Exploring ALP EFTs

Operator Basis Construction and Hilbert Series Techniques

Chang-Yuan Yao
In collaboration with Christophe Grojean, Jonathan Kley

Based on JHEP 11 (2023), 196 [arXiv:2307.08563]
Jun 13, 2024

DESY & Nankai University

Table of contents

Introduction to ALP EFTs

- **Axions**: Proposed to solve the strong CP problem in QCD.
- **ALPs**: Generalization of axions, potentially explaining dark matter and other phenomena.
- **Shift symmetry**: Goldstone nature under the spontaneously broken $U(1)_{\text{PQ}}$ symmetry in the PQ mechanism.
- **Shift breaking**: From a model building point of view, some breaking of the shift symmetry is allowed (Graham et al., 2015; Espinosa et al., 2015; Franceschini et al., 2016).
- **Operator basis**: Dim-5 (Georgi et al., 1986), Dim-6 (Bauer et al., 2017, 2019; Brivio et al., 2021; Bonilla et al., 2021), Dim-7 [incomplete] (Bauer et al., 2016, 2017).
- **What about Dim-8**: positivity bounds, matching calculations, mesons/nucleon decays.
- **How to build basis**: Hilbert series as a guide.

Hilbert Series Techniques

The *Hilbert series* is a mathematical tool that allows one to determine the number of independent invariants in a theory by considering the power series representation.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \sum_{r_1, \ldots, r_n} \sum_k c_{\mathbf{r}\,k}\, \phi_1^{r_1} \ldots \phi_n^{r_n} \mathcal{D}^k,$$

Hilbert Series Techniques

The *Hilbert series* is a mathematical tool that allows one to determine the number of independent invariants in a theory by considering the power series representation.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \sum_{r_1,\ldots,r_n} \sum_k c_{\mathbf{r}\,k}\, \phi_1^{r_1} \ldots \phi_n^{r_n} \mathcal{D}^k,$$

For instance,

$$\mathcal{H}_{\text{SMEFT}}^{\text{dim-6}} \supset 2L^\dagger e Q^\dagger u + \mathcal{D}H^2 u^\dagger d.$$

$$\mathcal{L}_{\text{SMEFT}}^{\text{dim-6}} \supset (\bar{L}^j e)\epsilon_{jk}(\bar{Q}^k u) + (\bar{L}^j \sigma_{\mu\nu} e)\epsilon_{jk}(\bar{Q}^k \sigma^{\mu\nu} u) + (\widetilde{H}^\dagger D_\mu H)(\bar{u}\gamma^\mu d).$$

Hilbert Series Techniques

The *Hilbert series* is a mathematical tool that allows one to determine the number of independent invariants in a theory by considering the power series representation.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \sum_{r_1,\ldots,r_n} \sum_k c_{\mathbf{r}\,k}\, \phi_1^{r_1} \ldots \phi_n^{r_n} \mathcal{D}^k,$$

For instance,

$$\mathcal{H}^{\text{dim-6}}_{\text{SMEFT}} \supset 2L^\dagger e Q^\dagger u + \mathcal{D} H^2 u^\dagger d.$$

$$\mathcal{L}^{\text{dim-6}}_{\text{SMEFT}} \supset (\bar{L}^j e)\epsilon_{jk}(\bar{Q}^k u) + (\bar{L}^j \sigma_{\mu\nu} e)\epsilon_{jk}(\bar{Q}^k \sigma^{\mu\nu} u) + (\widetilde{H}^\dagger D_\mu H)(\bar{u}\gamma^\mu d).$$

The *Hilbert series* is a mathematical tool that allows one to determine the number of independent invariants in a theory by considering the power series representation.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \sum_{r_1,\ldots,r_n} \sum_k c_{\mathbf{r}\,k}\, \phi_1^{r_1} \ldots \phi_n^{r_n} \mathcal{D}^k,$$

For instance,

$$\mathcal{H}^{\text{dim-6}}_{\text{SMEFT}} \supset 2L^\dagger e Q^\dagger u + \mathcal{D}H^2 u^\dagger d.$$

$$\mathcal{L}^{\text{dim-6}}_{\text{SMEFT}} \supset (\bar{L}^j e)\epsilon_{jk}(\bar{Q}^k u) + (\bar{L}^j \sigma_{\mu\nu} e)\epsilon_{jk}(\bar{Q}^k \sigma^{\mu\nu} u) + (\widetilde{H}^\dagger D_\mu H)(\bar{u}\gamma^\mu d).$$

Hilbert Series Techniques

The *Hilbert series* is a mathematical tool that allows one to determine the number of independent invariants in a theory by considering the power series representation.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \sum_{r_1,\ldots,r_n} \sum_k c_{\mathbf{r}\,k}\, \phi_1^{r_1} \ldots \phi_n^{r_n} \mathcal{D}^k,$$

For instance,

$$\mathcal{H}_{\text{SMEFT}}^{\text{dim-6}} \supset 2L^\dagger e Q^\dagger u + \mathcal{D}H^2 u^\dagger d.$$

$$\mathcal{L}_{\text{SMEFT}}^{\text{dim-6}} \supset (\bar{L}^j e)\epsilon_{jk}(\bar{Q}^k u) + (\bar{L}^j \sigma_{\mu\nu} e)\epsilon_{jk}(\bar{Q}^k \sigma^{\mu\nu} u) + (\widetilde{H}^\dagger D_\mu H)(\bar{u}\gamma^\mu d).$$

The calculation of the *Hilbert series* can be accomplished by using the orthonormality of group characters, i.e.,

$$\int d\mu_G(g)\, \chi_{\mathbf{R}}(g)\, \chi^*_{\mathbf{R}'}(g) = \delta_{\mathbf{R},\mathbf{R}'}\ ,$$

where $\chi_{\mathbf{R}}(g)$ is the character of representation $\mathbf{R}$ of a group G with $g \in G$, and $d\mu_G$ is the Haar measure.

Hilbert Series Techniques

By considering all possible tensor products, the orthonormality of the group characters allows one to project these products onto the group invariants. The generating function is called the plethystic exponential (PE) (an $U(1)$ example: Lehman and Martin, 2015).

$$\mathsf{PE}\left[\phi_{\mathbf{R}}\,\chi_{\mathbf{R}}(z)\right] = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r}(\pm 1)^{r+1}\phi_{\mathbf{R}}^{r}\,\chi_{\mathbf{R}}(z^{r})\right)\,,$$

Hilbert Series Techniques

By considering all possible tensor products, the orthonormality of the group characters allows one to project these products onto the group invariants. The generating function is called the plethystic exponential (PE) (an $U(1)$ example: Lehman and Martin, 2015).

$$\mathsf{PE}\left[\phi_{\mathbf{R}}\,\chi_{\mathbf{R}}(z)\right] = \exp\left(\sum_{r=1}^{\infty}\frac{1}{r}(\pm 1)^{r+1}\phi_{\mathbf{R}}^{r}\,\chi_{\mathbf{R}}(z^{r})\right)\,,$$

The *Hilbert series* can be obtained after the group integration

$$\mathcal{H}(\{\phi_i\}) = \int d\mu_G \prod_i \mathsf{PE}[\phi_i]\,.$$

Hilbert Series Techniques

By considering all possible tensor products, the orthonormality of the group characters allows one to project these products onto the group invariants. The generating function is called the plethystic exponential (PE) (an $U(1)$ example: Lehman and Martin, 2015).

$$\mathsf{PE}\left[\phi_{\mathbf{R}}\,\chi_{\mathbf{R}}(z)\right] = \exp\left(\sum_{r=1}^{\infty}\frac{1}{r}(\pm 1)^{r+1}\phi_{\mathbf{R}}^{r}\,\chi_{\mathbf{R}}(z^{r})\right)\,,$$

