CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Small-Instanton induced Flavor Invariants and the Axion Potential

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to appear in JHEP

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Higgs and Effective Field Theory - HEFT
Bologna, 13 June 2024

Preliminary & Outline of this talk

Strong CP problem & Axion solution

1.) QCD vacuum allows an effective(CP violating) term in the Lagrangian:

$$\mathcal{L} \supset \bar{\theta} \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

#Key feature: $\bar{\theta} = \theta_{\rm QCD} - \arg(\det M_q)$ received contributions from both Strong & Electroweak sectors => theta-bar expected to be O(1)

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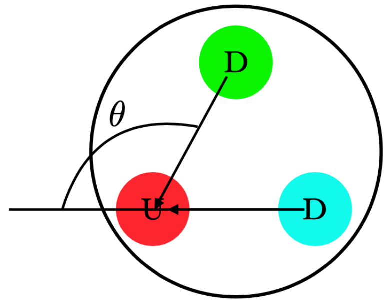

2.) Bound from Neutron EDM: $\bar{\theta} < 10^{-10}$

Strong CP problem: Why is theta-bar so small?

Alternative questions: why no CP-violation in QCD? What make theta-bar so small? (any mechanism behind?)

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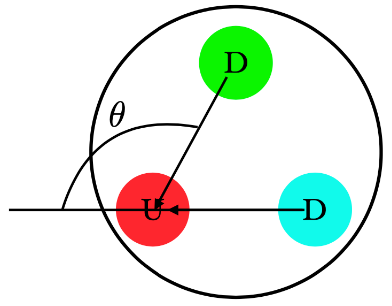

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3.) Axion solution: dynamically relaxes theta-bar to zero

$$\mathcal{L} \supset \left(\bar{\theta} + \frac{a}{f_a}\right) \frac{g_s^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}$$

Preliminary & Outline of this talk

- Instanton #101:

QCD θ-vacuum = Superposition of n-vacua (energy degenerate but topologically distinct)

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta}|n\rangle = \cdots |0\rangle + e^{-i\theta}|1\rangle + \cdots$$

Instanton describes the tunnelling effect between degenerate n-vacua

Instanton: localized objects in Euclidean spacetime, satisfying the Euclidean equation of motion with non-trivial topologies and therefore minimize the Euclidean action.

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Explicit SU(2) BPST instanton solution with Q = 1:
(Background field configuration)

$$\frac{g^2}{32\pi^2}\int d^4x\, G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}(x)\Big|_{\text{inst.}} = Q\,, \text{ where } Q \in \mathbb{Z}\,.$$

$$G^a_\mu(x)\Big|_{1-\text{inst.}} = 2\,\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}$$

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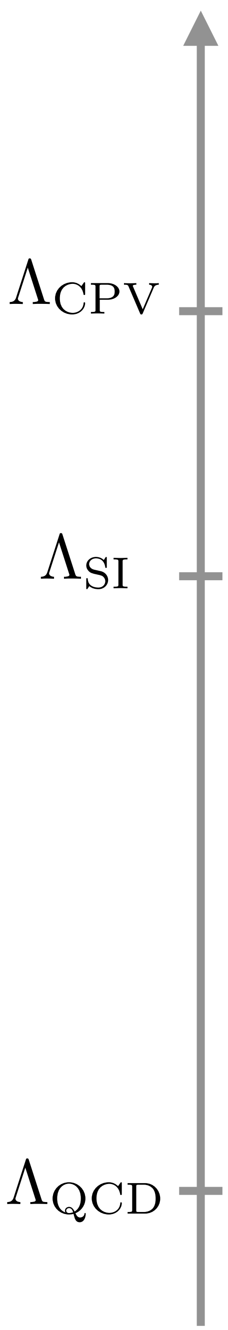
Characterized by a set of collective coordinates (family of solutions):

