AEPSHEP2024

An introduction to QCD

Xu Feng

Peking University

2024.06.16-19

Nuclear force and strong interaction

Interactions of matters

Nature of macroscopic forces

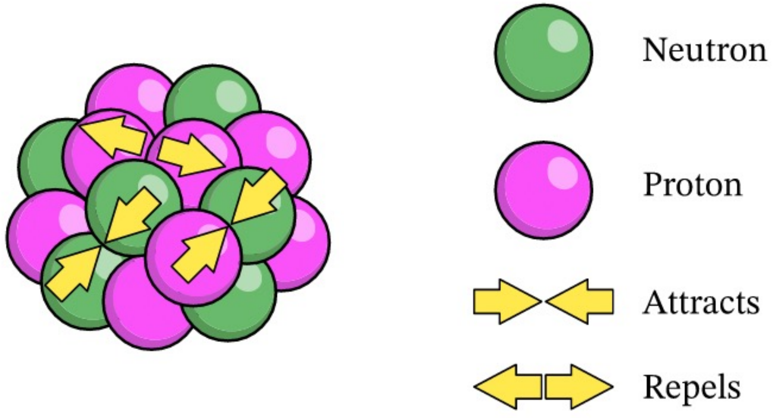
1687
Newton's law of Gravity
astronomical motion

1785
Coulomb's Law of Electricity
molecular & atomic structure

Interactions of matters

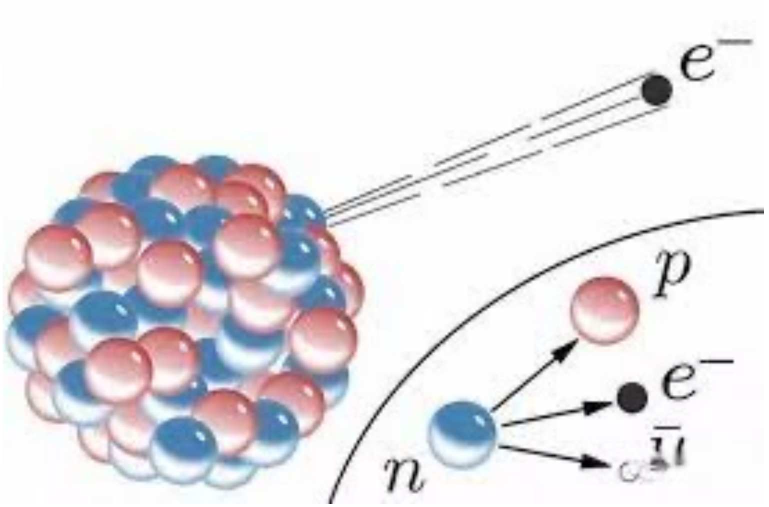
First Nobel Prize in Physics

1934
Yukawa's potential, nuclear force
Nuclear structure



1901

1934
Fermi's theory of weak interaction

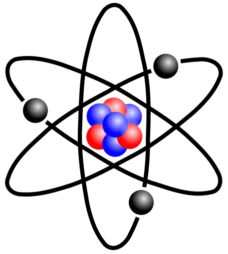


Strong interaction

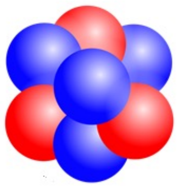
	Gravitation	Electromagnetic	Weak	Strong
Acts on	particles with mass and energy	particles with charge	quarks and leptons (decay)	quarks
Exchange particle	graviton (not yet observed)	photon, γ	W^+, W^- and Z^0	gluons, g, and mesons
Exchange particle mass	massless	massless	$M_{\mathrm{W}^\pm} = 80\,\mathrm{GeVc}^{-2}$, $M_{\mathrm{Z}} = 91\,\mathrm{GeVc}^{-2}$	gluons are massless
Relative strength	negligible, predicted about 10^{-41}	$\frac{1}{137}$	10^{-6}	1
Range	∞ decreasing $\propto \frac{1}{r^2}$	∞ decreasing $\propto \frac{1}{r^2}$	10^{-18} decreasing $\propto \frac{1}{r}$	10^{-15} increasing $\propto r$

Question:

α_s=O(1), α_{em}=1/137, why binding energy is 10^6 times larger?