The *Hilbert series* can be obtained after the group integration

$$\mathcal{H}(\{\phi_i\}) = \int d\mu_G \prod_i \mathsf{PE}[\phi_i]\,.$$

Procedure

1. introduce fields and their reps.
2. find group characters
3. calculate PE up to some order
4. perform group integration

Hilbert Series Techniques

By considering all possible tensor products, the orthonormality of the group characters allows one to project these products onto the group invariants. The generating function is called the plethystic exponential (PE) (an $U(1)$ example: Lehman and Martin, 2015).

$$\text{PE}\left[\phi_{\mathbf{R}}\, \chi_{\mathbf{R}}(z)\right] = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} (\pm 1)^{r+1} \phi_{\mathbf{R}}^{r}\, \chi_{\mathbf{R}}(z^r)\right) ,$$

The *Hilbert series* can be obtained after the group integration

$$\mathcal{H}(\{\phi_i\}) = \int d\mu_G \prod_i \text{PE}[\phi_i] \,.$$

Procedure

1. introduce fields and their reps.
2. find group characters
3. calculate PE up to some order
4. perform group integration

Problems

- EOM redundancy
- IBP redundancy

Conformal representations: (Henning et al., 2017)

Conformal Representation

single particle module

$$R_\phi = \begin{pmatrix} \phi \\ \partial_{\mu_1}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2\}}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}}\phi \\ \vdots \end{pmatrix}$$

$\{\dots\}$: symmetric, traceless

conformal representation

- symmetric: avoid field strength
- traceless: remove EOM $\partial^2\phi = m^2\phi$

$$\chi^{(d)}_{(n)}(x) = \begin{cases} \chi^{(d)}_{\mathrm{sym}^n(\square)}(x) & n < 2 \\ \chi^{(d)}_{\mathrm{sym}^n(\square)}(x) - \chi^{(d)}_{\mathrm{sym}^{n-2}(\square)}(x) & n \geq 2 \end{cases}$$

$$\widetilde{\chi}^{(d)}_{[\Delta;\underline{0}]}(q;x) = \sum_{n=0}^{\infty} q^{\Delta+n}\chi^{(d)}_{(n)}(x) = q^{\Delta}(1-q^2)P^{(d)}(q;x)$$

$(\partial^2\phi, \partial_\mu\partial^2\phi, \partial_{\mu_1}\partial_{\mu_2}\partial^2\phi, \cdots)$ is subtracted from R_ϕ

Conformal Representation

single particle module

$$R_\phi = \begin{pmatrix} \phi \\ \partial_{\mu_1}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2\}}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}}\phi \\ \vdots \end{pmatrix}$$

$\{\dots\}$: symmetric, traceless

conformal representation

- symmetric: avoid field strength
- traceless: remove EOM $\partial^2\phi = m^2\phi$

$$\chi^{(d)}_{(n)}(x) = \begin{cases} \chi^{(d)}_{\text{sym}^n(\Box)}(x) & n < 2 \\ \chi^{(d)}_{\text{sym}^n(\Box)}(x) - \chi^{(d)}_{\text{sym}^{n-2}(\Box)}(x) & n \geq 2 \end{cases}$$

$$\widetilde{\chi}^{(d)}_{[\Delta;\underline{0}]}(q;x) = \sum_{n=0}^{\infty} q^{\Delta+n}\chi^{(d)}_{(n)}(x) = q^{\Delta}(1-q^2)P^{(d)}(q;x)$$

$(\partial^2\phi, \partial_\mu\partial^2\phi, \partial_{\mu_1}\partial_{\mu_2}\partial^2\phi, \cdots)$ is subtracted from R_ϕ

multi-particle module

$$R^{\otimes n}_{\phi_{l'}} \sim \sum_{\mathcal{O}_l} \begin{pmatrix} \mathcal{O}_l \\ \partial\mathcal{O}_l \\ \partial^2\mathcal{O}_l \\ \vdots \end{pmatrix}$$

tensor product decomposition

scalar conformal primaries

$$\chi^{(d)}_{[\Delta;l]}(q;x) = \sum_{n=0}^{\infty} q^{\Delta+n}\chi^{(d)}_{\text{sym}^n(\Box)}(x)\chi^{(d)}_l(x)$$
$$= q^{\Delta}\chi^{(d)}_l(x)P^{(d)}(q;x),$$

$$P^{(d)}(q;x) \equiv \sum_{n=0}^{\infty} q^n \chi^{(d)}_{\text{sym}^n(\Box)}(x),$$

- Characters are proportional to P
- Multiplying by $1/P$ will remove IBP

Hilbert Series for Operator Basis

By decomposing the tensor products of the R_{ϕ_i} into conformal reps.
The operator basis is spanned by scalar, conformal primaries.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \int d\mu_{\text{Lorentz}} \int d\mu_{\text{gauge}} \frac{1}{P} \prod_i \text{PE}\left[\frac{\phi_i}{\mathcal{D}^{d_i}} \chi_i\right] + \Delta\mathcal{H}(\mathcal{D}, \{\phi_i\})$$

Hilbert Series for Operator Basis

By decomposing the tensor products of the R_{ϕ_i} into conformal reps.
The operator basis is spanned by scalar, conformal primaries.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \int d\mu_{\text{Lorentz}} \int d\mu_{\text{gauge}} \frac{1}{P} \prod_i \text{PE}\left[\frac{\phi_i}{\mathcal{D}^{d_i}} \chi_i\right] + \Delta\mathcal{H}(\mathcal{D}, \{\phi_i\})$$

Remove IBP

The function P is

$$P(\mathcal{D}, \alpha, \beta) = \frac{1}{(1-\mathcal{D}\alpha\beta)(1-\mathcal{D}/\alpha\beta)(1-\mathcal{D}\alpha/\beta)(1-\mathcal{D}\beta/\alpha)}$$

Hilbert Series for Operator Basis

By decomposing the tensor products of the R_{ϕ_i} into conformal reps.
The operator basis is spanned by scalar, conformal primaries.

$$\mathcal{H}(\mathcal{D}, \{\phi_i\}) = \int d\mu_{\text{Lorentz}} \int d\mu_{\text{gauge}} \frac{1}{P} \prod_i \text{PE}\left[\frac{\phi_i}{\mathcal{D}^{d_i}} \chi_i\right] + \Delta\mathcal{H}(\mathcal{D}, \{\phi_i\})$$

The function P is

$$P(\mathcal{D}, \alpha, \beta) = \frac{1}{(1 - \mathcal{D}\alpha\beta)(1 - \mathcal{D}/\alpha\beta)(1 - \mathcal{D}\alpha/\beta)(1 - \mathcal{D}\beta/\alpha)}$$

$\chi_i = \chi_{[d_i,(j_1,j_2)_i]} \chi_i^{\text{gauge}} \sim$ conformal $\otimes$ gauge. The conformal characters for the SM are (Henning et al., 2017)

$$\chi_{[1,(0,0)]}(\mathcal{D}, \alpha, \beta) = \mathcal{D}^{1} P(\mathcal{D}, \alpha, \beta)(1 - \mathcal{D}^2)$$

$$\chi_{[\frac{3}{2},(\frac{1}{2},0)]}(\mathcal{D}, \alpha, \beta) = \mathcal{D}^{\frac{3}{2}} P(\mathcal{D}, \alpha, \beta) \left(\alpha + \frac{1}{\alpha} - \mathcal{D}\left(\beta + \frac{1}{\beta}\right)\right)$$

$$\chi_{[\frac{3}{2},(0,\frac{1}{2})]}(\mathcal{D}, \alpha, \beta) = \mathcal{D}^{\frac{3}{2}} P(\mathcal{D}, \alpha, \beta) \left(\beta + \frac{1}{\beta} - \mathcal{D}\left(\alpha + \frac{1}{\alpha}\right)\right)$$

$$\chi_{[2,(1,0)]}(\mathcal{D}, \alpha, \beta) = \mathcal{D}^{2} P(\mathcal{D}, \alpha, \beta) \left(\alpha^2 + 1 + \frac{1}{\alpha^2} - \mathcal{D}\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) + \mathcal{D}^2\right)$$

$$\chi_{[2,(0,1)]}(\mathcal{D}, \alpha, \beta) = \mathcal{D}^{2} P(\mathcal{D}, \alpha, \beta) \left(\beta^2 + 1 + \frac{1}{\beta^2} - \mathcal{D}\left(\beta + \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\alpha}\right) + \mathcal{D}^2\right)$$

