Analytic solution
- $m = 1000 \text{ MeV}$

Express $\sqrt{\langle r^2 \rangle}$ and r_ρ in terms of R and E .

$$\sqrt{\langle r^2 \rangle} = \sqrt{\frac{R^2}{1 + \frac{1}{\rho R}} \left(\frac{1}{3} + \frac{1}{\rho R} + \frac{1}{\rho^2 R^2} + \frac{1}{2\rho^3 R^3} - \frac{1}{2\kappa^2 R^2} - \frac{1}{2\kappa^2 R^2} \frac{1}{\rho R} \right)}$$
(13a)

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}} = \frac{3}{2} \frac{\sqrt{1 + \frac{1}{\rho R}}}{\kappa}$$
(13b)

$$\rho \stackrel{\text{def}}{=} \sqrt{\frac{2m|E|}{\hbar^2}} = -\kappa \cot \kappa R \quad 0 < \rho R < \infty$$
(13c)

$$\kappa \stackrel{\text{def}}{=} \sqrt{\frac{2m(E + V_0(E, R))}{\hbar^2}} \quad \frac{\pi}{2} < \kappa R < \pi$$
(13d)

Deeply bound $\Rightarrow \rho R \rightarrow \infty$

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}} \quad (14)$$

$$\kappa R = \pi - \delta \quad \frac{1}{\rho R} = \frac{\delta}{\pi} \left(1 + \frac{\delta}{\pi} + \mathcal{O}(\delta^2) \right) \quad (15a)$$

$$\langle r^2 \rangle = \left(\frac{1}{3} - \frac{1}{2\pi^2} \right) R^2 \left(1 + \frac{2}{\rho R} + \mathcal{O}(\delta^2) \right) \quad (15b)$$

$$\langle p^2 \rangle = \frac{\pi^2 \hbar^2}{R^2} \left(1 - \frac{3}{\rho R} + \mathcal{O}(\delta^2) \right) \quad (15c)$$

$$r_{\langle p^2 \rangle} = \frac{3}{2\pi} R \left(1 + \frac{3}{2} \frac{1}{\rho R} + \mathcal{O}(\delta^2) \right) \quad (15d)$$

$$R = \frac{2\pi}{3} r_{\langle p^2 \rangle} - \frac{3}{2\rho} (1 + \mathcal{O}(\delta^1)) \quad (15e)$$

Shallowly bound $\Rightarrow \rho R \rightarrow 0$

$$r_{\langle p^2 \rangle} \stackrel{\text{def}}{=} \frac{3}{2} \frac{\hbar}{\sqrt{\langle p^2 \rangle}} \quad (16)$$

$$\kappa R = \frac{\pi}{2} + \delta, \quad \rho R = \frac{\pi}{2} \delta + \delta^2 + \mathcal{O}(\delta^3) \quad (17a)$$

$$\langle r^2 \rangle = \left[\frac{1}{2\rho^2 R^2} + \frac{1}{2\rho R} + \mathcal{O}(\delta^0) \right] R^2 \quad (17b)$$

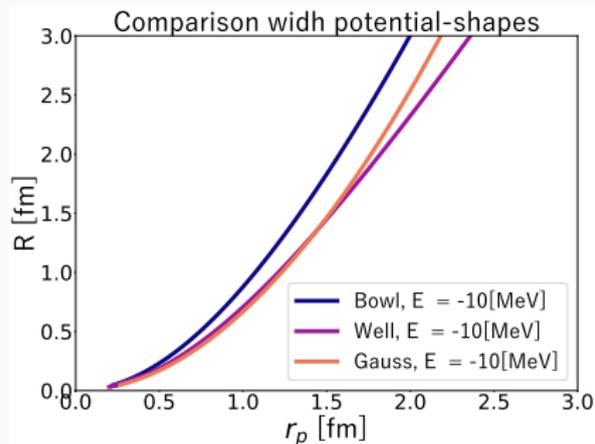
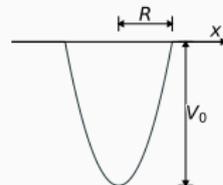
$$\langle p^2 \rangle = \frac{\pi^2}{4} \frac{\hbar^2}{R^2} \left(\rho R + \left(2 - \frac{\pi^2}{4} \right) \frac{2^2}{\pi^2} \rho^2 R^2 + \mathcal{O}(\delta^3) \right) \quad (17c)$$

$$r_{\langle p^2 \rangle} = \frac{3}{\pi} \frac{R}{\sqrt{\rho R}} \left[1 - \left(2 - \frac{\pi^2}{4} \right) \frac{2}{\pi^2} \rho R + \mathcal{O}(\delta^2) \right] \quad (17d)$$

