Effective field theory for quantum fields at finite temperature

Mikael Chala
(University of Granada)

based mostly on
MC, J.C. Criado, L. Gil and J. L. Miras; *2406.02667*

HEFT 2024, Bologna; June 14, 2024

I will argue that

We need EFT methods for studying QFT at finite temperature

Still lot of work to be done; expertise on SMEFT-like EFTs very valuable!

Short summary of field theory at finite temperature

$$\mathcal{Z} = \mathrm{Tr}(e^{-\beta H}) = \int \mathcal{D}\varphi \langle \varphi | e^{-\beta H} | \varphi \rangle$$

(For equilibrium physics) equivalent to a regular Euclidean field theory with periodic time

This gives rise to so-called Matsubara modes (equivalent to Kaluza-Klein excitations in extra-dimensions), which screen the masses and couplings of zero modes

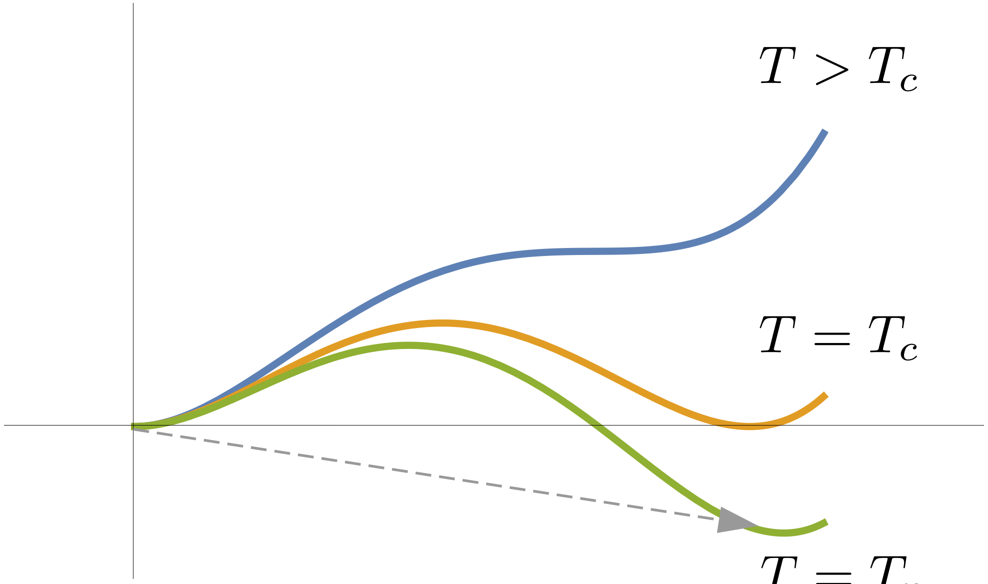
IR problems related to large hierarchy m<<T, rooted on Bose enhancement [Laine, '17]

$$n_b(\epsilon_k) = \frac{1}{(e^{\epsilon_k/T} - 1)} \rightarrow \frac{T}{\epsilon_k}$$

Phase transitions: theory and problems

A scalar field gets an effective temperature-dependent mass in its interaction with a thermal bath

$$V(\varphi) = \frac{1}{2}(m^2 + g^2T^2)\varphi^2 + \kappa\varphi^3 + \lambda\varphi^4$$



$$\mathcal{P} \sim e^{-\frac{S_3}{T}}$$

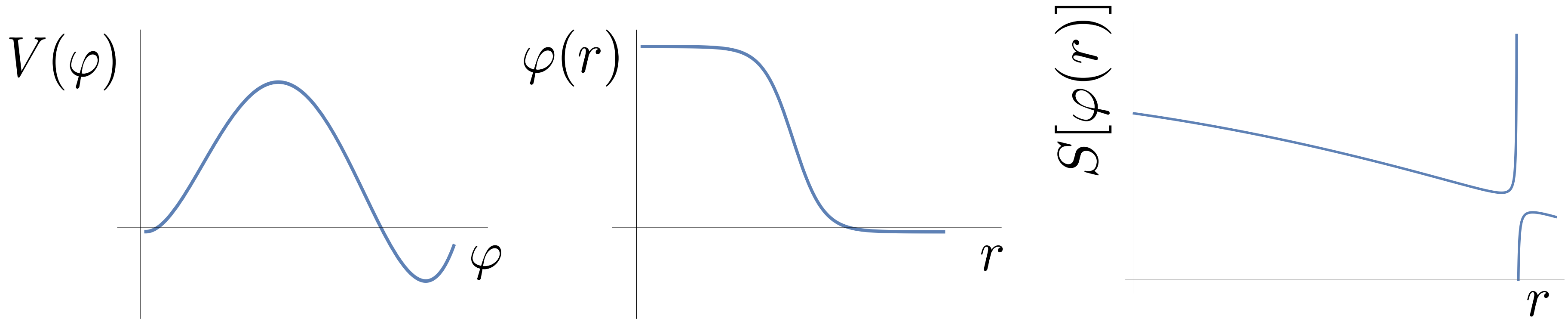
$$m_{\rm eff}^2 \ll T^2$$

Phase transitions: theory and problems

The effective action must be evaluated at the inhomogeneous bounce solution [Coleman '77]