Hilbert Series for Operator Basis

By decomposing the tensor products of the R_{ϕ_i} into conformal reps.
The operator basis is spanned by scalar, conformal primaries.

$$\mathcal{H}(\mathcal{D},\{\phi_i\}) = \int d\mu_{\text{Lorentz}} \int d\mu_{\text{gauge}} \frac{1}{P} \prod_i \text{PE}\left[\frac{\phi_i}{\mathcal{D}^{d_i}}\chi_i\right] + \Delta\mathcal{H}(\mathcal{D},\{\phi_i\})$$

The function P is

$$P(\mathcal{D},\alpha,\beta) = \frac{1}{(1-\mathcal{D}\alpha\beta)(1-\mathcal{D}/\alpha\beta)(1-\mathcal{D}\alpha/\beta)(1-\mathcal{D}\beta/\alpha)}$$

$\chi_i = \chi_{[d_i,(j_1,j_2)_i]}\chi_i^{\text{gauge}} \sim$ conformal $\otimes$ gauge. The conformal characters for the SM are (Henning et al., 2017)

Remove EOM
(Green's basis relevant)

$$\chi_{[1,(0,0)]}(\mathcal{D},\alpha,\beta) = \mathcal{D}\, P(\mathcal{D},\alpha,\beta)(1-\mathcal{D}^2)$$

$$\chi_{[\frac{3}{2},(\frac{1}{2},0)]}(\mathcal{D},\alpha,\beta) = \mathcal{D}^{\frac{3}{2}} P(\mathcal{D},\alpha,\beta)\left(\alpha+\frac{1}{\alpha}-\mathcal{D}\left(\beta+\frac{1}{\beta}\right)\right)$$

$$\chi_{[\frac{3}{2},(0,\frac{1}{2})]}(\mathcal{D},\alpha,\beta) = \mathcal{D}^{\frac{3}{2}} P(\mathcal{D},\alpha,\beta)\left(\beta+\frac{1}{\beta}-\mathcal{D}\left(\alpha+\frac{1}{\alpha}\right)\right)$$

$$\chi_{[2,(1,0)]}(\mathcal{D},\alpha,\beta) = \mathcal{D}^{2} P(\mathcal{D},\alpha,\beta)\left(\alpha^2+1+\frac{1}{\alpha^2}-\mathcal{D}\left(\alpha+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right)+\mathcal{D}^2\right)$$

$$\chi_{[2,(0,1)]}(\mathcal{D},\alpha,\beta) = \mathcal{D}^{2} P(\mathcal{D},\alpha,\beta)\left(\beta^2+1+\frac{1}{\beta^2}-\mathcal{D}\left(\beta+\frac{1}{\beta}\right)\left(\alpha+\frac{1}{\alpha}\right)+\mathcal{D}^2\right)$$

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} \not{a} \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

The conformal character of a shift-symmetric singlet scalar is

$$\begin{aligned} \chi_{\partial a}(\mathcal{D}, x) &= \sum_{n=1}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x) \\ &= \mathcal{D}^{d_a}\left(-1 + \sum_{n=0}^{\infty} \mathcal{D}^n \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^n \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x)\right) \\ &= \mathcal{D}\left(\left(1-\mathcal{D}^2\right) P(\mathcal{D}, x) - 1\right) . \end{aligned}$$

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

The conformal character of a shift-symmetric singlet scalar is

$$\begin{aligned}
\chi_{\partial a}(\mathcal{D}, x) &= \sum_{n=1}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x) \\
&= \mathcal{D}^{d_a}\left(-1 + \sum_{n=0}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x)\right) \\
&= \mathcal{D}\left(\left(1-\mathcal{D}^2\right) P\left(\mathcal{D}, x\right) - 1\right) .
\end{aligned}$$

Scaling dimension

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

The conformal character of a shift-symmetric singlet scalar is

$$\begin{aligned} \chi_{\partial a}(\mathcal{D}, x) &= \sum_{n=1}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x) \\ &= \mathcal{D}^{d_a}\left(-1 + \sum_{n=0}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x)\right) \\ &= \mathcal{D}\left(\left(1 - \mathcal{D}^2\right) P(\mathcal{D}, x) - 1\right) \, . \end{aligned}$$

Remove EOM

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

The conformal character of a shift-symmetric singlet scalar is

$$\begin{aligned} \chi_{\partial a}(\mathcal{D}, x) &= \sum_{n=1}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x) \\ &= \mathcal{D}^{d_a}\left(-1 + \sum_{n=0}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x)\right) \\ &= \mathcal{D}\left(\left(1-\mathcal{D}^2\right) P(\mathcal{D}, x) - 1\right) . \end{aligned}$$

Generate tower of derivatives

Shift-Symmetric Conformal Character

For the ALP, in order to implement the derivative coupling, we have to remove the scalar itself as a building block from the Hilbert series, amounting to removing the first entry from the single particle module. This yields

$$R_a = \begin{pmatrix} a \\ \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix} \longrightarrow R_{\partial a} = \begin{pmatrix} \partial_{\mu_1} a \\ \partial_{\{\mu_1}\partial_{\mu_2\}} a \\ \partial_{\{\mu_1}\partial_{\mu_2}\partial_{\mu_3\}} a \\ \vdots \end{pmatrix}$$

The conformal character of a shift-symmetric singlet scalar is

$$\begin{aligned} \chi_{\partial a}(\mathcal{D}, x) &= \sum_{n=1}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n+d_a} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x) \\ &= \mathcal{D}^{d_a}\left(-1 + \sum_{n=0}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^n\left(\frac{1}{2},\frac{1}{2}\right)}(x) - \sum_{n=2}^{\infty} \mathcal{D}^{n} \chi_{\mathsf{Sym}^{n-2}\left(\frac{1}{2},\frac{1}{2}\right)}(x)\right) \\ &= \mathcal{D}\left(\left(1-\mathcal{D}^2\right) P(\mathcal{D}, x) - 1\right) . \end{aligned}$$

Remove non-shift-symmetric part

Ingredients of the Hilbert Series

Depending on whether there is a shift symmetry for the axion, four types of EFTs are defined with corresponding spurions as follows

- aSMEFT$_{\text{PQ}}$: SMEFT extended with a shift-symmetric axion
  $\{\mathcal{D}, \partial a, Q, Q^\dagger, L, L^\dagger, H, H^\dagger, u, u^\dagger, d, d^\dagger, e, e^\dagger, B_L, B_R, W_L, W_R, G_L, G_R\}$,
- aSMEFT$_{\not{\text{PQ}}}$: SMEFT extended with a non-shift-symmetric axion
  $\{\mathcal{D}, a, Q, Q^\dagger, L, L^\dagger, H, H^\dagger, u, u^\dagger, d, d^\dagger, e, e^\dagger, B_L, B_R, W_L, W_R, G_L, G_R\}$,

Ingredients of the Hilbert Series

Depending on whether there is a shift symmetry for the axion, four types of EFTs are defined with corresponding spurions as follows