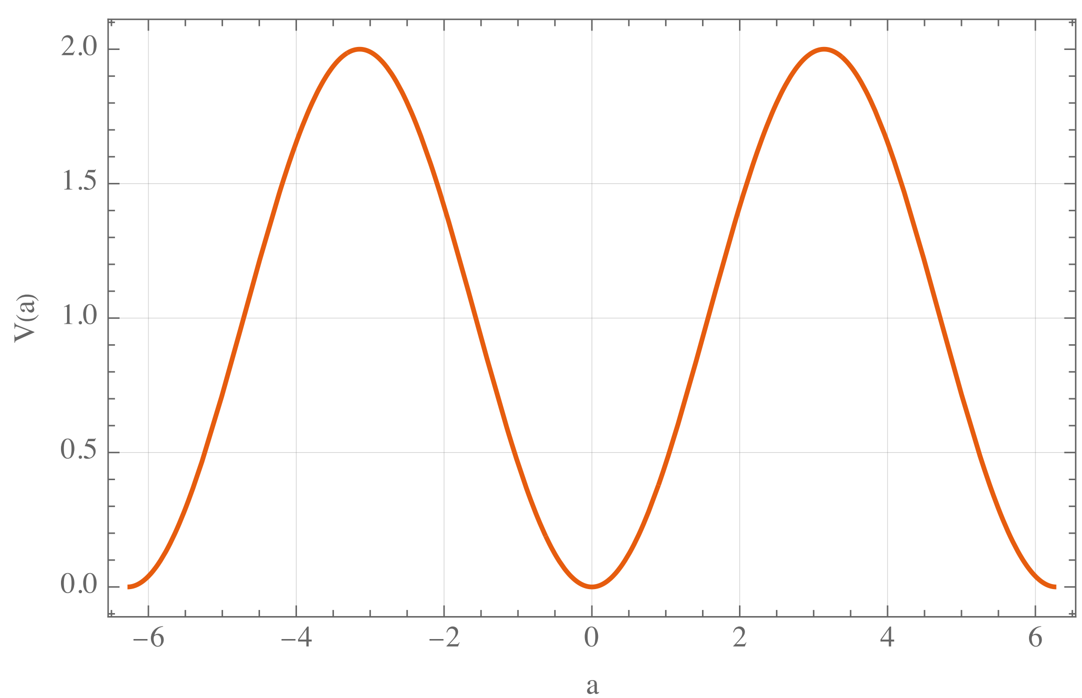
x_0 — Location of instanton

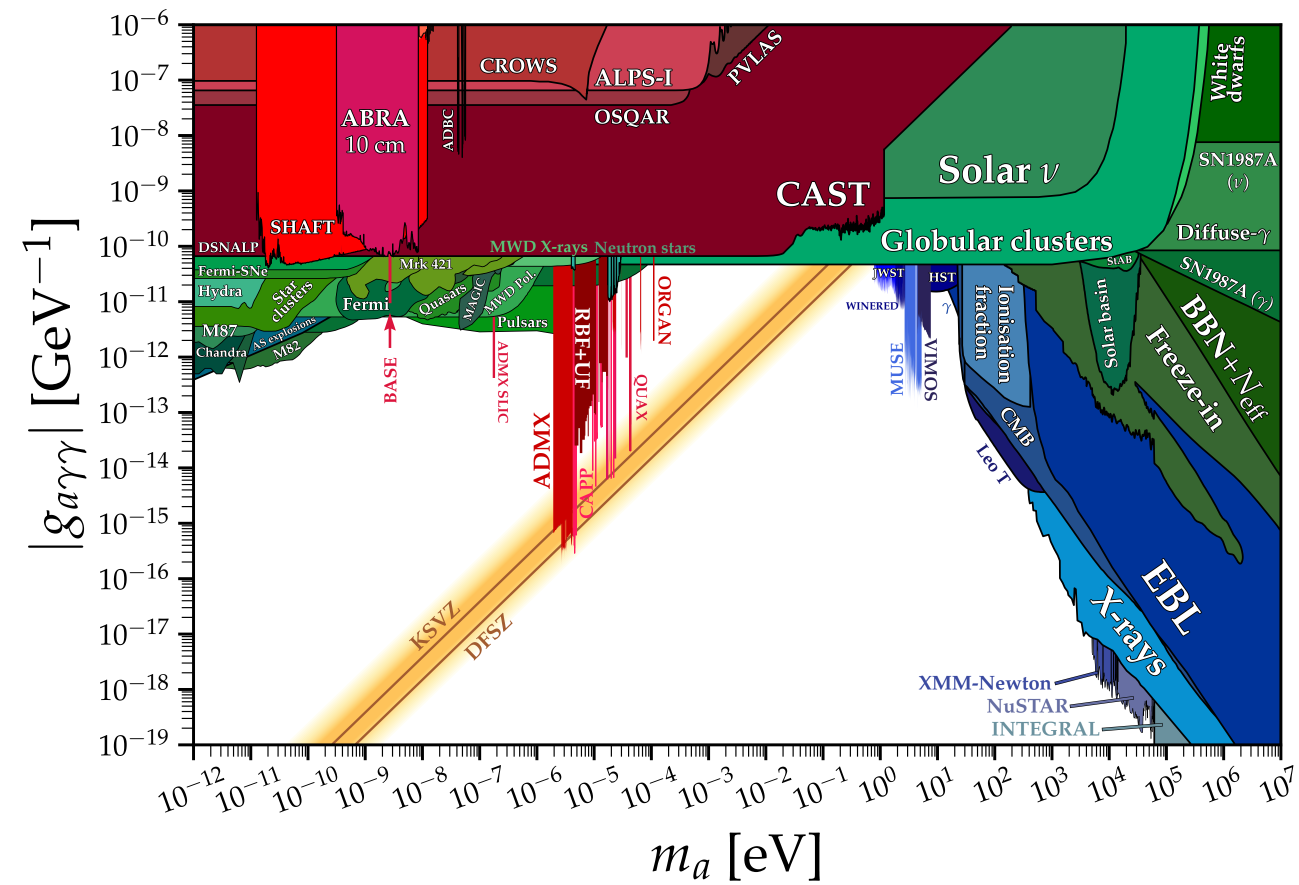
ρ — Size of instanton

Preliminary & Outline of this talk

- Axion potential: QCD contribution





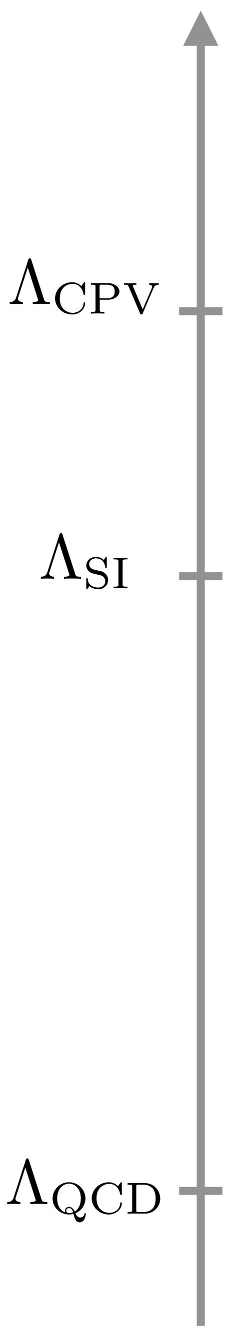


$$V(a) \sim m_\pi^2 f_\pi^2 \left[1 - \cos\frac{a}{f_a}\right] \rightarrow \langle \frac{a}{f_a} \rangle = 0$$

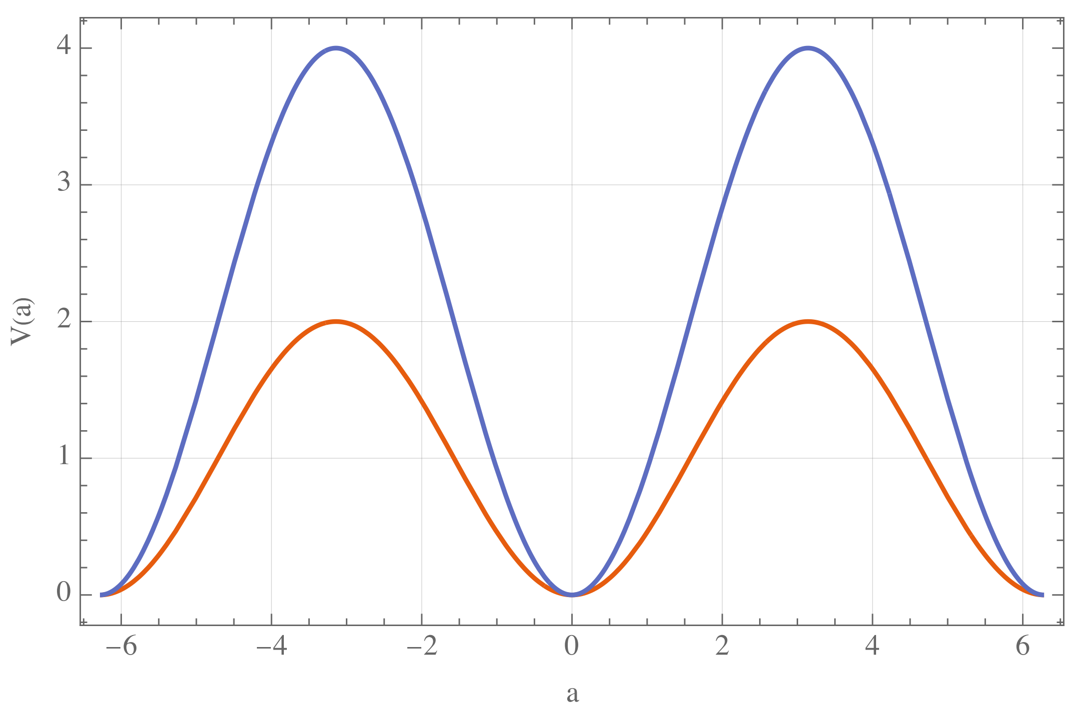
$$m_a^2 f_a^2 = \frac{m_u m_d}{(m_u + m_d)^2} m_\pi^2 f_\pi^2$$