This jeopardizes the computation of the effective action as an expansion in derivatives of the field [Berges et al '97, Strumia et al '99, Croon et al '09]

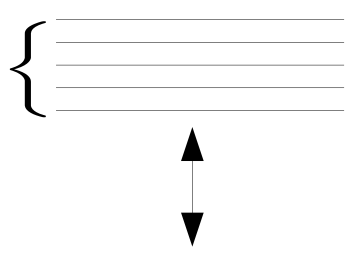
$$S_{\text{eff}} = \int \left[-V_{\text{eff}}(\phi) + \frac{1}{2}(\partial\phi)^2 (1 + \frac{g^2(\phi)}{192\pi^2 V''(\phi)}) \right]$$

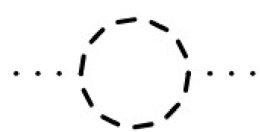


Dimensional-reduction solution

Build a 3-dimensional effective theory integrating out all Matsubara modes

$M \sim \pi T$





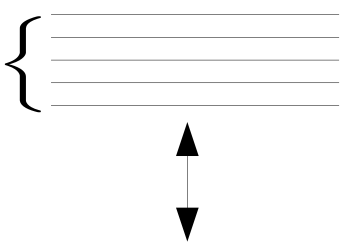
$$= \frac{(-3!\kappa)^2}{2} \sum\!\!\!\!\!\!\!\!\int{}' \frac{1}{Q^2+m^2} \frac{1}{(Q+P)^2+m^2}$$

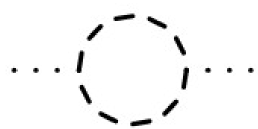
Dimensional-reduction solution

Build a 3-dimensional effective theory integrating out all Matsubara modes

Large logarithms avoided at the matching scale; summed within the EFT using renormalization group.

$M \sim \pi T$





$$= \frac{(-3!\kappa)^2}{2} \sum\!\!\!\!\!\!\!\int' \frac{1}{Q^2 + m^2} \frac{1}{(Q+P)^2 + m^2}$$

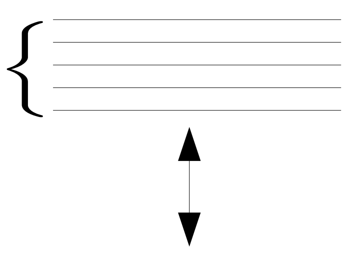
Dimensional-reduction solution

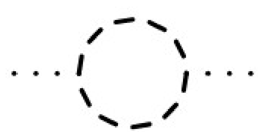
Build a 3-dimensional effective theory integrating out all Matsubara modes

Large logarithms avoided at the matching scale; summed within the EFT using renormalization group.

It solves other issues related to gauge dependence, double counting, ...

$M \sim \pi T$





$$= \frac{(-3!\kappa)^2}{2} \sum\!\!\!\!\!\!\int' \frac{1}{Q^2+m^2} \frac{1}{(Q+P)^2+m^2}$$

Properties of the 3-dimensional effective theory

Only bosons (no zero fermionic Matsubara modes) and Euclidean (it can be simulated on the lattice)

"Renormalizable" couplings are dimensionful

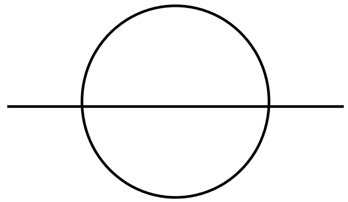
$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \kappa \varphi^3 + \lambda \varphi^4$$

$$[m] = 1$$
$$[\kappa] = 3/2$$
$$[\lambda] = 1$$

It is renormalized first at two loops

$$\text{div}_{\text{1-loop}} \sim \int \frac{d^3 k}{k^3}$$

In the absence of operators of dimension larger than 3, only the mass (and tadpoles) gets renormalized



State-of-the-art of dimensional reduction

Automation of matching for "renormalizable" terms in arbitrary theories completed [Ekstedt, Schicho, Tenkanen '22]. Two-loop sum-integrals solved recently [Davydychev, Navarrete and Schroder '23]