- aSMEFT$_{\text{PQ}}$: SMEFT extended with a shift-symmetric axion
  $\{\mathcal{D}, \partial a, Q, Q^\dagger, L, L^\dagger, H, H^\dagger, u, u^\dagger, d, d^\dagger, e, e^\dagger, B_L, B_R, W_L, W_R, G_L, G_R\}$,
- aSMEFT$_{\not{\text{PQ}}}$: SMEFT extended with a non-shift-symmetric axion
  $\{\mathcal{D}, a, Q, Q^\dagger, L, L^\dagger, H, H^\dagger, u, u^\dagger, d, d^\dagger, e, e^\dagger, B_L, B_R, W_L, W_R, G_L, G_R\}$,

	$SU(2)_L$	$SU(2)_R$	$SU(3)_c$	$SU(2)_W$	$U(1)_Y$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$1/2$
Q	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$
u	$\mathbf{2}$	$\mathbf{1}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-2/3$
d	$\mathbf{2}$	$\mathbf{1}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$1/3$
L	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
e	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	1
G_L	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{1}$	0
W_L	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0
B_L	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0

Table 1: The aSMEFT field content and their charges under Lorentz and gauge groups.

$$\chi_R^{\text{gauge}} = \chi_R^{U(1)}\chi_R^{SU(2)}\chi_R^{SU(3)}$$

$$\chi_Q^{U(1)}(x) = x^Q$$

$$\chi_{\mathbf{2}}^{SU(2)}(y) = \chi_{\bar{\mathbf{2}}}^{SU(2)}(y) = y + \frac{1}{y}$$

$$\chi_{\mathbf{ad}}^{SU(2)}(y) = y^2 + 1 + \frac{1}{y^2}$$

$$\chi_{\mathbf{3}}^{SU(3)}(z_1, z_2) = z_1 + \frac{z_2}{z_1} + \frac{1}{z_2},$$

$$\chi_{\bar{\mathbf{3}}}^{SU(3)}(z_1, z_2) = z_2 + \frac{z_1}{z_2} + \frac{1}{z_1}$$

$$\chi_{\mathbf{ad}}^{SU(3)}(z_1, z_2) = z_1 z_2 + \frac{z_2^2}{z_1} + \frac{z_1^2}{z_2} + 2 + \frac{z_1}{z_2^2} + \frac{z_2}{z_1^2} + \frac{1}{z_1 z_2}$$

Hilbert Series and Isolation Condition

Hilbert series of $\text{aSMEFT}_{\text{PQ}}$

$$
\begin{aligned}
\mathcal{H}_5^{\text{PQ}} &= \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger + \partial a\, LL^\dagger + \partial a\, ee^\dagger + 3aX^2\,, \\
\mathcal{H}_6^{\text{PQ}} &= (\partial a)^2 HH^\dagger\,, \\
\mathcal{H}_7^{\text{PQ}} &= \partial a\, QQ^\dagger B_L + \partial a\, QQ^\dagger B_R + \partial a\, QQ^\dagger G_L + \partial a\, QQ^\dagger G_R + \partial a\, QQ^\dagger W_L + \partial a\, QQ^\dagger W_R \\
&\quad + \partial a\, uu^\dagger B_L + \partial a\, uu^\dagger B_R + \partial a\, uu^\dagger G_L + \partial a\, uu^\dagger G_R + \partial a\, dd^\dagger B_L + \partial a\, dd^\dagger B_R \\
&\quad + \partial a\, dd^\dagger G_L + \partial a\, dd^\dagger G_R + \partial a\, LL^\dagger B_L + \partial a\, LL^\dagger B_R + \partial a\, LL^\dagger W_L + \partial a\, LL^\dagger W_R \\
&\quad + \partial a\, ee^\dagger B_L + \partial a\, ee^\dagger B_R + 2\partial a\, QQ^\dagger HH^\dagger + \partial a\, uu^\dagger HH^\dagger + \partial a\, dd^\dagger HH^\dagger \\
&\quad + 2\partial a\, LL^\dagger HH^\dagger + \partial a\, ee^\dagger HH^\dagger + \partial a\, B_L HH^\dagger \mathcal{D} + \partial a\, B_R HH^\dagger \mathcal{D} + \partial a\, W_L HH^\dagger \mathcal{D} \\
&\quad + \partial a\, W_R HH^\dagger \mathcal{D} + \partial a\, H^2 H^{\dagger 2} \mathcal{D} + 2\partial a\, QuH\mathcal{D} + 2\partial a\, Q^\dagger u^\dagger H^\dagger \mathcal{D} + 2\partial a\, QdH^\dagger \mathcal{D} \\
&\quad + 2\partial a\, Q^\dagger d^\dagger H\mathcal{D} + 2\partial a\, LeH^\dagger \mathcal{D} + 2\partial a\, L^\dagger e^\dagger H\mathcal{D}\,, \\
&\ \ \vdots \\
\mathcal{H}_{15}^{\text{PQ}} &= \ldots
\end{aligned}
$$

Hilbert Series and Isolation Condition

Hilbert series of aSMEFT$_{\mathsf{PQ}}$

$$\mathcal{H}_5^{\mathsf{PQ}} = \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger + \partial a\, LL^\dagger + \partial a\, ee^\dagger + 3aX^2\,,$$

$$\mathcal{H}_6^{\mathsf{PQ}} = (\partial a)^2 HH^\dagger\,,$$

$$\begin{aligned}\mathcal{H}_7^{\mathsf{PQ}} =\ & \partial a\, QQ^\dagger B_L + \partial a\, QQ^\dagger B_R + \partial a\, QQ^\dagger G_L + \partial a\, QQ^\dagger G_R + \partial a\, QQ^\dagger W_L + \partial a\, QQ^\dagger W_R \\ & + \partial a\, uu^\dagger B_L + \partial a\, uu^\dagger B_R + \partial a\, uu^\dagger G_L + \partial a\, uu^\dagger G_R + \partial a\, dd^\dagger B_L + \partial a\, dd^\dagger B_R \\ & + \partial a\, dd^\dagger G_L + \partial a\, dd^\dagger G_R + \partial a\, LL^\dagger B_L + \partial a\, LL^\dagger B_R + \partial a\, LL^\dagger W_L + \partial a\, LL^\dagger W_R \\ & + \partial a\, ee^\dagger B_L + \partial a\, ee^\dagger B_R + 2\partial a\, QQ^\dagger HH^\dagger + \partial a\, uu^\dagger HH^\dagger + \partial a\, dd^\dagger HH^\dagger \\ & + 2\partial a\, LL^\dagger HH^\dagger + \partial a\, ee^\dagger HH^\dagger + \partial a\, B_L HH^\dagger\mathcal{D} + \partial a\, B_R HH^\dagger\mathcal{D} + \partial a\, W_L HH^\dagger\mathcal{D} \\ & + \partial a\, W_R HH^\dagger\mathcal{D} + \partial a\, H^2 H^{\dagger 2}\mathcal{D} + 2\partial a\, QuH\mathcal{D} + 2\partial a\, Q^\dagger u^\dagger H^\dagger\mathcal{D} + 2\partial a\, QdH^\dagger\mathcal{D} \\ & + 2\partial a\, Q^\dagger d^\dagger H\mathcal{D} + 2\partial a\, LeH^\dagger\mathcal{D} + 2\partial a\, L^\dagger e^\dagger H\mathcal{D}\,, \\ & \vdots \end{aligned}$$

$$\mathcal{H}_{15}^{\mathsf{PQ}} = \ldots$$

Peccei–Quinn breaking isolation condition

$$\mathcal{H}_n^{\not{\mathsf{PQ}}} = a\,\mathcal{H}_{n-1}^{\not{\mathsf{PQ}}} + a\,\mathcal{H}_{n-1}^{\mathsf{SMEFT}} + \mathcal{H}_n^{\mathsf{PQ}}(\partial a \to a\mathcal{D})\,, \qquad n > 5$$

