Addressing the sign-problem in Euclidean path integrals with radial basis function neural networks

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Contents

- **Path integrals and its numerical challenges**
- **Neural network model**
  - For quantum mechanics
  - For quantum field theory
- **Results**
  - Sign problem
  - Silver baze region

G.Balassa, arXiv:2510.01695
(RBF applied to complex scalar fields at finite densities, sign problem, silver blaze)

G.Balassa, arXiv:2509.18785
(RBF applied to φ^4 in 1+1D, phase transitions, renormalized masses)

G.Balassa, arXiv:2509.16953
(MLP-RBF applied to quantum mechanics)

Path integrals at finite densities

- Euclidean path integral formalism
  - Wick rotation → imaginary time → $S = \int_{-\infty}^{+\infty} d^D x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)),$

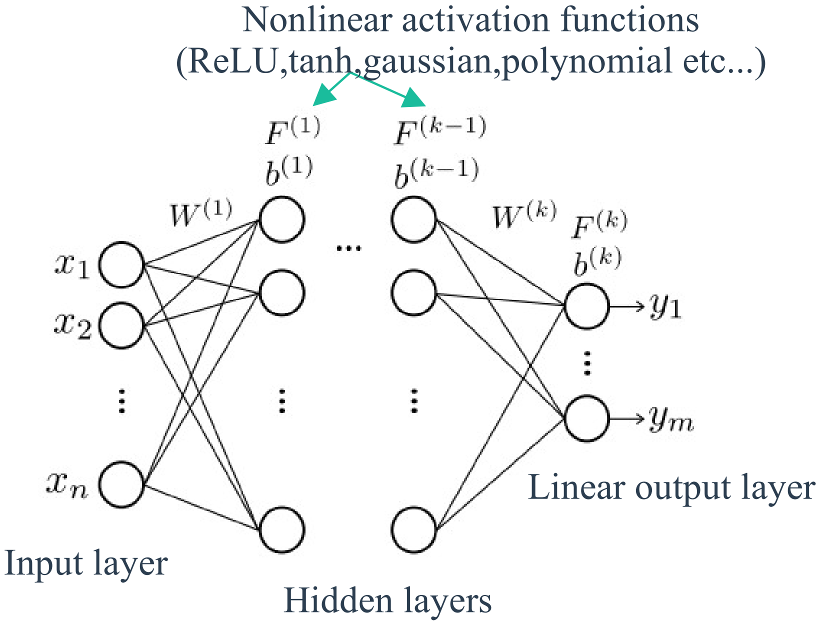
$$\mathcal{Z}_0 = \int \mathcal{D}\phi(x) \, e^{-S(\phi(x))}.$$

- Correlators → renormalized masses/couplings/…
- Thermodynamics (free energy, heat capacity, entropy, pressure,...)

- Key challenges:
  - Nonlinear interactions → no closed form solutions
  - Large dimensional integral→ importance sampling
    - Positive definite probability measure is needed
    - Finite density→ complex → sign problem, silver blaze,...
  - Many other problems e.g. fermion doubling, gauge invariance, chiral symmetry,...

Neural network model

- **Many different constructions**
  - MLP, RBF, CNN, SVM, INN,…
- **Important aspects:**
  - In essence it is a multiparameter fit ...but with good generalization capabilities $\longrightarrow$ very good for system identification
  - Universal approximator
  - MIMO, SISO, MISO, etc… systems
  - Supervised, unsupervised learning
  - Can identify nonlinear dynamical systems (MLP $\leftrightarrow$ Volterra kernels)



$$\mathbf{y} = \mathbf{F}^{(k)}(\mathbf{b}^k + \mathbf{W}^k\mathbf{F}^{k-1}(\mathbf{b}^{k-1} + \mathbf{W}^{k-1}\mathbf{F}^{k-2}(\ldots\ldots\mathbf{F}^1(\mathbf{b}^1 + \mathbf{w}^1\mathbf{x})))),$$

Warm-up: RBF for quantum mechanics

- **In 1D-QM with a general V(x) potential:**

$$K(x_f, x_i, T) = \int\limits_{\substack{x(T)=x_f \\ x(0)=x_i}} \mathcal{D}x(\tau) \exp\left(-\int_0^T d\tau \left(\frac{m}{2}\dot{x}^2 + V(x)\right)\right),$$

  - ’Well-defined’ boundaries
  - ’Well defined’ paths

  $\Big\}$ for training the network

- **Expand the interaction terms with an ’RBF ’ network:** ***(MLP by structure RBF by heart)***