State-of-the-art of dimensional reduction

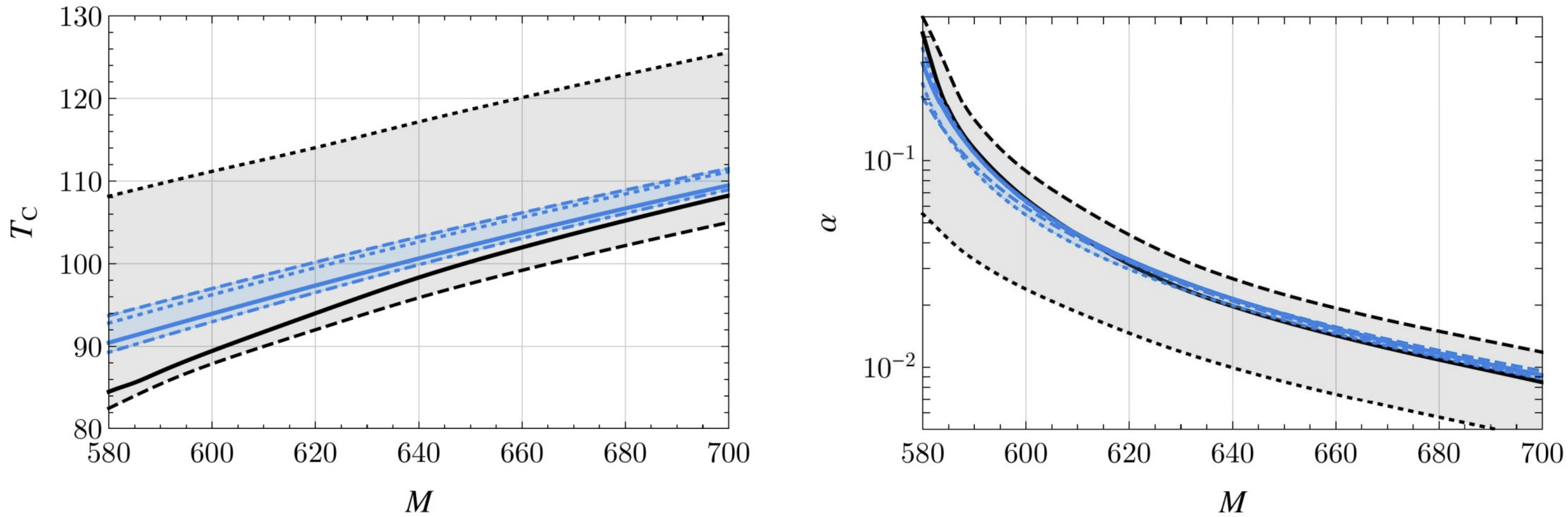
Automation of matching for “renormalizable” terms in arbitrary theories completed [Ekstedt, Schicho, Tenkanen ‘22]. Two-loop sum-integrals solved recently [Davydychev, Navarrete and Schroder ‘23]

Application to the study of phase transitions (including crossover of the SM): [Kajantie et al ‘95, Andersen ‘96, Niemi et al ‘05, D’Onofrio et al ‘16, Brauner et al ‘17, Croon et al ‘21, Gould ‘21, Hirvonen ‘22, ...]

State-of-the-art of dimensional reduction

Automation of matching for “renormalizable” terms in arbitrary theories completed [Ekstedt, Schicho, Tenkanen ‘22]. Two-loop sum-integrals solved recently [Davydychev, Navarrete and Schroder ‘23]

Application to the study of phase transitions (including crossover of the SM): [Kajantie et al ‘95, Andersen ‘96, Niemi et al ‘05, D’Onofrio et al ‘16, Brauner et al ‘17, Croon et al ‘21, Gould ‘21, Hirvonen ‘22, ...]



What (to the best of my knowledge) we are missing

The 3D EFT (including higher-dimensional operators) of most models, including the SM!

Renormalization of the 3D EFT of models such as the SM

Tools for counting/generating operators in general 3D EFTs

Tools for automating matching/running in general EFTs

Application of other techniques (e.g. functional methods, geometry, ...) providing more insight into these 3D EFTs

Applying all this to describe better phase transitions, gravitational waves, ...