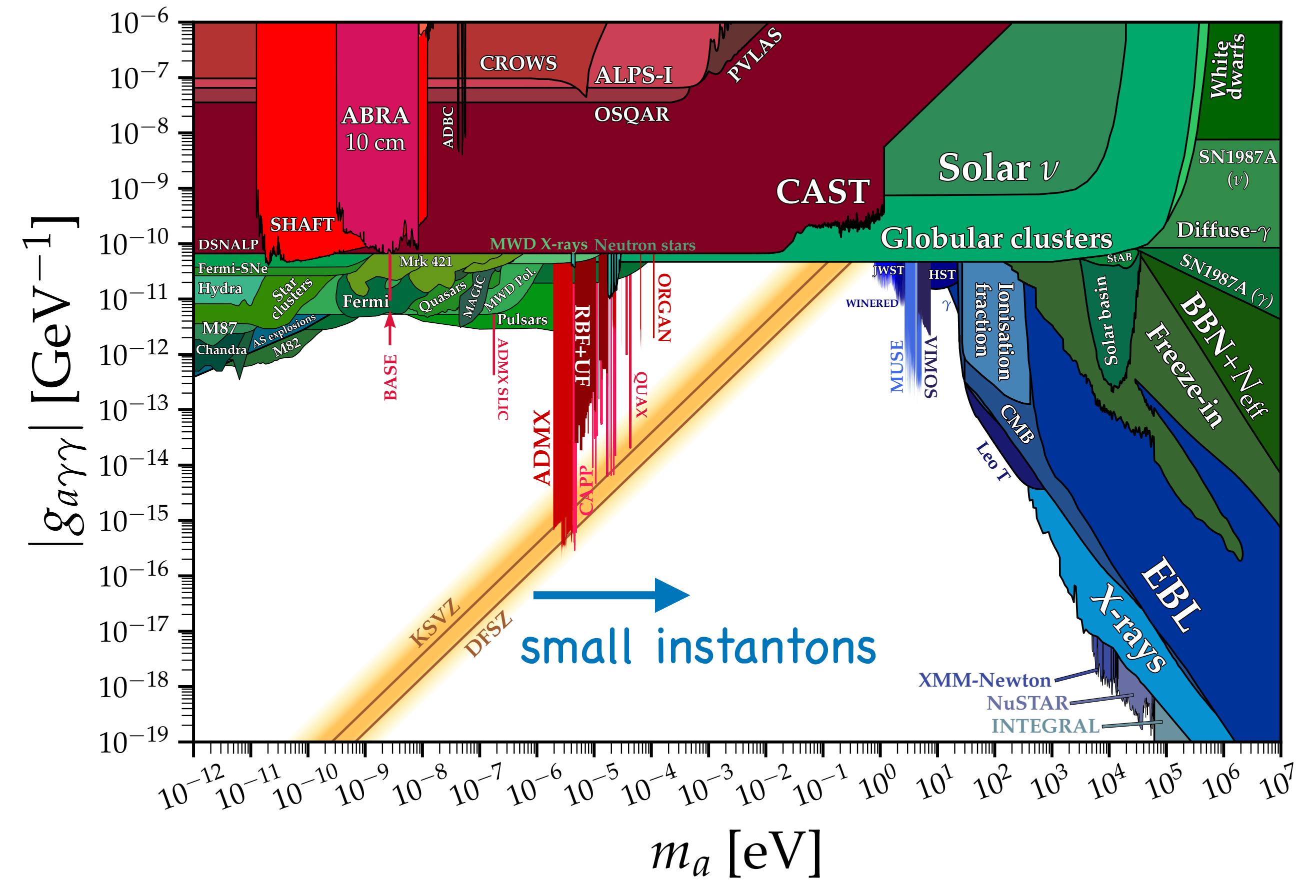
Preliminary & Outline of this talk

- Axion potential: UV aligned contribution



Small instantons $\rho = 1/\Lambda$



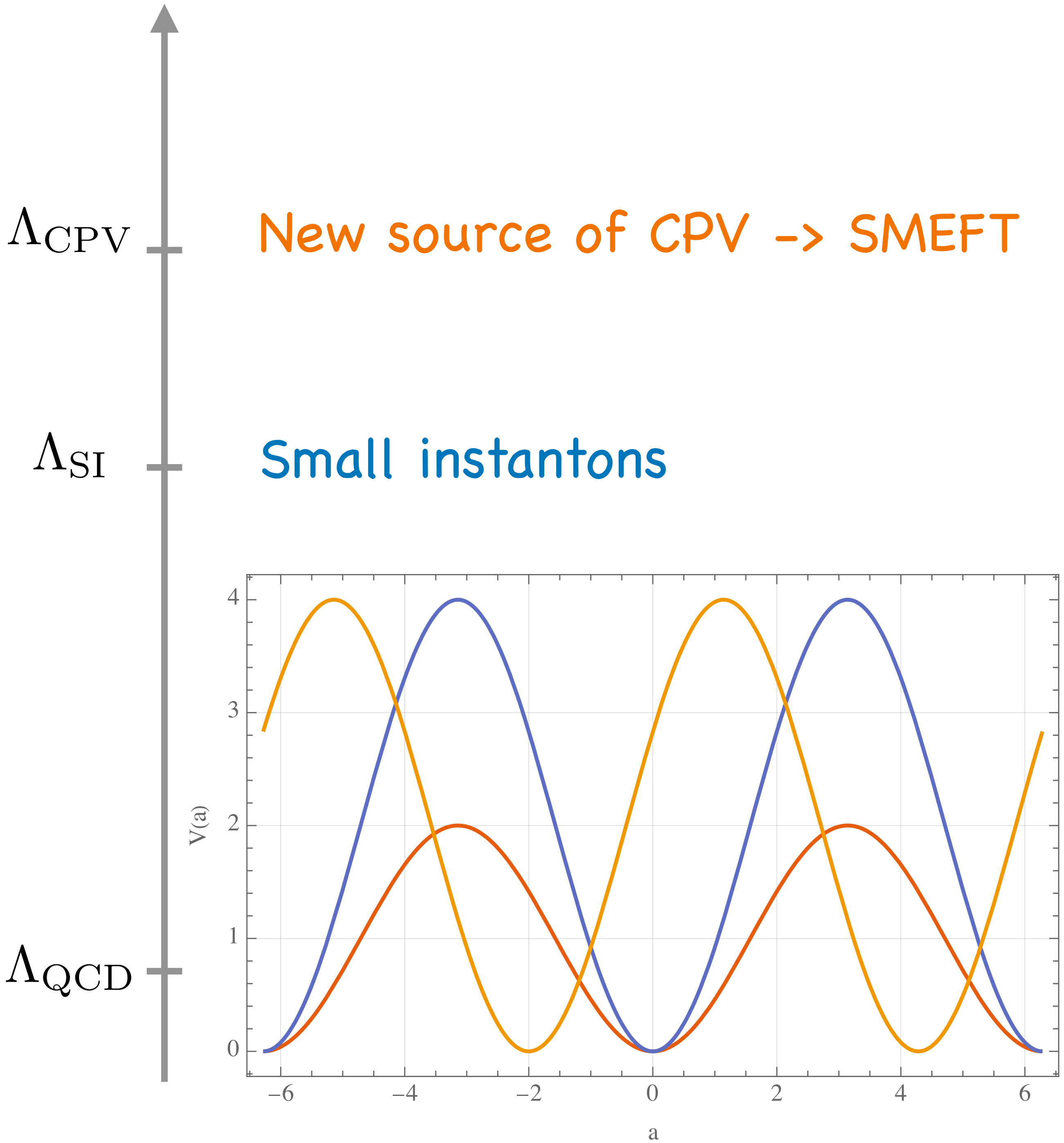


$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\frac{a}{f_a}\right]$$

Enhance axion mass & solve strong CP problem

Preliminary & Outline of this talk

- Axion potential: UV misaligned contribution



$$V(a) \sim (m_\pi^2 f_\pi^2 + \Lambda_{UV}^4)\left[1 - \cos\left(\frac{a}{f_a} + \delta_{CPV}\right)\right]$$