Binding energy: 10 eV



Binding energy: 10 MeV

Exercise: binding energy

Virial Theorem in Statistics Mechanics

If the force between any two particles of the system results from a potential energy $V(r)=\alpha r^n$ that is proportional to some power n of the interparticle distance r, then

$$\langle T \rangle = n \langle V \rangle$$

- Given an atom, we have kinetic energy of the electron $T=p^2/(2m)$ and $V=-\alpha_{em}/r$
- Using Heisenberg's uncertainty principle, the distance r can be determined, and the binding energy can be calculated
- Think about how to determine the binding energy for a nucleus

Cosmic Evolution and Energy Storage

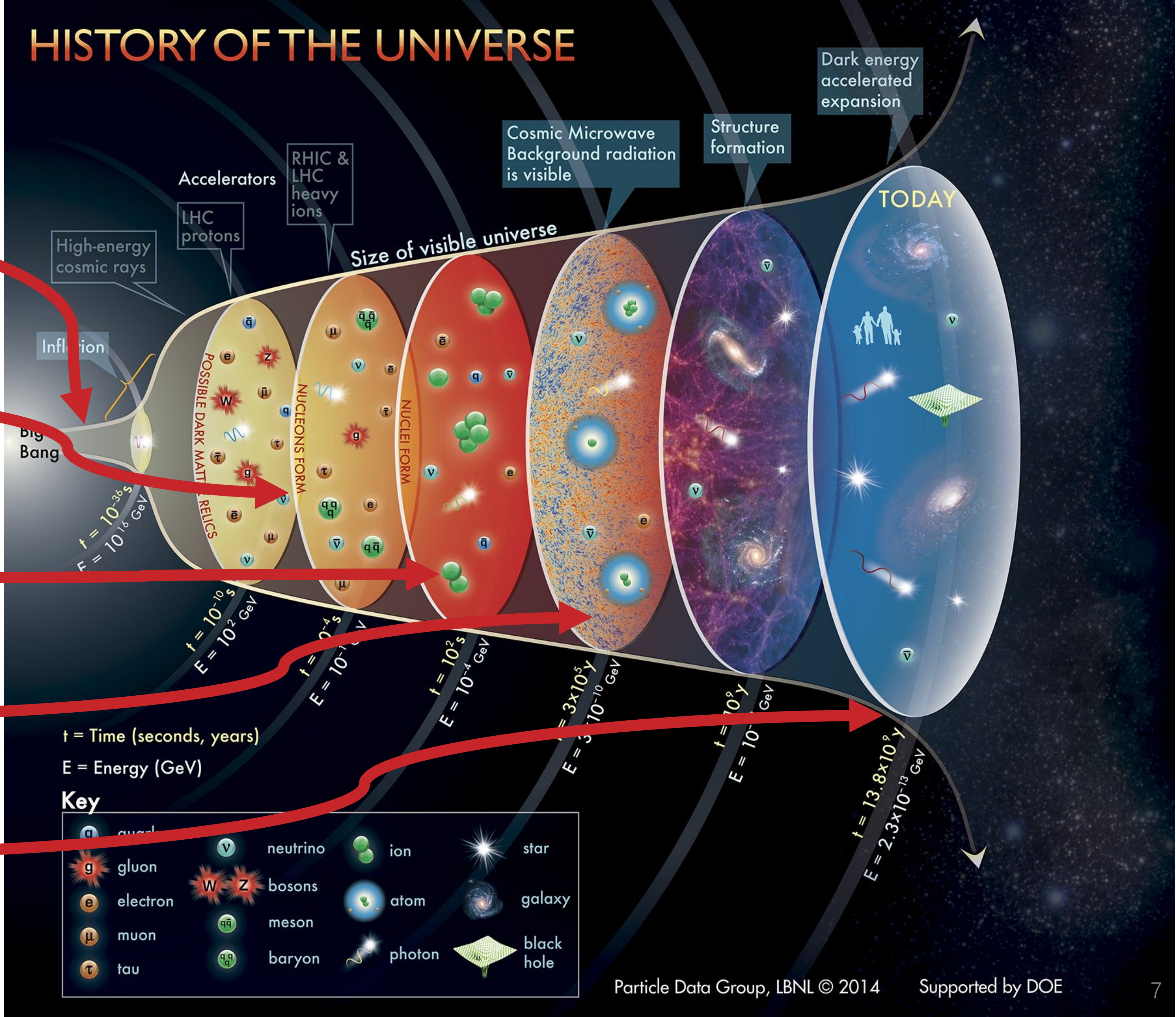
Big Bang

Quark-Gluon Plasma

Nucleus

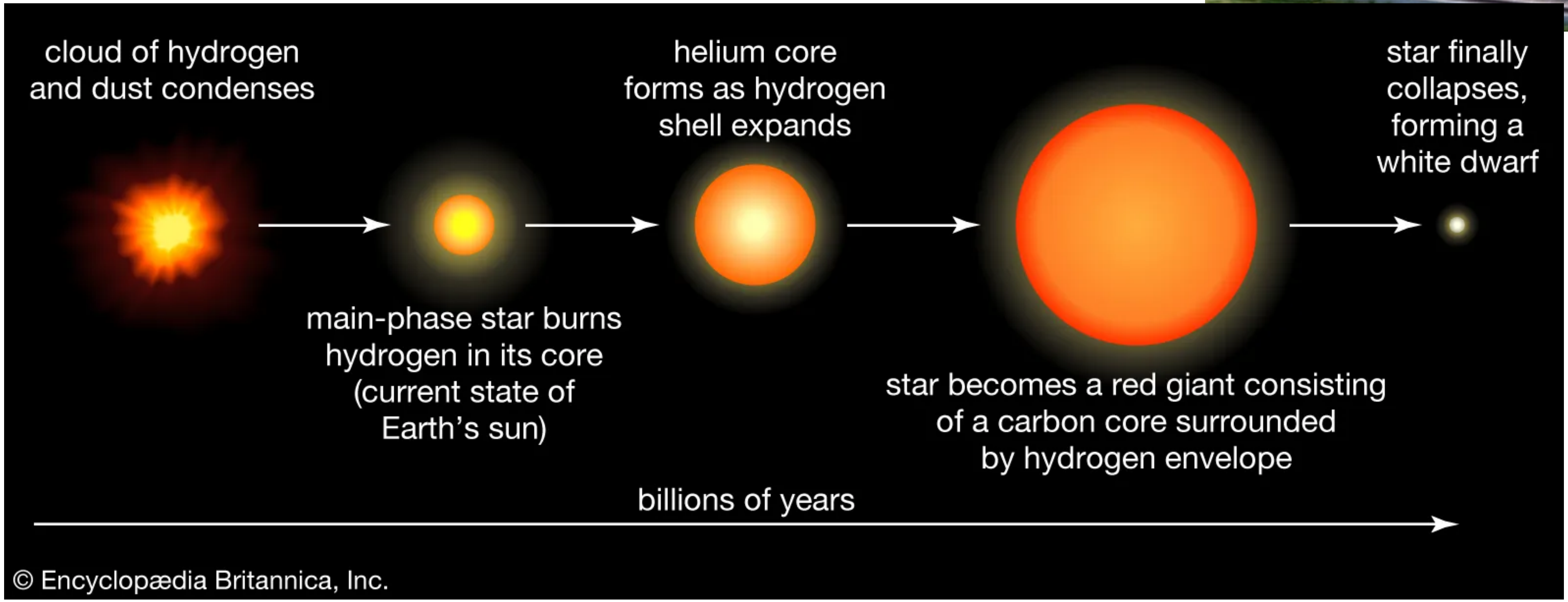
Atom

Today's universe



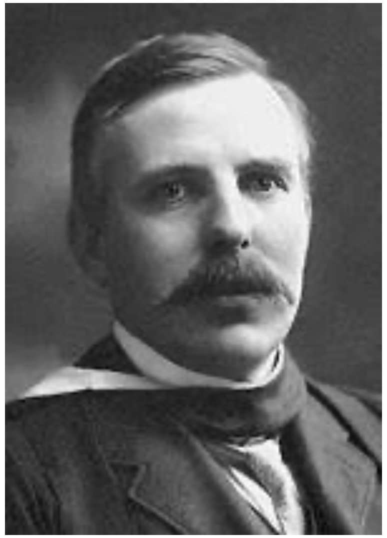
Cosmic Evolution and Energy Storage

- Generate > 99% mass in visible universe
- Evolution of stars
- Solar energy
- Nuclear power plant



Some History

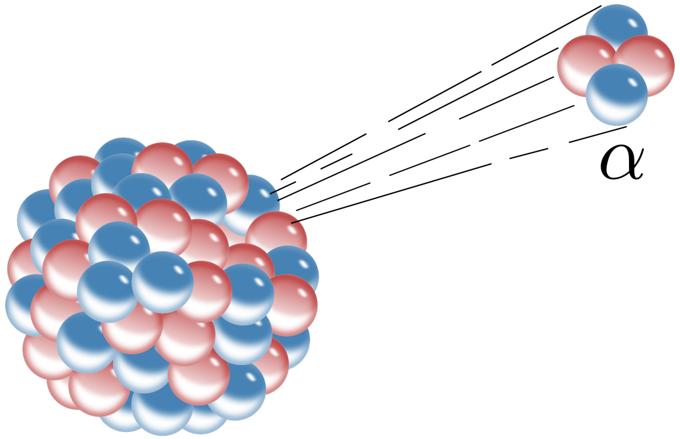
Rutherford: Open the Door to Nucleus World with α Particles