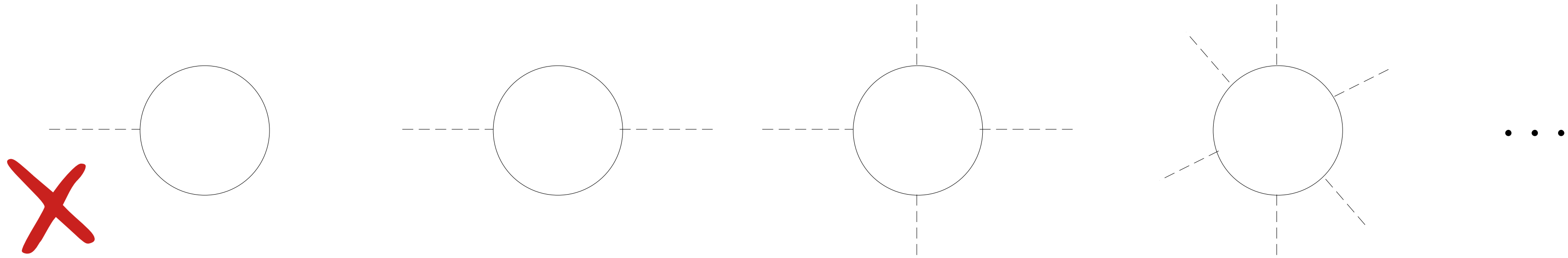
Further progress in dimensional reduction

[MC, Criado, Gil and Miras; *2406.02667*]

Strong phase transitions produce gravitational waves [see L. Gil's talk]. Observable in the strong regime, v/T > 1 (α > 0.1). Unavoidable effective operators not negligible!

Strategy: Consider a simple model, perform the matching to higher-order, compute the parameters, compare with results without higher-order terms

$$L = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \kappa\varphi^3 - \lambda\varphi^4 + i\bar{\psi}D\psi + g\varphi\bar{\psi}\psi$$



The effective field theory

The problem of building an off-shell basis of the effective field theory of a set of fields to a specified dimension is solved [Criado '19, Fonseca '19]

In our case, it reads:

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m_3^2\phi^2 + \kappa_3\phi^3 + \lambda_3\phi^4 \\
& + \alpha_{61}\phi^6 + \beta_{61}\partial^2\phi\partial^2\phi + \beta_{62}\phi^3\partial^2\phi \\
& + \alpha_{81}\phi^8 + \alpha_{82}\phi^2\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + \beta_{81}\phi\partial^6\phi + \beta_{82}\phi^3\partial^4\phi + \beta_{83}\phi^2\partial^2\phi\partial^2\phi + \beta_{84}\phi^5\partial^2\phi \\
& + \cdots
\end{aligned}
$$

All the beta operators, however, can be removed upon field redefinitions [see J. Mira's talk]

The effective field theory

The problem of building an off-shell basis of the effective field theory of a set of fields to a specified dimension is solved [Criado '19, Fonseca '19]

In our case, it reads:

$$K_3 = 1 + \frac{g^2}{12\pi^2}\,, \quad m_3^2 = m^2 + \frac{g^2T^2}{6}\,, \quad \kappa_3 = \kappa\sqrt{T}\,, \quad \lambda_3 = \lambda T$$

$$\alpha_{61} = -\frac{7\zeta(3)g^6}{192\pi^4}\,, \qquad \beta_{61} = -\frac{7\zeta(3)g^2}{384\pi^4T^2}\,, \qquad \beta_{62} = \frac{35\zeta(3)g^4}{576\pi^4T}\,;$$

$$\alpha_{81} = \frac{31\zeta(5)g^8}{2048\pi^6T}\,, \qquad \alpha_{82} = -\frac{31\zeta(5)g^4}{10240\pi^6T^3}\,, \qquad \beta_{81} = -\frac{31\zeta(5)g^2}{10240\pi^6T^4}\,,$$

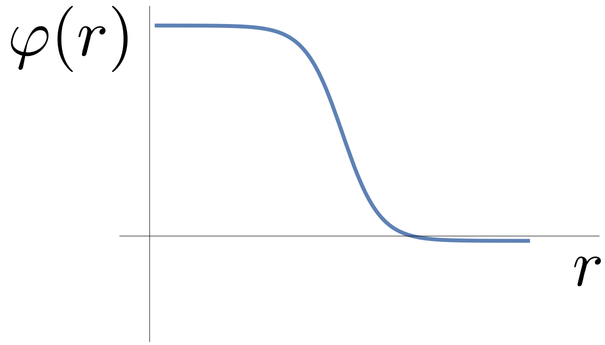
$$\beta_{82} = \frac{217\zeta(5)g^4}{20480\pi^6T^3}\,, \qquad \beta_{83} = \frac{279\zeta(5)g^4}{20480\pi^6T^3}\,, \qquad \beta_{84} = -\frac{217\zeta(5)g^6}{5120\pi^6T^2}\,.$$

All the beta operators, however, can be removed upon field redefinitions [see J. Mira's talk]

How to compute the bounce?