Can destroy the Axion solution

Preliminary & Outline of this talk

- SMEFT Flavour invariants: Systematically capture the CPV phases of SMEFT operators

CP-violation in the SM is parametrised by

$$J_4 = \mathrm{Im}\left(\mathrm{Tr}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right]\right)$$

Jarlskog 85

Bernabeu, Branco, Gronau 85

Construct Jarlskog-like CPV invariants for the SMEFT: Example $\mathcal{O}_{uH} = |H|^2 \bar{Q}\tilde{H}u$

$$\mathrm{ImTr}\left(X_u^a X_d^b X_u^c X_d^d C_{uH} Y_u^\dagger\right) \qquad X_{u,d} = Y_{u,d} Y_{u,d}^\dagger$$

Has 9 complex parameters

Bonnefoy, Gendy, Grojean, Ruderman

2112.03889, 2302.07288

Preliminary & Outline of this talk

- Small instanton & Axion potential: UV misaligned contributions

Small instantons generate axion potential of the form:

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2 \longrightarrow \langle\frac{a}{f_a}\rangle \equiv \theta_{\text{ind}} = -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

Induced by CP-violating operator

This talk

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Induced by CP-violating operator

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Coefficients in the potential can be computed from following correlators:

$$\chi(0) = -i\lim_{k\to 0}\int d^4x\, e^{ikx}\left\langle 0\left|T\left\{\frac{g^2}{32\pi^2}G\tilde{G}(x)\,,\,\frac{g^2}{32\pi^2}G\tilde{G}(0)\right\}\right|0\right\rangle\Bigg|_{\text{1-(a.-)inst.}}$$

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Evaluating these correlation functions within perturbative regime and one-(anti)instanton approximation. Making connection with SMEFT flavour invariants => Simplify the calculations

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- Core technique 1: Path Integral & Instanton background

$$\chi_{\mathcal{O}}(0) = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \middle| T\left\{ \frac{g^2}{32\pi^2} G\tilde{G}(x)\,,\, \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \right\} \middle| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)}\, d\bar{\xi}_f^{(0)}\right)$$

$$\times \int \mathcal{D}\varphi\, e^{-S_0[\varphi] - S_{\rm int}[\varphi_{\rm I}, \varphi]} \int d^4x \frac{g^2}{32\pi^2} G\tilde{G}(x) \times \frac{C_{\mathcal{O}}}{\Lambda^2_{\not{CP}}} \mathcal{O}[\varphi_{\rm I}, \varphi](0) \Big|_{\text{1-(a.-)inst.}}$$

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Fields with instanton solutions (e.g. gluon, quark): φ_I

=> Expand the fields in their eigenmodes, replace zero mode wave function by instanton solutions, and integrate out non-zero modes:

Instanton density

$$\int \mathcal{D}\varphi_{\rm I}\, e^{-S_E[\varphi_{\rm I}]} \to e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)}\, d\bar{\xi}_f^{(0)}\right)$$

Path integral measure of zero modes

=> Integration over collective coordinates

't Hooft 76

Shifman, Vainshtein, Zakharov 79

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Fields without instanton solutions: φ

=> Integrate over without performing the eigenmode expansion

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

Core technique 2: Fermion zero mode & Grassmann integral

Fermion eigenmode expansion & fermion zero-mode solutions:

$$\psi_f(x) = \sum_k \xi_f^{(k)} \psi^{(k)}(x)\,; \quad \bar{\psi}_f(x) = \sum_k \bar{\xi}_f^{(k)} \bar{\psi}^{(k)}(x)$$

$$-i\not{D}\big|_{\text{1-inst.}} \psi^{(0)}(x) = 0. \longrightarrow \psi^{(0)}(x)\Big|_{\text{1-inst.}} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\pi} \frac{\rho}{\left[(x-x_0)^2 + \rho^2\right]^{3/2}} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \varphi_{\alpha m} = \epsilon^{\alpha m}$$

Fermion zero modes & Grassmann integral give rise to determinant-like structures:

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} = \det A\,,$$

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} \xi_1 B \xi_2 = \frac{1}{2} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} B_{i_3 j_3}$$

$$\det A = \frac{1}{n!} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} A_{i_1 j_1} \cdots A_{i_n j_n}$$

=> First hint of determinant-like flavour invariants will appear in the final result

Small instanton & Axion potential: Evaluating the correlator $\chi_{\mathcal{O}}(0)$

- Core technique 3: Determinant-like flavour invariants

Considering non-perturbative effects => Use $\theta_{\rm QCD}$ as a spurion:

	$U(3)_{\rm Q}$	$U(3)_{\rm u}$	$U(3)_{\rm d}$	$U(3)_{\rm L}$	$U(3)_{\rm e}$
$e^{i\theta_{\rm QCD}}$	$\mathbf{1}_{+6}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm u}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm d}$	$\mathbf{3}_{+1}$	$\mathbf{1}_{0}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
$Y_{\rm e}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$

SM has one more CP-odd flavour invariant:

$$J_\theta = {\rm Im}[e^{-i\theta_{\rm QCD}} \det(Y_{\rm u} Y_{\rm d})]$$

Built invariants featuring $\theta_{\rm QCD}$ for CP-violating SMEFT operators:

$$\mathcal{O}^{(1)}_{quqd} = \bar{Q}u\bar{Q}d$$

$$\mathcal{I}(C^{(1,8)}_{\rm quqd}) = {\rm Im}\left[e^{-i\theta_{\rm QCD}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{{\rm u},Aa}Y_{{\rm u},Bb}C^{(1,8)}_{{\rm quqd},CcDd}Y_{{\rm d},Ee}Y_{{\rm d},Ff}\right]$$