Operator Counting and N_f Dependence

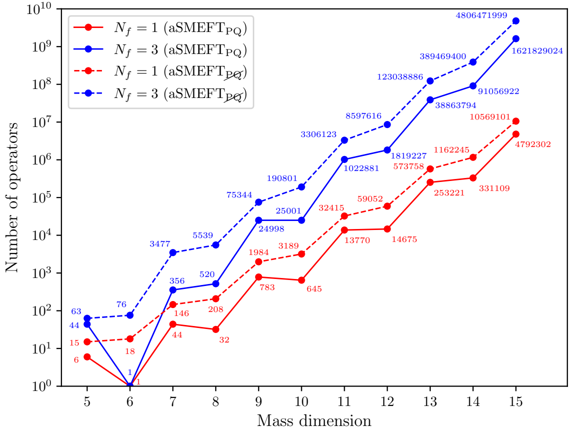

Figure 1: The number of operators in the aSMEFT with and without a shift symmetry for the ALP plotted against the mass dimension for $N_f = 1$ and $N_f = 3$ number of flavors.

Constructing Operator Bases

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$

Constructing Operator Bases

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$
$$\mathcal{O}_5^{\mathsf{PQ}} \supset \partial_\mu a\, \bar{Q}\gamma^\mu Q\,,\ \partial_\mu a\, \bar{u}\gamma^\mu u\,,\ \partial_\mu a\, \bar{d}\gamma^\mu d$$

Constructing Operator Bases

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$
$$\mathcal{O}_5^{\mathsf{PQ}} \supset \partial_\mu a\, \bar{Q}\gamma^\mu Q\,,\ \partial_\mu a\, \bar{u}\gamma^\mu u\,,\ \partial_\mu a\, \bar{d}\gamma^\mu d$$

- Hard example

$$\mathcal{H}_8^{\mathsf{PQ}} \supset (\partial a)^2 B_L^2 + (\partial a)^2 B_L B_R + (\partial a)^2 B_R^2$$

Constructing Operator Bases

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$
$$\mathcal{O}_5^{\mathsf{PQ}} \supset \partial_\mu a\, \bar{Q}\gamma^\mu Q\,,\ \partial_\mu a\, \bar{u}\gamma^\mu u\,,\ \partial_\mu a\, \bar{d}\gamma^\mu d$$

- Hard example

$$\mathcal{H}_8^{\mathsf{PQ}} \supset (\partial a)^2 B_L^2 + (\partial a)^2 B_L B_R + (\partial a)^2 B_R^2$$

However, one can naively build 4 operators:

$$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho},\ \partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} \widetilde{B}_{\nu\rho}$$

Constructing Operator Bases

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$
$$\mathcal{O}_5^{\mathsf{PQ}} \supset \partial_\mu a\, \bar{Q}\gamma^\mu Q\,,\ \partial_\mu a\, \bar{u}\gamma^\mu u\,,\ \partial_\mu a\, \bar{d}\gamma^\mu d$$

- Hard example

$$\mathcal{H}_8^{\mathsf{PQ}} \supset (\partial a)^2 B_L^2 + (\partial a)^2 B_L B_R + (\partial a)^2 B_R^2$$

However, one can naively build 4 operators:

$$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho},\ \partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} \widetilde{B}_{\nu\rho}$$

We can use the Schouten identity to derive

$$T_{\mu\nu} X^{\mu\rho} \widetilde{X}^{\nu}{}_{\rho} = \frac{1}{4} T^\mu_\mu X_{\nu\rho} \widetilde{X}^{\nu\rho}$$

where $T^{\mu\nu}$ is generic tensor (which we identify with $\partial^\mu a \partial^\nu a$) and $X_{\mu\nu}$ is an anti-symmetric tensor.

- Simple example

$$\mathcal{H}_5^{\mathsf{PQ}} \supset \partial a\, QQ^\dagger + \partial a\, uu^\dagger + \partial a\, dd^\dagger$$
$$\mathcal{O}_5^{\mathsf{PQ}} \supset \partial_\mu a\, \bar{Q}\gamma^\mu Q\,,\ \partial_\mu a\, \bar{u}\gamma^\mu u\,,\ \partial_\mu a\, \bar{d}\gamma^\mu d$$

- Hard example

$$\mathcal{H}_8^{\mathsf{PQ}} \supset (\partial a)^2 B_L^2 + (\partial a)^2 B_L B_R + (\partial a)^2 B_R^2$$

However, one can naively build 4 operators:

$$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho},\ \partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho},\ \partial_\mu a \partial^\nu a\, B^{\mu\rho} \widetilde{B}_{\nu\rho}$$

We can use the Schouten identity to derive

One is redundant!

$$T_{\mu\nu} X^{\mu\rho} \widetilde{X}^{\nu}{}_{\rho} = \frac{1}{4} T^\mu_\mu X_{\nu\rho} \widetilde{X}^{\nu\rho}$$

where $T^{\mu\nu}$ is generic tensor (which we identify with $\partial^\mu a \partial^\nu a$) and $X_{\mu\nu}$ is an anti-symmetric tensor.

$(\partial a)^2 X^2$		$(\partial a)^2 \psi^2 D$	
$\mathcal{O}^{(1)}_{\partial a^2 B}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho}$	$\mathcal{O}_{\partial a^2 LD}$	$\partial_\mu a \partial_\nu a\, (\bar{L}\gamma^\mu \overleftrightarrow{D}^\nu L)$
$\mathcal{O}^{(2)}_{\partial a^2 B}$	$\partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho}$	$\mathcal{O}_{\partial a^2 eD}$	$\partial_\mu a \partial_\nu a\, (\bar{e}\gamma^\mu \overleftrightarrow{D}^\nu e)$
$\mathcal{O}_{\partial a^2 \tilde{B}}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho}$	$\mathcal{O}_{\partial a^2 QD}$	$\partial_\mu a \partial_\nu a\, (\bar{Q}\gamma^\mu \overleftrightarrow{D}^\nu Q)$
$\mathcal{O}^{(1)}_{\partial a^2 W}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} W^{I,\nu\rho}$	$\mathcal{O}_{\partial a^2 uD}$	$\partial_\mu a \partial_\nu a\, (\bar{u}\gamma^\mu \overleftrightarrow{D}^\nu u)$
$\mathcal{O}^{(2)}_{\partial a^2 W}$	$\partial_\mu a \partial^\nu a\, W^{I,\mu\rho} W^I_{\nu\rho}$	$\mathcal{O}_{\partial a^2 dD}$	$\partial_\mu a \partial_\nu a\, (\bar{d}\gamma^\mu \overleftrightarrow{D}^\nu d)$
$\mathcal{O}^{(2)}_{\partial a^2 \tilde{W}}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} \widetilde{W}^{I,\nu\rho}$	$(\partial a)^2 \psi^2 H +$ h.c.	
$\mathcal{O}^{(1)}_{\partial a^2 G}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} G^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 eH}$	$\partial_\mu a \partial^\mu a\, \bar{L} H e$
$\mathcal{O}^{(2)}_{\partial a^2 G}$	$\partial_\mu a \partial^\nu a\, G^{a,\mu\rho} G^a_{\nu\rho}$	$\mathcal{O}_{\partial a^2 uH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H u$
$\mathcal{O}_{\partial a^2 \tilde{G}}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} \widetilde{G}^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 dH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H d$
$(\partial a)^4$		$(\partial a)^2 H^2 D^2$	
$\mathcal{O}_{\partial a^4}$	$\partial_\mu a \partial^\mu a \partial_\nu a \partial^\nu a$	$\mathcal{O}^{(1)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial^\mu a D_\nu H^\dagger D^\nu H$
$(\partial a)^2 H^4$		$\mathcal{O}^{(2)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial_\nu a D^\mu H^\dagger D^\nu H$
$\mathcal{O}_{\partial a^2 H^4}$	$\partial_\mu a \partial^\mu a \lvert H\rvert^4$		
$\not{B}$ and $\not{L}$ terms			
$\partial a\, \psi^4 +$ h.c.		$\partial a\, \psi^2 H^2 D +$ h.c.	
$\mathcal{O}_{\partial a Ldu}$	$\partial_\mu a\, (\bar{L}^c L)(\bar{d}\gamma^\mu u)$	$\mathcal{O}^{(1)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c H)(\tilde{H}^\dagger D^\mu L)$
$\mathcal{O}_{\partial a LQd}$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{L} d_\alpha)(\bar{Q}^c_\beta \gamma^\mu d_\gamma)$	$\mathcal{O}^{(2)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c D^\mu H)(\tilde{H}^\dagger L)$
$\mathcal{O}_{\partial a ed}$ $(\star)$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{d}^c_\alpha d_\beta)(\bar{e}\gamma^\mu d_\gamma)$		