Well known methods/codes for computing the bounce solution in the presence of a "standard" kinetic term

$$S_3 = \int dr\, r^2 [\frac{1}{2} m^2 \varphi^2 + \frac{1}{2} (\partial\varphi)^2 + V(\varphi)]$$



$$\ddot{\varphi} + \frac{2}{r}\varphi = V'(\varphi)$$

$$\dot{\varphi}(0) = 0\,, \quad \lim_{r\to\infty} \varphi(r) = 0$$

Neither the bounce nor the effective action are physical; only the value of S_3 at extrema is (naive computations unphysical!)

Perturbative bounce solution

We have a perturbative expansion, so let's use it consistently

$$\varphi_c = \varphi_c^{(0)} + \epsilon\varphi_c^{(1)} + \epsilon^2\varphi_c^{(2)} + \cdots\,, \quad S_3 = S_3^{(0)} + \epsilon S_3^{(1)} + \epsilon^2 S_3^{(2)} + \cdots$$

Perturbative bounce solution

We have a perturbative expansion, so let's use it consistently

$$\varphi_c = \varphi_c^{(0)} + \epsilon\varphi_c^{(1)} + \epsilon^2\varphi_c^{(2)} + \cdots\,, \quad S_3 = S_3^{(0)} + \epsilon S_3^{(1)} + \epsilon^2 S_3^{(2)} + \cdots$$

$$S_3[\varphi_c] = S_3^{(0)}[\varphi_c^{(0)}] + \epsilon S_3^{(1)}[\varphi_c^0] + \epsilon^2 \left\{ S_3^{(2)}[\varphi_c^{(0)}] + 2\pi \int_0^\infty drr^2 \varphi_c^{(1)} \frac{\delta\mathcal{L}^{(1)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} \right\} + \mathcal{O}(\epsilon^3)$$

$$+\epsilon \int \varphi_1 \frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} \qquad +\epsilon^2 \int \varphi_2 \frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}}$$

Perturbative bounce solution

We have a perturbative expansion, so let's use it consistently

$$\varphi_c = \varphi_c^{(0)} + \epsilon\varphi_c^{(1)} + \epsilon^2\varphi_c^{(2)} + \cdots\,, \qquad S_3 = S_3^{(0)} + \epsilon S_3^{(1)} + \epsilon^2 S_3^{(2)} + \cdots$$

$$S_3[\varphi_c] = S_3^{(0)}[\varphi_c^{(0)}] + \epsilon S_3^{(1)}[\varphi_c^0] + \epsilon^2\left\{S_3^{(2)}[\varphi_c^{(0)}] + 2\pi\int_0^\infty dr r^2 \varphi_c^{(1)} \frac{\delta\mathcal{L}^{(1)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}}\right\} + \mathcal{O}(\epsilon^3)$$

$$+\epsilon\int\varphi_1\frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} \qquad\qquad +\epsilon^2\int\varphi_2\frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}}$$