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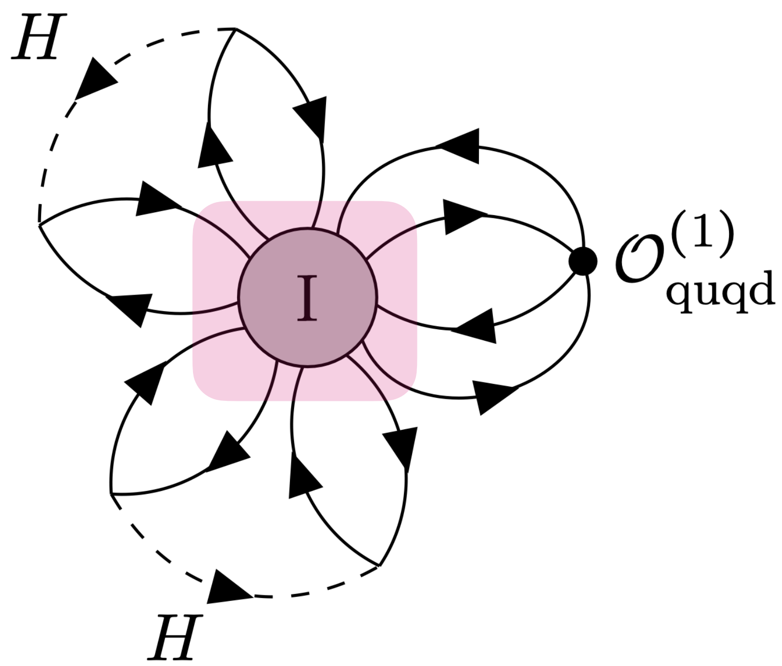
Topological Susceptibilities & Flavor invariants: Four-quark operator



$$\chi^{(1)}_{\rm quqd}(0)^{\rm 1-inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm quqd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times\, e^{\int d^4x (\bar{Q} Y_{\rm u} \widetilde{H} u + \bar{Q} Y_{\rm d} H d + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm quqd}}{\Lambda^2_{\not{CP}}} \bar{Q}u\bar{Q}d(0) + {\rm h.c.} \right) ,$$

Q = 1

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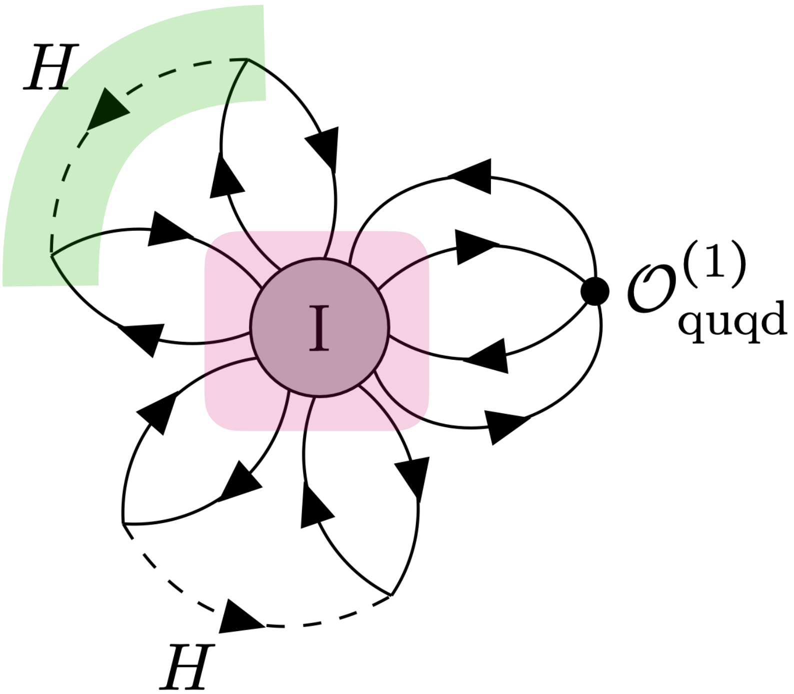

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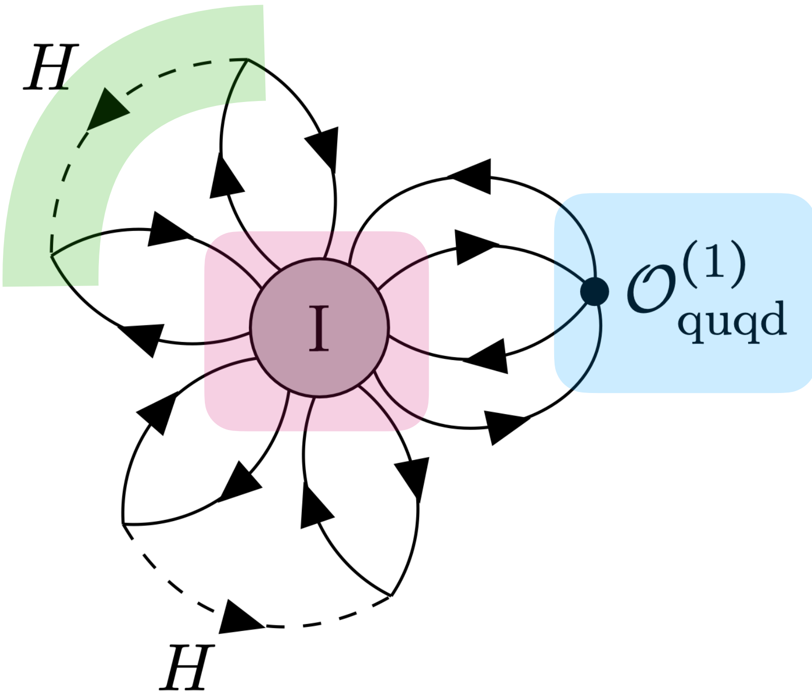

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$$= e^{-i\theta_{\rm QCD}}\int d^4x_0\int\frac{d\rho}{\rho^5}d_N(\rho)\int\mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]}\int\prod_{f=1}^{3}\left(\rho^2\, d\xi^{(0)}_{u_f}\, d\xi^{(0)}_{d_f}\, d^2\bar{\xi}^{(0)}_{Q_f}\right)$$

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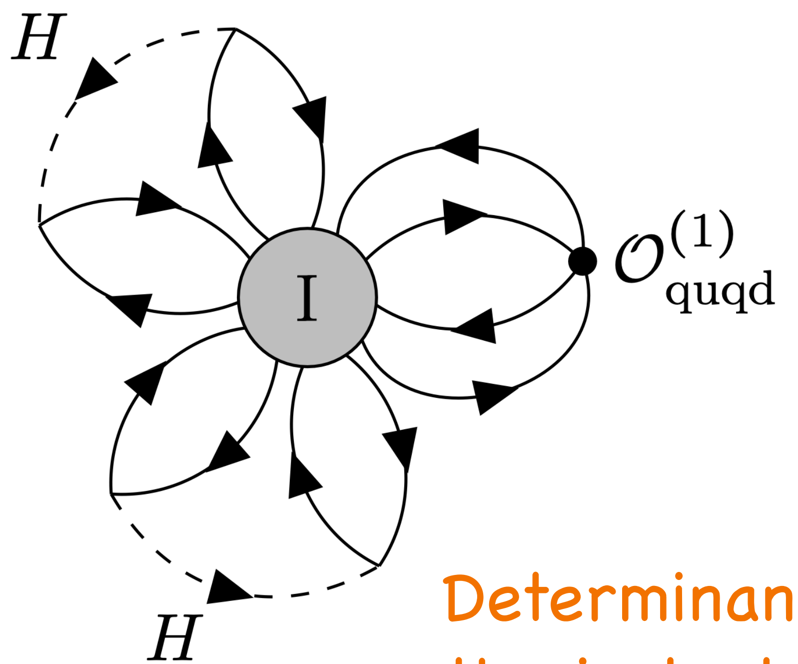
Topological Susceptibilities & Flavor invariants: Four-quark operator