Table 2: Operators in the $\mathrm{aSMEFT_{PQ}}$ at mass dimension 8

aSMEFT and aLEFT Operator Bases

$(\partial a)^2 X^2$		$(\partial a)^2 \psi^2 D$	
$\mathcal{O}^{(1)}_{\partial a^2 B}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho}$	$\mathcal{O}_{\partial a^2 LD}$	$\partial_\mu a \partial_\nu a\, (\bar{L}\gamma^\mu \overleftrightarrow{D}^\nu L)$
$\mathcal{O}^{(2)}_{\partial a^2 B}$	$\partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho}$	$\mathcal{O}_{\partial a^2 eD}$	$\partial_\mu a \partial_\nu a\, (\bar{e}\gamma^\mu \overleftrightarrow{D}^\nu e)$
$\mathcal{O}_{\partial a^2 \tilde{B}}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho}$	$\mathcal{O}_{\partial a^2 QD}$	$\partial_\mu a \partial_\nu a\, (\bar{Q}\gamma^\mu \overleftrightarrow{D}^\nu Q)$
$\mathcal{O}^{(1)}_{\partial a^2 W}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} W^{I,\nu\rho}$	$\mathcal{O}_{\partial a^2 uD}$	$\partial_\mu a \partial_\nu a\, (\bar{u}\gamma^\mu \overleftrightarrow{D}^\nu u)$
$\mathcal{O}^{(2)}_{\partial a^2 W}$	$\partial_\mu a \partial^\nu a\, W^{I,\mu\rho} W^I_{\nu\rho}$	$\mathcal{O}_{\partial a^2 dD}$	$\partial_\mu a \partial_\nu a\, (\bar{d}\gamma^\mu \overleftrightarrow{D}^\nu d)$
$\mathcal{O}^{(2)}_{\partial a^2 \tilde{W}}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} \widetilde{W}^{I,\nu\rho}$	$(\partial a)^2 \psi^2 H +$ h.c.	
$\mathcal{O}^{(1)}_{\partial a^2 G}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} G^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 eH}$	$\partial_\mu a \partial^\mu a\, \bar{L} H e$
$\mathcal{O}^{(2)}_{\partial a^2 G}$	$\partial_\mu a \partial^\nu a\, G^{a,\mu\rho} G^a_{\nu\rho}$	$\mathcal{O}_{\partial a^2 uH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H u$
$\mathcal{O}_{\partial a^2 \tilde{G}}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} \widetilde{G}^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 dH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H d$
$(\partial a)^4$		$(\partial a)^2 H^2 D^2$	
$\mathcal{O}_{\partial a^4}$	$\partial_\mu a \partial^\mu a \partial_\nu a \partial^\nu a$	$\mathcal{O}^{(1)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial^\mu a D_\nu H^\dagger D^\nu H$
$(\partial a)^2 H^4$		$\mathcal{O}^{(2)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial_\nu a D^\mu H^\dagger D^\nu H$
$\mathcal{O}_{\partial a^2 H^4}$	$\partial_\mu a \partial^\mu a \lvert H \rvert^4$		
$\not{B}$ and $\not{L}$ terms			
$\partial a\, \psi^4 +$ h.c.		$\partial a\, \psi^2 H^2 D +$ h.c.	
$\mathcal{O}_{\partial a L d u}$	$\partial_\mu a\, (\bar{L}^c L)(\bar{d}\gamma^\mu u)$	$\mathcal{O}^{(1)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c H)(\tilde{H}^\dagger D^\mu L)$
$\mathcal{O}_{\partial a L Q d}$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{L} d_\alpha)(\bar{Q}^c_\beta \gamma^\mu d_\gamma)$	$\mathcal{O}^{(2)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c D^\mu H)(\tilde{H}^\dagger L)$
$\mathcal{O}_{\partial a e d}$ $(\star)$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{d}^c_\alpha d_\beta)(\bar{e}\gamma^\mu d_\gamma)$		

Table 2: Operators in the $\mathrm{aSMEFT}_{\mathrm{PQ}}$ at mass dimension 8

In addition, we construct operator basis of $\mathrm{aSMEFT}_{\mathrm{PQ}}$, $\mathrm{aSMEFT}_{\not{\mathrm{PQ}}}$, $\mathrm{aLEFT}_{\mathrm{PQ}}$, $\mathrm{aLEFT}_{\not{\mathrm{PQ}}}$ up to dimension 8.