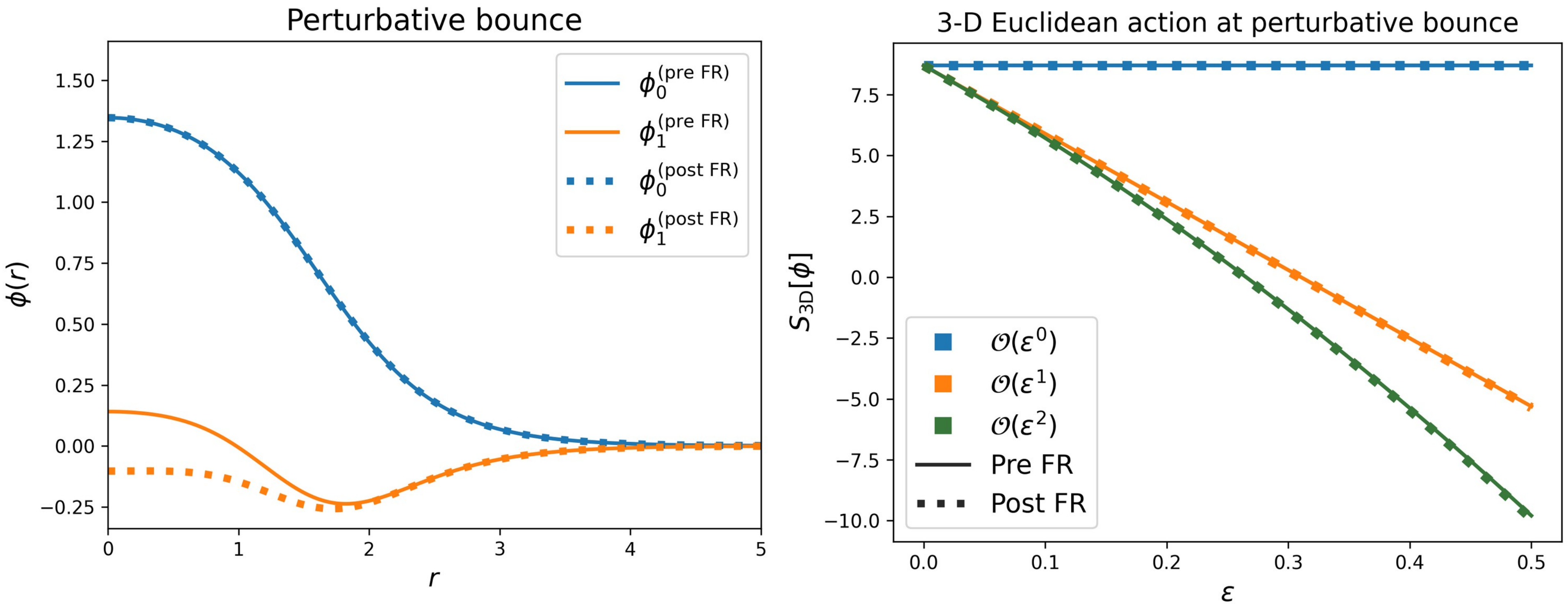
Then require that $\displaystyle \frac{\delta}{\delta\varphi}S_3\bigg|_{\varphi_c} = 0$

$$\frac{\delta}{\delta\varphi}S_3^{(0)}\bigg|_{\varphi_c^{(0)}} = 0$$

$$\ddot{\varphi}_c^{(1)} + \frac{2}{r}\dot{\varphi}_c^{(1)} - V^{(0)''}(\varphi_c^{(0)}) - \frac{1}{4\pi r^2}\frac{\delta\mathcal{L}^{(1)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} = 0$$

Perturbative bounce solution

$S_3[\varphi_c]$ computed this way is physical (i.e. invariant under field redefinitions; physical observables independent of how matching is performed)

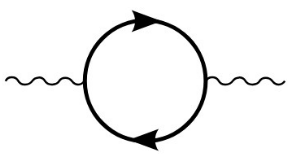


Further progress in dimensional reduction

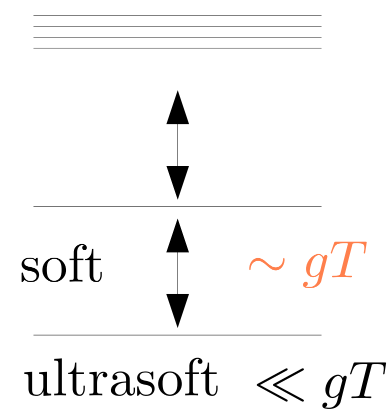
3-dimensional theory of the SM? Leading part used to determine the nature of electroweak phase transition [D'Onofrio and Rummukainen '16]

Partial results for higher-dimensional operators [Moore '95, Laine et al '18]. Complications associated to the presence of gauge bosons: further scalars (temporal components), Debye masses larger than m_{eff} [**MC**, Ekstedt and Guedes 'work in progress]

$$\mathcal{L}^{(2)}_{\text{SM3D}} = m_\phi^2 |\phi|^2 + \frac{1}{2} m_{B_0}^2 B_0^2$$



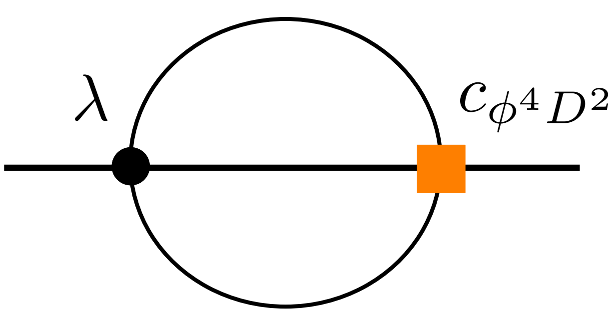
$$\mathcal{L}^{(4)}_{\text{SM3D}} = k_\phi (D_r\phi)^\dagger (D^r\phi) + \frac{k_{B_0}}{2}(D_r B_0)(D^r B_0) - \frac{k_B}{4} B_{rs}B^{rs}$$
$$+ \lambda_{\phi^4}|\phi|^4 + \lambda_{B_0^4} B_0^4 + \lambda_{\phi^2 B_0^2}|\phi|^2 B_0^2 \,,$$