Determinant-like flavour invariants naturally arise in the instanton calculations

$$\chi^{(1)}_{\rm quqd}(0)^{1-{\rm inst.}} = \frac{1}{4\Lambda^2_{\not{CP}}}\Big[e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n}Y_{{\rm u},i_1 j_1}Y_{{\rm u},i_2 j_2}C^{(1)}_{{\rm quqd},mnop}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p}Y_{{\rm d},k_1 l_1}Y_{{\rm d},k_2 l_2}$$

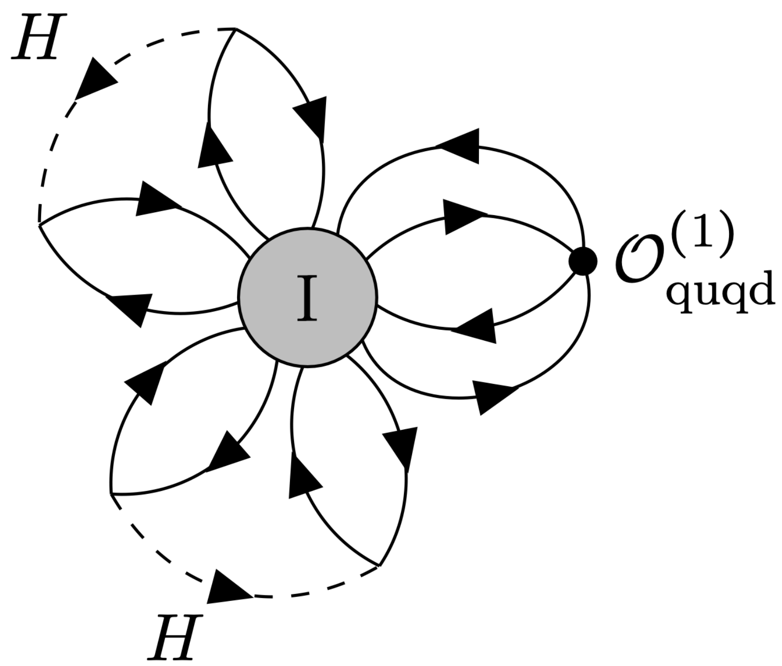
$$+e^{-i\theta_{\rm QCD}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n}Y_{{\rm u},i_1 j_1}Y_{{\rm u},i_2 j_2}C^{(1)}_{{\rm quqd},onmp}\epsilon^{k_1 k_2 o}\epsilon^{l_1 l_2 p}Y_{{\rm d},k_1 l_1}Y_{{\rm d},k_2 l_2}\Big]\int d^4x_0\int\frac{d\rho}{\rho^5}d_N(\rho)\rho^6$$

$$\times\underbrace{\int\mathcal{D}H\mathcal{D}H^\dagger\,e^{-S_0[H,H^\dagger]}\Big[\int d^4x_1 d^4x_2(\bar\psi^{(0)}H_I^\dagger\epsilon^{IJ}P_R\psi^{(0)})(x_1)\,(\bar\psi^{(0)}\epsilon_{JK}H^K P_R\psi^{(0)})(x_2)\Big]^2}_{=2!\,[\int d^4x_1 d^4x_2(\bar\psi^{(0)}P_R\psi^{(0)})(x_1)\Delta_H(x_1-x_2)\epsilon_{IJ}\epsilon^{JI}(\bar\psi^{(0)}P_R\psi^{(0)})(x_2)]^2\equiv 2!\,\mathcal{I}^2}$$

$$\times\left(\epsilon_{MN}\epsilon^{MN}\bar\psi^{(0)}P_R\psi^{(0)}\,\bar\psi^{(0)}P_R\psi^{(0)}\right)(0)\int d^4x\frac{G\widetilde{G}(x)}{32\pi^2}\,.$$

Fermion zero modes

Topological Susceptibilities & Flavor invariants: Four-quark operator

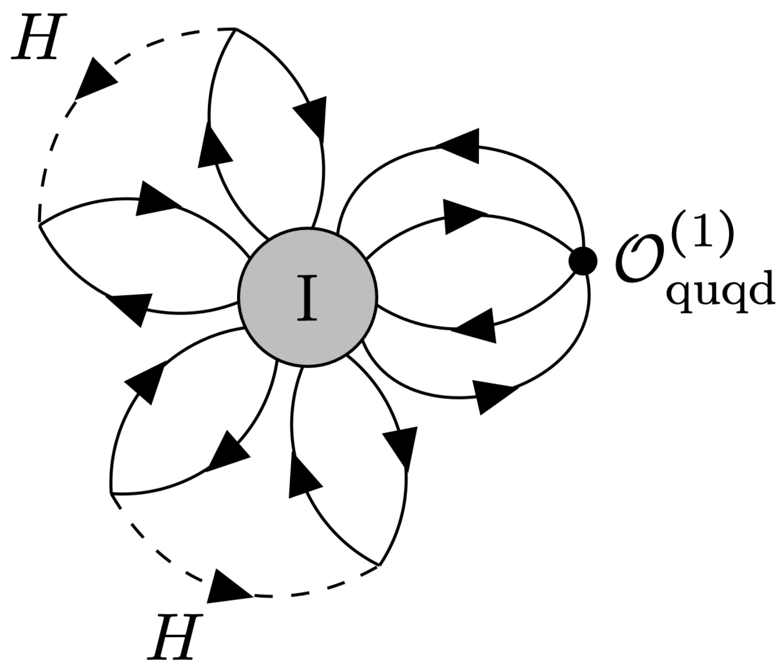


Plugging explicit form of fermion zero modes
Integrate over loop momenta,
collective coordinates

$$\chi^{(1)\,(\text{UV})}_{\text{quqd}}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\text{quqd}}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\text{quqd}}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\, \frac{2!}{(6\pi^2)^2}\, \frac{2}{5\pi^2\rho^2}$$

Contraction of Yukawa matrices
encapsulated in the Flavour invariants

Topological Susceptibilities & Flavor invariants: Four-quark operator



Can also use Instanton Naive Dimensional Analysis (NDA), result up to $\mathcal{O}(1)$

Csáki, D'Agnolo, Kuflik, Ruhdorfer (2311.09285)

$$\chi^{(1)\,(\mathrm{UV})}_{\text{ququd}}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\text{ququd}}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\text{ququd}}\right)\right)\int \frac{d\rho}{\rho^5} d_N(\rho)\frac{1}{(256\pi^6)\rho^2}$$

Contraction of Yukawa matrices encapsulated in the Flavour invariants

Combining Flavour invariants & Instanton NDA
=> quickly estimate 't Hooft flower diagrams

Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

- Now start from the Flavour invariants:

$$\mathrm{Im}\left(I^{(1)}_{\mathrm{lequ}}\right) = \mathrm{Im}\left[e^{-i\theta_{\mathrm{QCD}}}\epsilon^{i_1 i_2 m}\epsilon^{j_1 j_2 n} Y_{\mathrm{u},i_1 j_1} Y_{\mathrm{u},i_2 j_2} C^{(1)}_{\mathrm{lequ},opmn} Y^{\dagger}_{\mathrm{e},po} \det Y_{\mathrm{d}}\right] = \mathcal{I}^{0}_{0000}\left(C^{(1)}_{\mathrm{lequ}}\right)$$