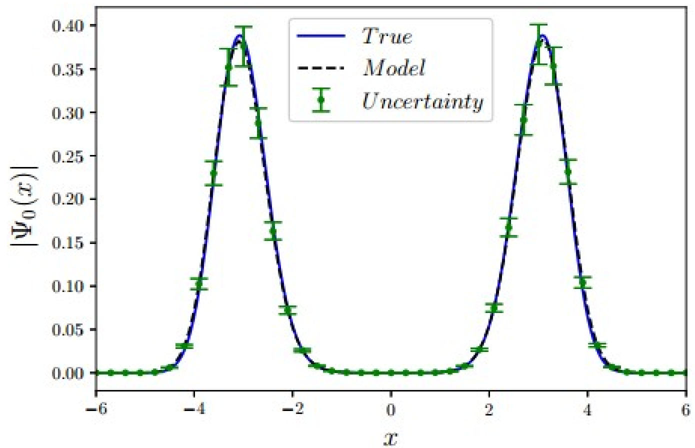
$$\exp\left(-\int_0^T dt\, V(x)\right) \approx d_0 + \sum_{k=1}^{M} \hat{a}_k(T) \exp\left(-\int_0^T d\tau \left[A_k\left(x + \frac{B_k}{2A_k}\right)^2\right]\right),$$

Free particle

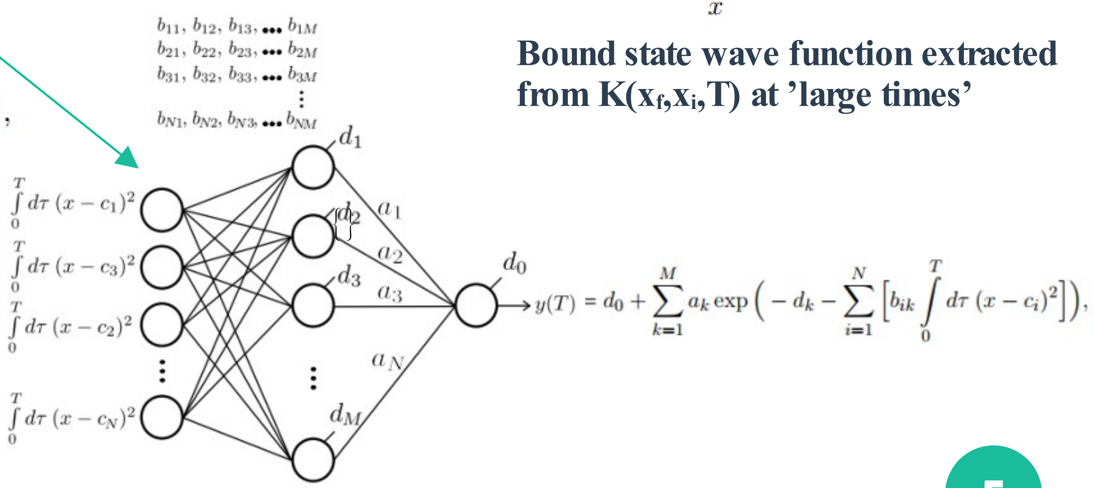
$$K(x_f, x_i, T) \approx d_0 \int\limits_{\substack{x(T)=x_f \\ x(0)=x_i}} \mathcal{D}x(\tau) \exp\left(-\int_0^T dt \left(\frac{m}{2}\dot{x}^2\right)\right) +$$

$$+ \sum_{k=1}^{M} \hat{a}_k(T) \int\limits_{\substack{x(T)=x_f \\ x(0)=x_i}} \mathcal{D}x(\tau) \exp\left(-\int_0^T d\tau \left[\frac{m}{2}\dot{x}^2 + A_k\left(x + \frac{B_k}{2A_k}\right)^2\right]\right),$$

sum of M harmonic oscillators



Bound state wave function extracted from $K(x_f,x_i,T)$ at ’large times’



Interacting complex scalar fields at finite μ

- Field theory → 'global' network is not that straightforward → local 'RBF' at each lattice site
- **μ** couples to the conserved charge → non hermitian evolution

$$S = \int d^4x |\partial_\mu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\phi^* \partial_4 \phi - \phi \partial_4 \phi^*) + \lambda(\phi^* \phi)^2,$$

$$\longrightarrow \quad S = \sum_x S_{0,x}(\mu) + S_{1,x}(m, \lambda),$$

$$S_{0,x}(\mu) = 4 \sum_{a=1,2} \phi_{a,x}^2 - \sum_{i=1}^{3} \sum_{a=1,2} \phi_{a,x}\phi_{a,x+\hat{i}} - \cosh(\mu) \sum_{a=1,2} \phi_{a,x}\phi_{a,x+\hat{4}} + \underline{i \sinh(\mu) \left[\phi_{1,x}\phi_{2,x+\hat{4}} - \phi_{2,x}\phi_{1,x+\hat{4}}\right]},$$

$$S_{1,x}(m, \lambda) = \frac{m^2}{2} \sum_{a=1,2} \phi_{a,x}^2 + \frac{\lambda}{4} \underline{\left(\sum_{a=1,2} \phi_{a,x}^2\right)^2}.$$