Further progress in dimensional reduction

Potentially interesting effects [**MC**, Ekstedt and Guedes 'work in progress]

$$
\begin{aligned}
\mathcal{L}^{(6)}_{\rm SM3D} = {} & r_{\phi^2 D^4} D^2\phi^\dagger D^2\phi + c^{(1)}_{\phi^4 D^2}|\phi|^2\Box|\phi|^2 + c^{(2)}_{\phi^4 D^2}(\phi^\dagger D^r\phi)^\dagger(\phi^\dagger D_r\phi) \\
& + r^{(3)}_{\phi^4 D^2}|\phi|^2 D_r\phi^\dagger D^r\phi + r^{(4)}_{\phi^4 D^2}|\phi|^2 D_r(\phi^\dagger \mathrm{i}\overleftrightarrow{D}^r\phi) + c_{\phi^6}|\phi|^6 \\
& + r_{B_0^2 D^4}(D^2 B_0)^2 + r_{B_0^4 D^2} D_r(B_0^2) D^r(B_0^2) + c_{B_0^6} B_0^6 + c^{(1)}_{\phi^2 B_0^2 D^2} D_r\phi^\dagger D^r\phi B_0^2 \\
& + r^{(2)}_{\phi^2 B_0^2 D^2}|\phi|^2(D_r B_0)(D^r B_0) + r^{(3)}_{\phi^2 B_0^2 D^2} D_r|\phi|^2 D^r(B_0^2) + c_{\phi^2 B_0^4}|\phi|^2 B_0^4 + c_{\phi^4 B_0^2}|\phi|^4 B_0^2 \\
& + r_{B^2 D^2}(D_r B^{rs})(D^p B_{ps}) + c_{\phi^2 B^2}|\phi|^2 B_{rs}B^{rs} + r_{\phi B D} D_r B^{rs}(\phi^\dagger \mathrm{i}\overleftrightarrow{D}_r\phi) + c_{B_0^2 B^2} B_0^2 B_{rs} B^{rs}
\end{aligned}
$$



$$\longrightarrow (D_r\phi)^\dagger(D^r\phi)$$

(Speculative) skyrmions in the SM

Static field configurations. Finite energy implies $\lim_{r\to\infty} \partial_\theta \varphi = 0$

This is equivalent to taking fields $\varphi : \mathcal{S}^n \to M$

These field configurations can be grouped into homotopy classes related by $\pi_n(M)$. Fields in different classes can not be deformed into each other (they have different topological charge); they are stable

In the electroweak sector of the SM, the Goldstone collected in

$$U = e^{i\frac{\sigma_a G^a}{f}} \in S^3\,, \qquad \pi_3(\mathcal{S}_3) = \mathbb{Z}$$

(Speculative) skyrmions in the SM

These electroweak skyrmions have been shown to exist in the SM in the presence of four-derivative interactions [Criado, Khoze, Spannowsky ‘20]

This can be understood upon studying re-scaled solitons (Derrick’s theorem):

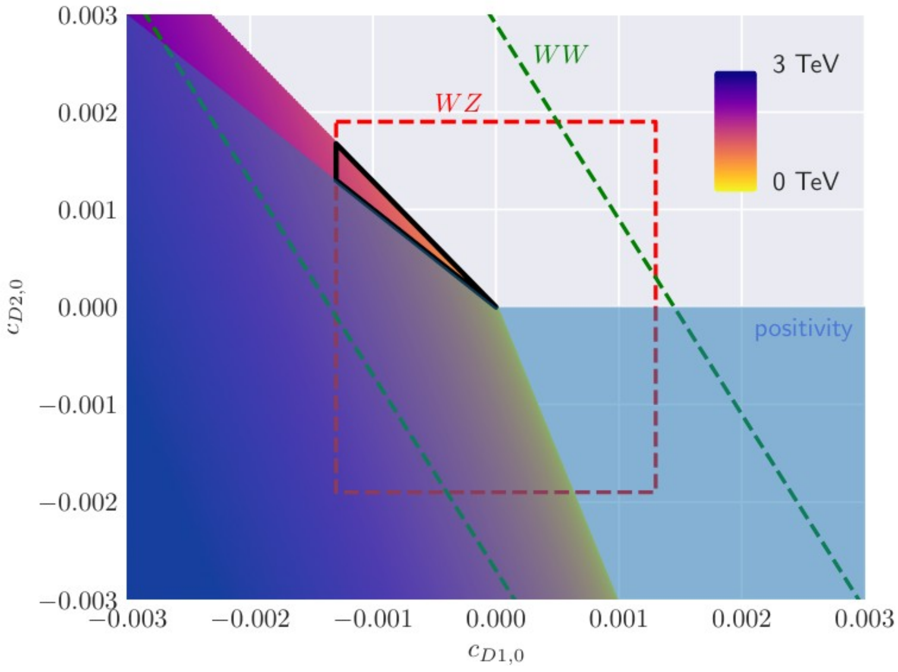
$$L = (\partial\phi)^2 - V(\phi) \qquad\qquad E[\phi_\lambda] = \lambda^{2-n} E_2[\phi] + \lambda^{-n} E_0[\phi]$$