Trace-like contractions

Topological Susceptibilities & Flavor invariants: Semi-leptonic operator

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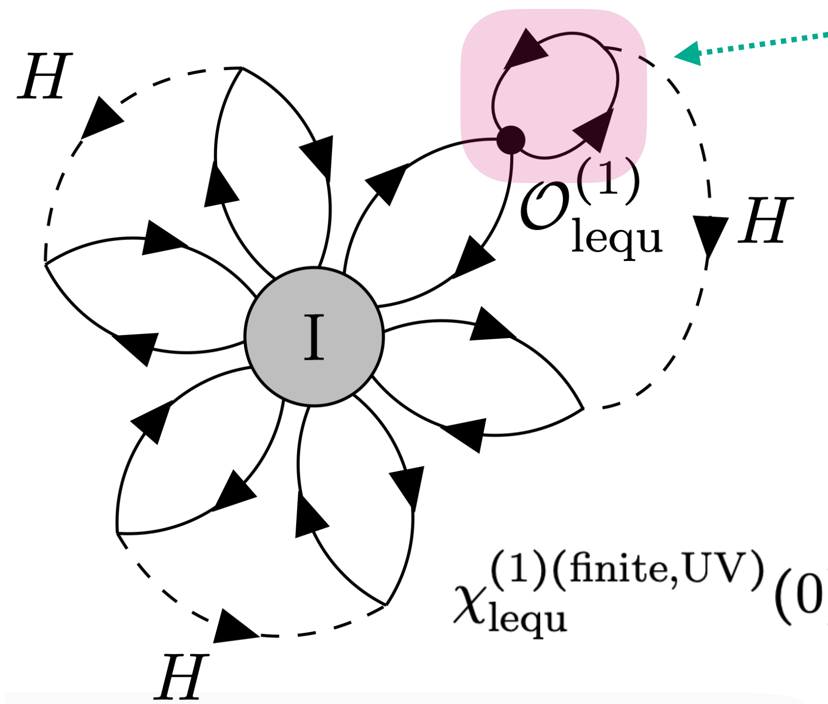
Trace-like contractions



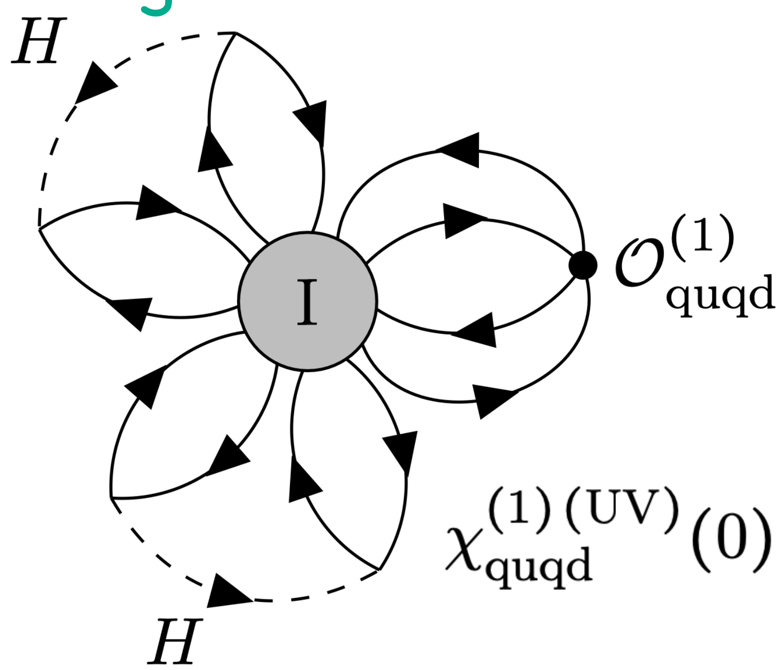
1-loop suppression
induced by leptonic fields

$$\chi^{(1)(\mathrm{finite,UV})}_{\mathrm{lequ}}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\mathcal{I}^{0}_{0000}\left(C^{(1)}_{\mathrm{lequ}}\right)\int \frac{d\rho}{\rho^5} d_N(\rho)\frac{3!}{(6\pi^2)^2}\frac{11 + 30\left(\log\left(\rho\Lambda_{\not{CP}}\right) + \gamma_{\mathrm{E}} - \log 2\right)}{600\pi^4\rho^2}$$

- Anticipating how CPV SMEFT operators participate in the instanton computations
- Classifying the leading effects from the Wilson coefficients

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton



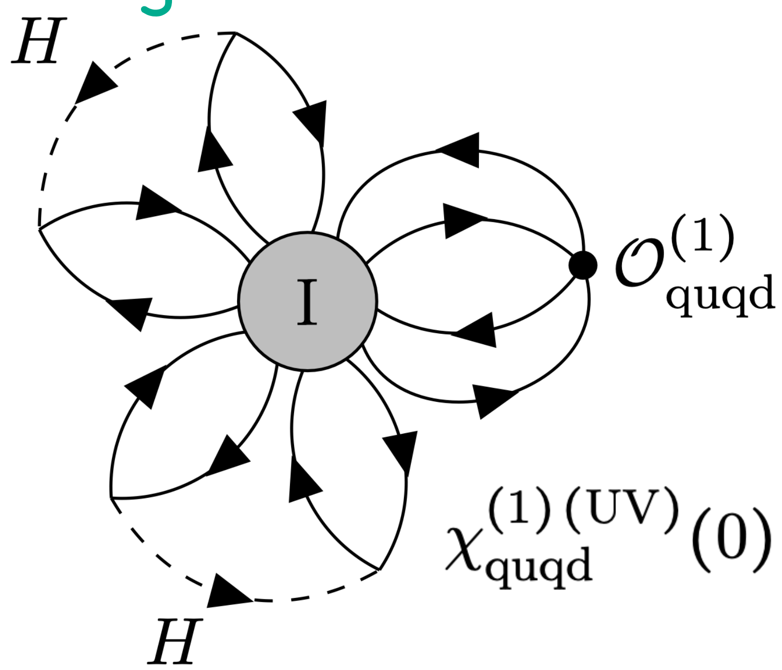
$$\chi^{(1)\,(\mathrm{UV})}_{\rm ququd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

ρ-integral is IR divergent

=> Need a physical IR cut-off

Topological Susceptibilities & Flavor invariants: Four-quark operator

#Integration over the size of instanton



$$\chi^{(1)\,({\rm UV})}_{\rm quqd}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm quqd}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\,\frac{2!}{(6\pi^2)^2}\,\frac{2}{5\pi^2\rho^2}$$

ρ-integral is IR divergent
=> Need a physical IR cut-off

- Possible UV completion of small-instantons:

Product of gauge groups

Higgsing $SU(3)_1 \times \cdots \times SU(3)_k \to SU(3)_c$ via Bifundamental scalars σ, vev $<\sigma>$

$$d_N(\rho) \to d_N(\rho)\, e^{-2\pi^2\rho^2 \sum |\langle\sigma\rangle|^2}$$

C. Csáki, M. Ruhdorfer, Y. Shirman (1912.02197)

5D instantons

Uplift BPST instanton to a compact extra dimension of size R

$$d_N(\rho) \to d_N(\rho)\, e^{R/\rho}$$

T. Gherghetta, V. V. Khoze, A. Pomarol, Y. Shirman (2001.05610)

Bounds from non-measurement of theta-induced: Four-quark operator

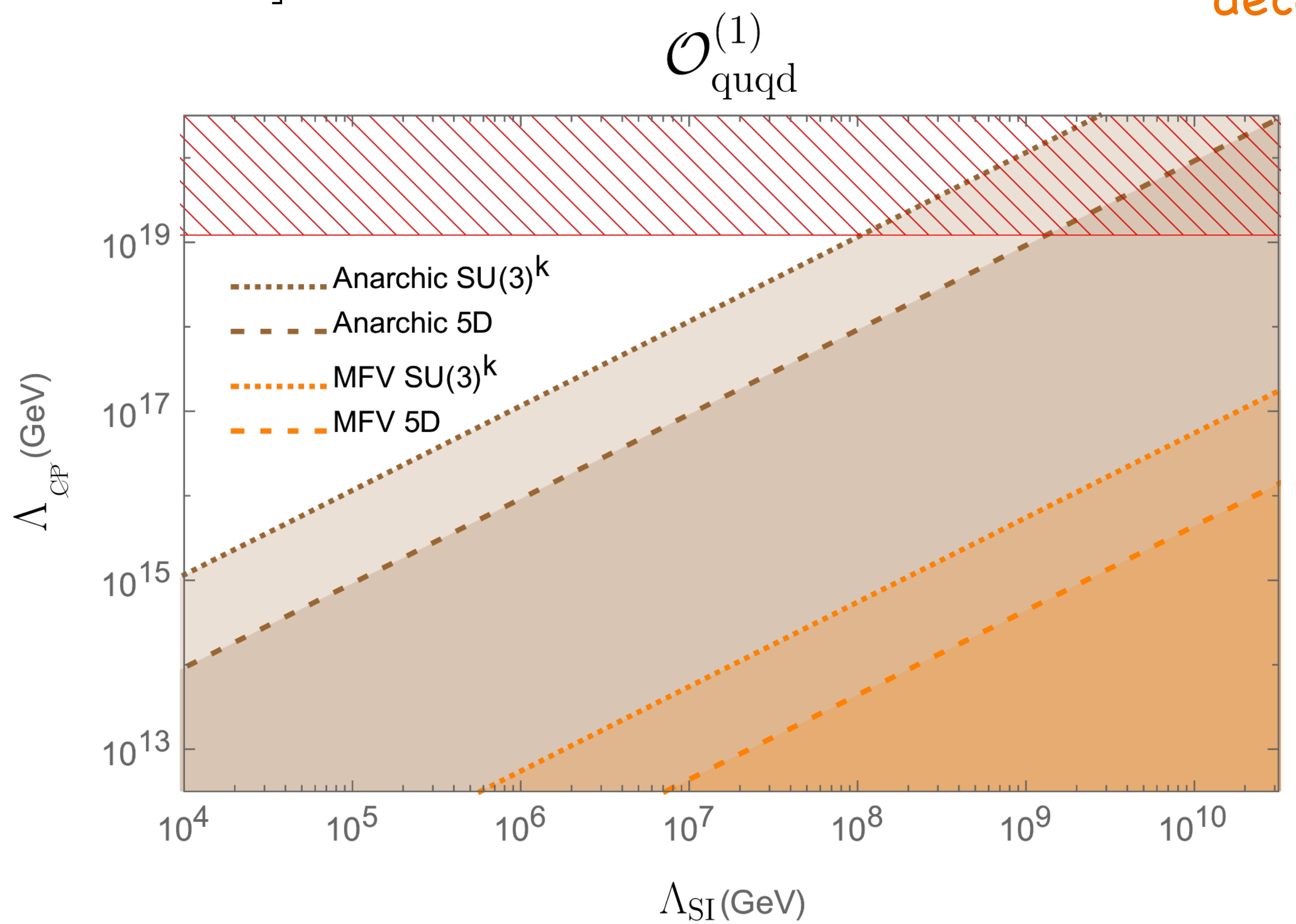
- Product of gauge groups

$$\theta_{\rm ind} = \frac{16\pi^2}{5(b_0-6)K_\theta}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\rm ququd}\right)\right)\frac{\Lambda^2_{\rm SI}}{\Lambda^2_{\not{CP}}}$$

$$K_\theta = {\rm Re}\left[e^{-i\theta_{\rm QCD}}\det\left(Y_{\rm u}Y_{\rm d}\right)\right]$$

Test different flavour scenarios

Finite ratio in the decoupling limit



Conclusions

- Enhancing the axion mass via small-instanton also (accidentally)enhances CPV effects that misalign potential
  => Dangerous effects which spoil Axion solution
  => The quality of Axion solution depends on UV scenarios

- The estimation of these effects can be made easier with the help of determinant-like invariants

- We can also use the invariants to study the contributions of all remaining CP-odd SMEFT operators

Backup slides

BPST instanton

$$G^a_\mu(x)\big|_{1\text{-inst.}} = 2\,\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}\,, \qquad \eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu}, & \mu,\nu\in\{1,2,3\} \\ -\delta_{a\nu}, & \mu=0 \\ +\delta_{a\mu}, & \nu=0 \\ 0, & \mu=\nu=0 \end{cases}$$

An important property of the one-(anti-)instanton solution is that it satisfies the *(anti-) self-dual* equation

$$G^a_{\mu\nu} = \pm\tilde{G}^a_{\mu\nu}\,, \tag{C.6}$$

and thus, due to the Bianchi identity, automatically solves the gluon equation of motion $D^\mu G^a_{\mu\nu} = D^\mu \tilde{G}^a_{\mu\nu} = 0$. With all of these properties, the one-(anti)-instanton solution then yields the finite QCD classical action

$$S^{\text{1-inst.}}_{\text{YM}} = \int d^4x\left(\frac{1}{4}G^A_{\mu\nu}G^{A,\mu\nu} + i\theta_{\text{QCD}}\frac{g^2}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\right)\Bigg|_{\text{1-(a.-)inst.}} = \frac{8\pi^2}{g^2} \pm i\theta_{\text{QCD}}\,. \tag{C.7}$$

Instanton density

$$d_N(\rho) = C[N] \left(\frac{8\pi^2}{g^2}\right)^{2N} e^{-8\pi^2/g^2(1/\rho)}$$

$$C[N] = \frac{C_1\, e^{-C_2 N}}{(N-1)!(N-2)!}\, e^{0.292 N_f}$$

$$\frac{8\pi^2}{g^2(1/\rho)} = \frac{8\pi^2}{g_0^2(\Lambda_{\rm UV})} - b_0 \log \rho\Lambda_{\rm UV}\,, \qquad b_0 = \frac{11}{3}N - \frac{2}{3}N_f$$

Bounds from non-measurement of theta-induced: Semi-leptonic operator