- RBF expansion of the interaction → sum of quadratic path integrals
  - Fixed (A) RBF widths (always a possible choice)
  - k-dependent (c_k) centers
  - a_k linear weight factors

$\left.\right\}$ **'Unsupervised learning'**

$$Z \approx \int \prod d\phi_{1,x} d\phi_{2,x} \, e^{-S_{0,x}(\mu)} \cdot F[\phi_{a,x}] \quad \longrightarrow \quad F[\phi_{a,x}] = \sum_{k=1}^{K} a_k \, e^{-A \sum_a \phi_{a,x}^2 + 2A \sum_a c_{a,k}\phi_{a,x} - A \sum_a c_{a,k}^2},$$

Interacting complex scalar fields at finite μ

- Diagonalize (Fourier transform):
  - Clear separation in the quadratic parts due to the fixed 'A' widths
  - Mixing of momentum modes in the linear shifts due to the k-dependent RBF centers

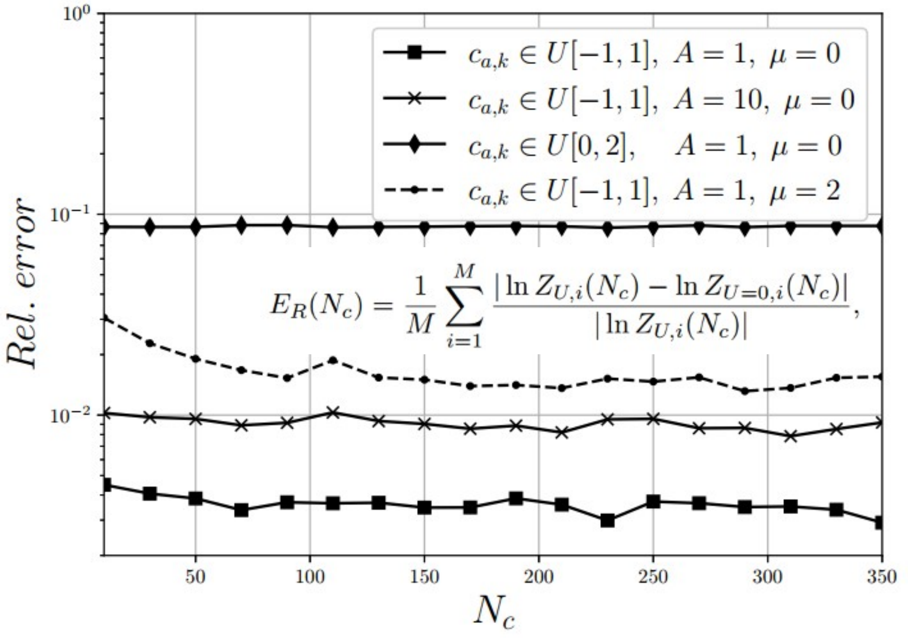
$$Z \approx \sum_{k \in \{1,\dots,K\}^{N^4}} \hat{a}_k \int \mathcal{D}\phi \, e^{-\frac{1}{2}\phi^T W \phi + 2A(\hat{c}_k^T \phi) - A(1^T \hat{c}_k^2)},$$

$$Z \approx \sum_{k \in \{1,\dots,K\}^{N^4}} \hat{a}_k \int \mathcal{D}\widetilde{\Phi} \, e^{-\frac{1}{2}\widetilde{\Phi}^T (U^T W U)\widetilde{\Phi} + 2A(U^T \hat{c}_k)^T \widetilde{\Phi} - A(1^T \hat{c}_k)},$$

Choosing 'good' RBF parameters:

- **Symmetric centers around 0 and some corresponding 'A' widths**

$$Z^{rbf} = \prod_p \sum_{k=1}^{K} a_k \int \mathcal{D}\widetilde{\Phi}_p \exp\Big[-\frac{1}{2} \sum_{a,b} \widetilde{\Phi}_{a,-p} \widetilde{W}_{ab,p} \widetilde{\Phi}_{b,p} + 2A \sum_a c_{a,k} \widetilde{\Phi}_{a,p} - A \sum_a c_{a,k}^2 \Big]$$



Phase transition

- Phase transition (Bose condensation) from the effective potential (momentum independent part of the generating functional)

$$V_{eff}(\phi_1) \propto \int d\langle\phi_1\rangle J(\langle\phi_1\rangle),$$

- Add constant background fields:

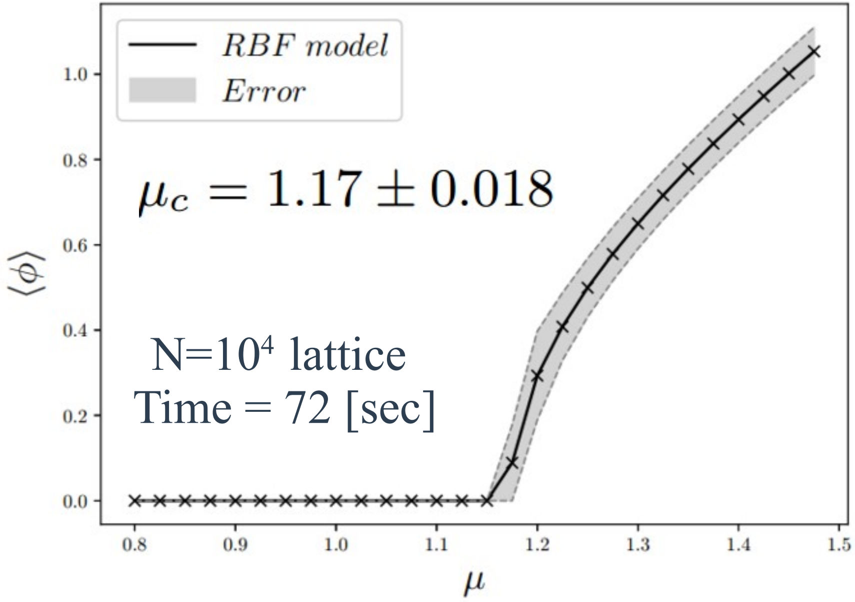
  J→ <φ>$_J$ → J(<φ>) → V_{eff}

$$S_{1,x} = \frac{m^2}{2}\sum_{a=1,2}\phi_{a,x}^2 + \frac{\lambda}{4}\Big(\sum_{a=1,2}\phi_{a,x}^2\Big)^2 + \sum_{a=1,2} J_a\phi_{a,x}$$

- Minimum of V_{eff} corresponds to <vev>
  - $\mu<\mu_c$: <vev> = 0
  - $\mu>\mu_c$: <vev> > 0 ==== Bose condensation

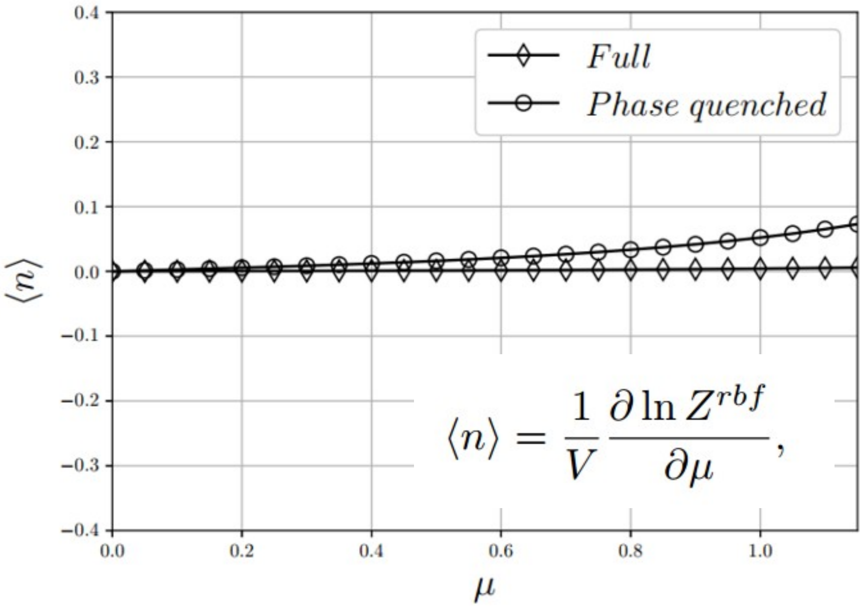
G. Aarts, Phys. Rev. Lett. 102, 10.1103/physrevlett.102.131601 (2009).

(From complex Langevin dynamics → μ_c≈1.16)



Silver blaze phenomenon

- Silver blaze: physical observables remain independent of μ at T=0, up to a critical threshold.
- Reason: severe cancellations due to the imaginary parts in the path integral.
- Any correct method shuld reproduce this behaviour.
- Phase quenched⟶ neglect the off-diagonal terms in the nonhermitian quadratic matrix ~ no imaginary parts⟶ no cancellation.



Summary and future plans

- Approximate path integrals with RBF networks (global, or local) → the approximation can be solved in a closed form
  - Very fast results (phase transition points, renormalized masses, …)
  - No sign problem, no problem at the silver blaze region
- Future and on-going works:
  - Grassman fields → needs special care (Berezin integrals…)
  - Gauge theories → how to approximate the plaquet action…
  - <u>Apply to QCD</u> (the final boss)

Thank you for your attention!