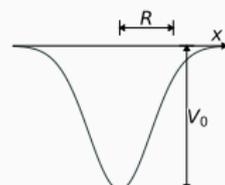
$$R = \frac{\pi^2}{9} \rho r_{\langle p^2 \rangle}^2 \left[1 + \left(2 - \frac{\pi^2}{4} \right) \frac{2}{9} \pi \rho^2 r_{\langle p^2 \rangle}^2 + \mathcal{O}(\delta^3) \right] \quad (17e)$$

$$(17f)$$

Numerical calculation of Bowl-Shaped and Gaussian potential

A little different definition of R 

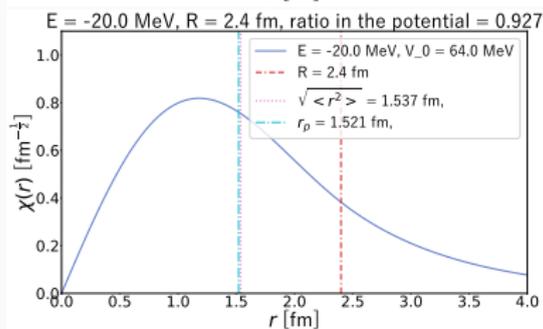
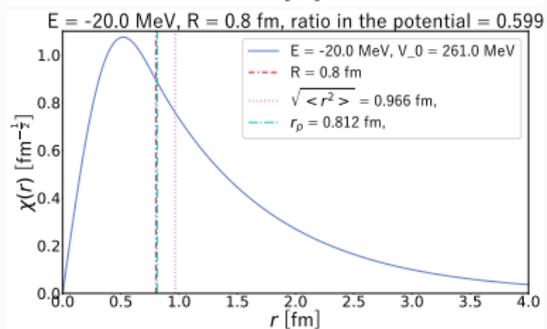
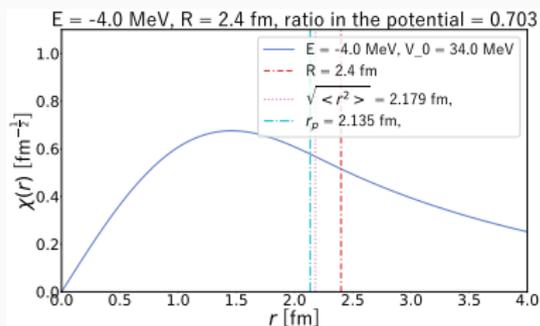
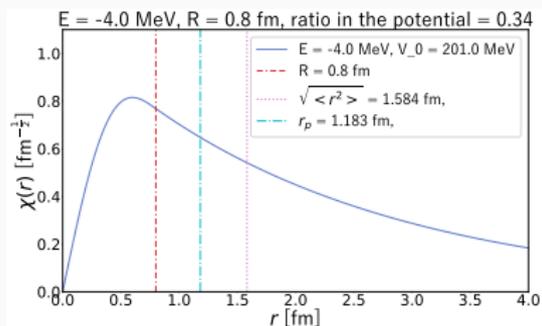
Bowl-shape



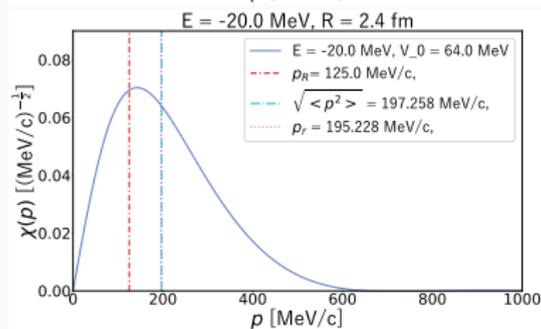
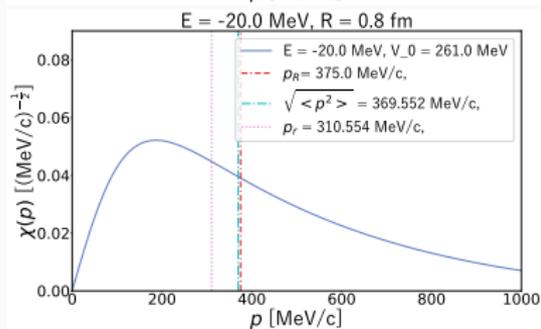
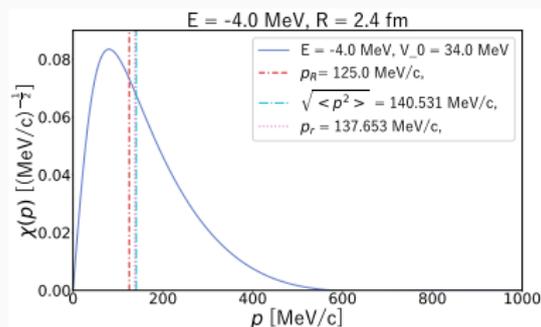
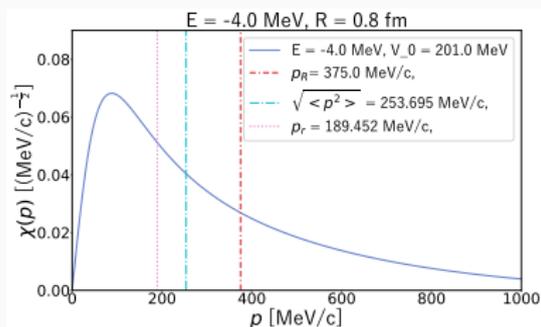
Gaussian

Similar relations

- Multiplying R by a constant factor \Rightarrow More match
- Potential shape dependence is small.



Shallowly bound \Rightarrow WF spreads out



Shallowly bound \Rightarrow High momentum tail