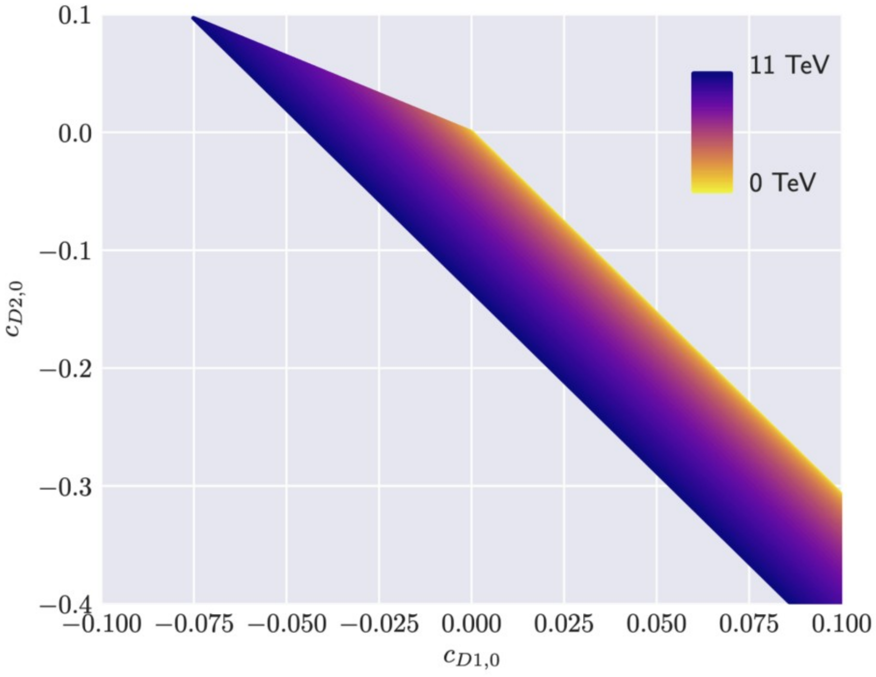
For having a minimum at $\lambda = 1$:

$$\begin{aligned} 0 &= \frac{d}{d\lambda} E[\phi_\lambda]\Big|_{\lambda=1} = (2-n)E_2[\phi] - nE_0[\phi], \\ 0 &< \frac{d^2}{d\lambda^2} E[\phi_\lambda]\Big|_{\lambda=1} = (2-n)(2-n-1)E_2[\phi] + n(n+1)E_0[\phi] \end{aligned} \qquad \Rightarrow (2-n)E_2 > 0$$

(Speculative) skyrmions in the SM

These electroweak skyrmions have been shown to exist in the SM in the presence of four-derivative interactions [Criado, Khoze, Spannowsky '20]





(Speculative) skyrmions in the SM

At finite temperature, these four-derivative interactions exist necessarily

Previous results on the skyrmion region can not be used, because they assume custodial-symmetric four-derivative interactions. These are not the ones in the SM at finite temperature:

$$\mathcal{L}_{\varphi^4 D^4} \sim I_{4,0,0}\, |Y_t|^4 \left[-\frac{1}{5}\, \mathcal{O}^{(1)}_{\varphi^4 D^4} + \frac{1}{2}\, \mathcal{O}^{(2)}_{\varphi^4 D^4} - \frac{1}{5}\, \mathcal{O}^{(3)}_{\varphi^4 D^4} \right]$$

We aim to work at finding skyrmions in this case, at temperatures below the electroweak phase transition

Outlook

Dimensional-reduction is the most appropriate description of systems at finite temperature

Up to now, the higher-point/higher-derivative terms in the 3-dimensional theory have been mostly ignored

We derived the correct way of computing physical quantities taking these interactions into consideration

For strong phase transitions, the effect of these interactions are as large as those ensuing from variations of renormalization scale

Thank you!