aSMEFT and aLEFT Operator Bases

$(\partial a)^2 X^2$		$(\partial a)^2 \psi^2 D$	
$\mathcal{O}^{(1)}_{\partial a^2 B}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} B^{\nu\rho}$	$\mathcal{O}_{\partial a^2 LD}$	$\partial_\mu a \partial_\nu a\, (\bar{L}\gamma^\mu \overleftrightarrow{D}^\nu L)$
$\mathcal{O}^{(2)}_{\partial a^2 B}$	$\partial_\mu a \partial^\nu a\, B^{\mu\rho} B_{\nu\rho}$	$\mathcal{O}_{\partial a^2 eD}$	$\partial_\mu a \partial_\nu a\, (\bar{e}\gamma^\mu \overleftrightarrow{D}^\nu e)$
$\mathcal{O}_{\partial a^2 \tilde{B}}$	$\partial_\mu a \partial^\mu a\, B_{\nu\rho} \widetilde{B}^{\nu\rho}$	$\mathcal{O}_{\partial a^2 QD}$	$\partial_\mu a \partial_\nu a\, (\bar{Q}\gamma^\mu \overleftrightarrow{D}^\nu Q)$
$\mathcal{O}^{(1)}_{\partial a^2 W}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} W^{I,\nu\rho}$	$\mathcal{O}_{\partial a^2 uD}$	$\partial_\mu a \partial_\nu a\, (\bar{u}\gamma^\mu \overleftrightarrow{D}^\nu u)$
$\mathcal{O}^{(2)}_{\partial a^2 W}$	$\partial_\mu a \partial^\nu a\, W^{I,\mu\rho} W^I_{\nu\rho}$	$\mathcal{O}_{\partial a^2 dD}$	$\partial_\mu a \partial_\nu a\, (\bar{d}\gamma^\mu \overleftrightarrow{D}^\nu d)$
$\mathcal{O}^{(2)}_{\partial a^2 \tilde{W}}$	$\partial_\mu a \partial^\mu a\, W^I_{\nu\rho} \widetilde{W}^{I,\nu\rho}$	$(\partial a)^2 \psi^2 H + \text{h.c.}$	
$\mathcal{O}^{(1)}_{\partial a^2 G}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} G^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 eH}$	$\partial_\mu a \partial^\mu a\, \bar{L} H e$
$\mathcal{O}^{(2)}_{\partial a^2 G}$	$\partial_\mu a \partial^\nu a\, G^{a,\mu\rho} G^a_{\nu\rho}$	$\mathcal{O}_{\partial a^2 uH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H u$
$\mathcal{O}_{\partial a^2 \tilde{G}}$	$\partial_\mu a \partial^\mu a\, G^a_{\nu\rho} \widetilde{G}^{a,\nu\rho}$	$\mathcal{O}_{\partial a^2 dH}$	$\partial_\mu a \partial^\mu a\, \bar{Q} H d$
$(\partial a)^4$		$(\partial a)^2 H^2 D^2$	
$\mathcal{O}_{\partial a^4}$	$\partial_\mu a \partial^\mu a \partial_\nu a \partial^\nu a$	$\mathcal{O}^{(1)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial^\mu a D_\nu H^\dagger D^\nu H$
$(\partial a)^2 H^4$		$\mathcal{O}^{(2)}_{\partial a^2 DH^2}$	$\partial_\mu a \partial_\nu a D^\mu H^\dagger D^\nu H$
$\mathcal{O}_{\partial a^2 H^4}$	$\partial_\mu a \partial^\mu a \lvert H \rvert^4$		
$\not{B}$ and $\not{L}$ terms			
$\partial a\, \psi^4 + \text{h.c.}$		$\partial a\, \psi^2 H^2 D + \text{h.c.}$	
$\mathcal{O}_{\partial a L d u}$	$\partial_\mu a\, (\bar{L}^c L)(\bar{d}\gamma^\mu u)$	$\mathcal{O}^{(1)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c H)(\tilde{H}^\dagger D^\mu L)$
$\mathcal{O}_{\partial a L Q d}$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{L} d_\alpha)(\bar{Q}^c_\beta \gamma^\mu d_\gamma)$	$\mathcal{O}^{(2)}_{\partial a LHD}$	$\partial_\mu a\, (\bar{L}^c D^\mu H)(\tilde{H}^\dagger L)$
$\mathcal{O}_{\partial a e d}$ $(\star)$	$\epsilon^{\alpha\beta\gamma} \partial_\mu a\, (\bar{d}^c_\alpha d_\beta)(\bar{e}\gamma^\mu d_\gamma)$		

Table 2: Operators in the $\text{aSMEFT}_{\text{PQ}}$ at mass dimension 8

$\text{aSMEFT}_{\text{PQ}}$, $\text{aSMEFT}_{\not{\text{PQ}}}$ are also constructed by the Young tensor method: (Song et al., 2024) [See Zhe's talk]

In addition, we construct operator basis of $\text{aSMEFT}_{\text{PQ}}$, $\text{aSMEFT}_{\not{\text{PQ}}}$, aLEFT_{PQ}, $\text{aLEFT}_{\not{\text{PQ}}}$ up to dimension 8.

Mathematica Package for Hilbert Series

(Grojean, Kley, and Yao, arxiv:2024.xxxx)
CHINCHILLA: A Mathematica package for the construction of invariants using the Hilbert series

Code **H**elping with the **IN**variant **C**onstruction using the **HIL**bert series **LA**nguage

Mathematica Package for Hilbert Series

(Grojean, Kley, and Yao, arxiv:2024.xxxx)
CHINCHILLA: A Mathematica package for the construction of invariants using the Hilbert series

Code **H**elping with the **IN**variant **C**onstruction using the **HIL**bert series **LA**nguage

Existing packages

- ABC4EFT (Li et al., 2022)
- AutoEFT (Harlander et al., 2023)
- BasisGen (Criado, 2019)
- DECO (Calò et al., 2023)
- DEFT (Gripaios and Sutherland, 2019)
- ECO (Marinissen et al., 2020)
- GrIP (Banerjee et al., 2020)
- Sym2Int (Fonseca, 2020)

⋮

Mathematica Package for Hilbert Series

(Grojean, Kley, and Yao, arxiv:2024.xxxx)
CHINCHILLA: A Mathematica package for the construction of invariants using the Hilbert series

Code **H**elping with the **IN**variant **C**onstruction using the **HIL**bert series **LA**nguage

Existing packages

- ABC4EFT (Li et al., 2022)
- AutoEFT (Harlander et al., 2023)
- BasisGen (Criado, 2019)
- DECO (Calò et al., 2023)
- DEFT (Gripaios and Sutherland, 2019)
- ECO (Marinissen et al., 2020)
- GrIP (Banerjee et al., 2020)
- Sym2Int (Fonseca, 2020)

⋮

`CHINCHILLA`

- For generic problems
- Operator basis
- Green's basis
- Charge and Parity symmetry
- Flavor invariants
- Discrete symmetry
- Covariants

Usage of the Mathematica Package

SMEFT Hilbert series by CHINCHILLA

In[1]:=
```
SetSymmetries[{"Conformal"->Conformal,"SU2"->SU[2],"SU3"->SU[3],"U1"->U[1]}];
```

In[2]:=
```
AddSpurion[H,"Conformal"->{0,0},"SU2"->2,"SU3"->1,"U1"->1/2];
AddSpurion[Hd,"Conformal"->{0,0},"SU2"->2,"SU3"->1,"U1"->-1/2];
AddSpurion[Q,"Conformal"->{1/2,0},"SU2"->2,"SU3"->3,"U1"->1/6,"Flavor"->Nf];
AddSpurion[Qd,"Conformal"->{0,1/2},"SU2"->2,"SU3"->-3,"U1"->-1/6,"Flavor"->Nf];
AddSpurion[u,"Conformal"->{1/2,0},"SU2"->1,"SU3"->-3,"U1"->-2/3,"Flavor"->Nf];
AddSpurion[ud,"Conformal"->{0,1/2},"SU2"->1,"SU3"->3,"U1"->2/3,"Flavor"->Nf];
AddSpurion[d,"Conformal"->{1/2,0},"SU2"->1,"SU3"->-3,"U1"->1/3,"Flavor"->Nf];
AddSpurion[dd,"Conformal"->{0,1/2},"SU2"->1,"SU3"->3,"U1"->-1/3,"Flavor"->Nf];
AddSpurion[L,"Conformal"->{1/2,0},"SU2"->2,"SU3"->1,"U1"->-1/2,"Flavor"->Nf];
AddSpurion[Ld,"Conformal"->{0,1/2},"SU2"->2,"SU3"->1,"U1"->1/2,"Flavor"->Nf];
AddSpurion[e,"Conformal"->{1/2,0},"SU2"->1,"SU3"->1,"U1"->1,"Flavor"->Nf];
AddSpurion[ed,"Conformal"->{0,1/2},"SU2"->1,"SU3"->1,"U1"->-1,"Flavor"->Nf];
AddSpurion[Bl,"Conformal"->{1,0},"SU2"->1,"SU3"->1,"U1"->0];
AddSpurion[Br,"Conformal"->{0,1},"SU2"->1,"SU3"->1,"U1"->0];
AddSpurion[Wl,"Conformal"->{1,0},"SU2"->3,"SU3"->1,"U1"->0];
AddSpurion[Wr,"Conformal"->{0,1},"SU2"->3,"SU3"->1,"U1"->0];
AddSpurion[Gl,"Conformal"->{1,0},"SU2"->1,"SU3"->8,"U1"->0];
AddSpurion[Gr,"Conformal"->{0,1},"SU2"->1,"SU3"->8,"U1"->0];
```

In[3]:=
```
HilbertSeries[8] (*"EOM" -> True, "IBP" -> True, "Kernel" -> 4, "CP" -> "Odd"*)
```

Conclusion

- ALP operator basis is in need for EFT studies.
- Hilbert series is a useful tool for operator counting and construction.
- EOM and IBP redundancies are effectively removed by using conformal characters.
- Shift-symmetric interaction can be easily implemented in the conformal character.
- We have constructed shift-symmetric and non-shift-symmetric operator basis for aSMEFT and aLEFT up to dimension 8.
- `CHINCHILLA` is developed to compute the Hilbert series, and will be available to use soon.

Questions?

Backup: Dim-5 Hilbert series

The *Hilbert series* with flavor dependence is given by

$$\mathcal{H}_5^{\mathsf{PQ}} = N_f^2 \partial a\, QQ^\dagger + N_f^2 \partial a\, uu^\dagger + N_f^2 \partial a\, dd^\dagger + N_f^2 \partial a\, LL^\dagger + N_f^2 \partial a\, ee^\dagger + \partial a\, HH^\dagger \mathcal{D} - \partial a\, B_L \mathcal{D} - \partial a\, B_R \mathcal{D} - \partial a\, \mathcal{D}^3 \,.$$

The minus terms can be canceled by the $\Delta\mathcal{H}$.

$$\Delta\mathcal{H} = \partial a\, B_L \mathcal{D} + \partial a\, B_R \mathcal{D} + \partial a\, \mathcal{D}^3 \,.$$

The Higgs current term can be removed by a global hypercharge transformation on the Higgs field which is not captured in our Hilbert series approach.

$$\mathcal{O}_{\partial a H} = \partial^\mu a \left(H^\dagger i \overleftrightarrow{D}_\mu H \right) .$$

This applies to the operators of type $\partial_\mu a\, \bar{\psi} \gamma^\mu \psi$ where flavor diagonal parts of the Wilson coefficients can be removed by moving the derivative to the fermions by IBP and using the conservation of baryon and lepton family number $\partial_\mu j^\mu_B = \partial_\mu j^\mu_{L_i} = 0$.
In addition, there are operators of the form $aF\widetilde{F}$ at mass dimension 5 which do not appear in the Hilbert series. This is due to the fact that we use ∂a and F as a building block. Therefore, we should add them by hands, and the final flavor dependent Hilbert series has the following form

$$\mathcal{H}_5^{\mathsf{PQ}} = (N_f^2 - 1) \partial a\, QQ^\dagger + N_f^2 \partial a\, uu^\dagger + N_f^2 \partial a\, dd^\dagger + (N_f^2 - N_f) \partial a\, LL^\dagger + N_f^2 \partial a\, ee^\dagger + 3aX^2 \,.$$

Upalaparna Banerjee, Joydeep Chakrabortty, Suraj Prakash, and Shakeel Ur Rahaman. Characters and group invariant polynomials of (super)fields: road to "Lagrangian". *Eur. Phys. J. C*, 80(10):938, 2020. doi: 10.1140/epjc/s10052-020-8392-x.

Martin Bauer, Matthias Neubert, and Andrea Thamm. Analyzing the CP Nature of a New Scalar Particle via S->Zh Decay. *Phys. Rev. Lett.*, 117:181801, 2016. doi: 10.1103/PhysRevLett.117.181801.

Martin Bauer, Matthias Neubert, and Andrea Thamm. Collider Probes of Axion-Like Particles. *JHEP*, 12:044, 2017. doi: 10.1007/JHEP12(2017)044.

Martin Bauer, Mathias Heiles, Matthias Neubert, and Andrea Thamm. Axion-Like Particles at Future Colliders. *Eur. Phys. J. C*, 79(1):74, 2019. doi: 10.1140/epjc/s10052-019-6587-9.

J. Bonilla, I. Brivio, M. B. Gavela, and V. Sanz. One-loop corrections to ALP couplings. *JHEP*, 11:168, 2021. doi: 10.1007/JHEP11(2021)168.

I. Brivio, O. J. P. Éboli, and M. C. Gonzalez-Garcia. Unitarity constraints on ALP interactions. *Phys. Rev. D*, 104(3):035027, 2021. doi: 10.1103/PhysRevD.104.035027.

Simon Calò, Coenraad Marinissen, and Rudi Rahn. Discrete symmetries and efficient counting of operators. *JHEP*, 05:215, 2023. doi: 10.1007/JHEP05(2023)215.

Juan Carlos Criado. BasisGen: automatic generation of operator bases. *Eur. Phys. J. C*, 79(3):256, 2019. doi: 10.1140/epjc/s10052-019-6769-5.

J. R. Espinosa, C. Grojean, G. Panico, A. Pomarol, O. Pujolàs, and G. Servant. Cosmological Higgs-Axion Interplay for a Naturally Small Electroweak Scale. *Phys. Rev. Lett.*, 115(25):251803, 2015. doi: 10.1103/PhysRevLett.115.251803.

Renato M. Fonseca. Enumerating the operators of an effective field theory. *Phys. Rev. D*, 101(3):035040, 2020. doi: 10.1103/PhysRevD.101.035040.

Roberto Franceschini, Gian F. Giudice, Jernej F. Kamenik, Matthew McCullough, Francesco Riva, Alessandro Strumia, and Riccardo Torre. Digamma, what next? *JHEP*, 07:150, 2016. doi: 10.1007/JHEP07(2016)150.

Howard Georgi, David B. Kaplan, and Lisa Randall. Manifesting the Invisible Axion at Low-energies. *Phys. Lett. B*, 169:73–78, 1986. doi: 10.1016/0370-2693(86)90688-X.

Peter W. Graham, David E. Kaplan, and Surjeet Rajendran. Cosmological Relaxation of the Electroweak Scale. *Phys. Rev. Lett.*, 115(22):221801, 2015. doi: 10.1103/PhysRevLett.115.221801.

Ben Gripaios and Dave Sutherland. DEFT: A program for operators in EFT. *JHEP*, 01:128, 2019. doi: 10.1007/JHEP01(2019)128.

Christophe Grojean, Jonathan Kley, and Chang-Yuan Yao. CHINCHILLA: A Mathematica package for the construction of invariants using the Hilbert series. To appear, 2024.

R. V. Harlander, T. Kempkens, and M. C. Schaaf. Standard model effective field theory up to mass dimension 12. *Phys. Rev. D*, 108 (5):055020, 2023. doi: 10.1103/PhysRevD.108.055020.

Brian Henning, Xiaochuan Lu, Tom Melia, and Hitoshi Murayama. Operator bases, S-matrices, and their partition functions. *JHEP*, 10:199, 2017. doi: 10.1007/JHEP10(2017)199.

Landon Lehman and Adam Martin. Hilbert Series for Constructing Lagrangians: expanding the phenomenologist's toolbox. *Phys. Rev. D*, 91:105014, 2015. doi: 10.1103/PhysRevD.91.105014.

Hao-Lin Li, Zhe Ren, Ming-Lei Xiao, Jiang-Hao Yu, and Yu-Hui Zheng. Operators for generic effective field theory at any dimension: on-shell amplitude basis construction. *JHEP*, 04:140, 2022. doi: 10.1007/JHEP04(2022)140.

Coenraad B. Marinissen, Rudi Rahn, and Wouter J. Waalewijn. ..., 83106786, 114382724, 1509048322, 2343463290, 27410087742, ... efficient Hilbert series for effective theories. *Phys. Lett. B*, 808: 135632, 2020. doi: 10.1016/j.physletb.2020.135632.

Huayang Song, Hao Sun, and Jiang-Hao Yu. Effective field theories of axion, ALP and dark photon. *JHEP*, 01:161, 2024. doi: 10.1007/JHEP01(2024)161.