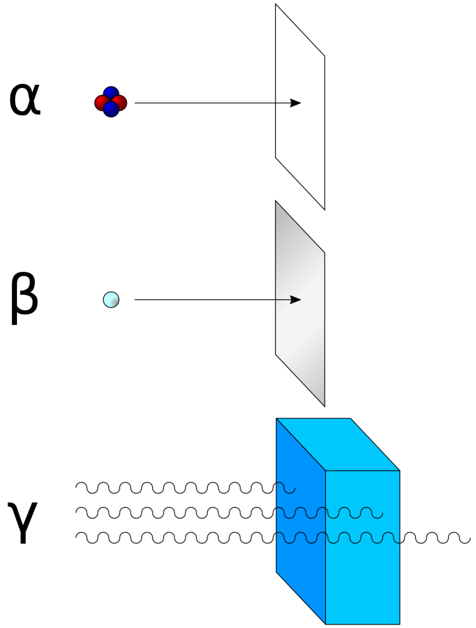


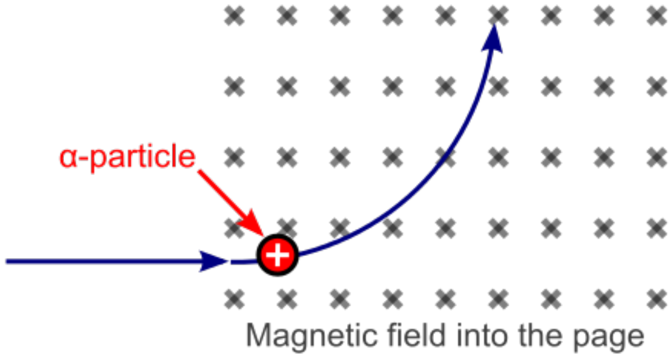
1908

Ernest Rutherford (1871-1937)
Father of Radiochemistry
In 1899, the discovery of natural α radiation

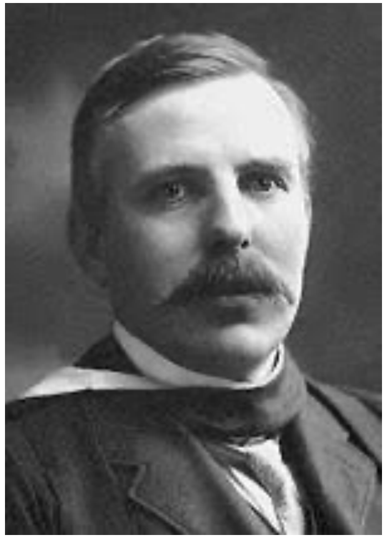
→ Nobel Prize in Chemistry in 1908







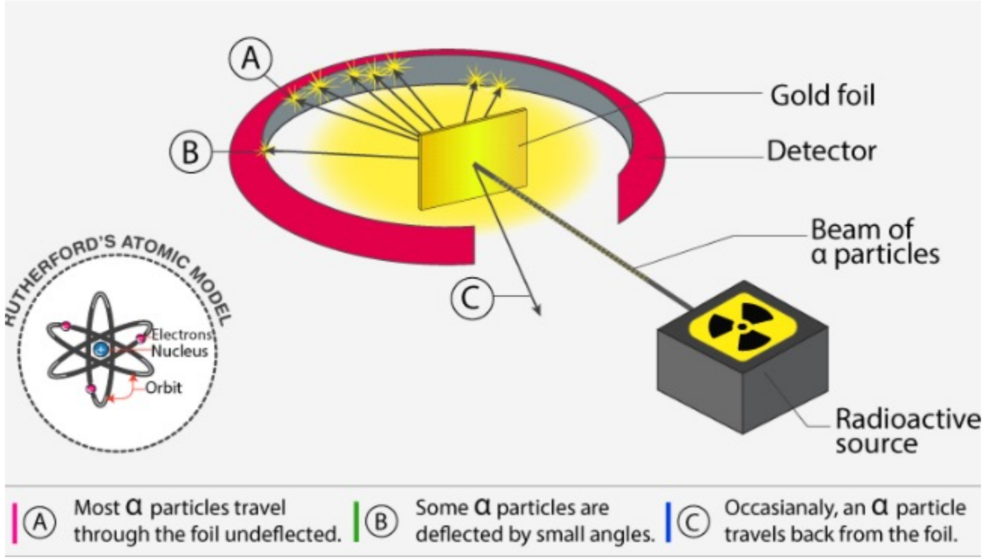
Rutherford: Open the Door to Nucleus World with α Particles



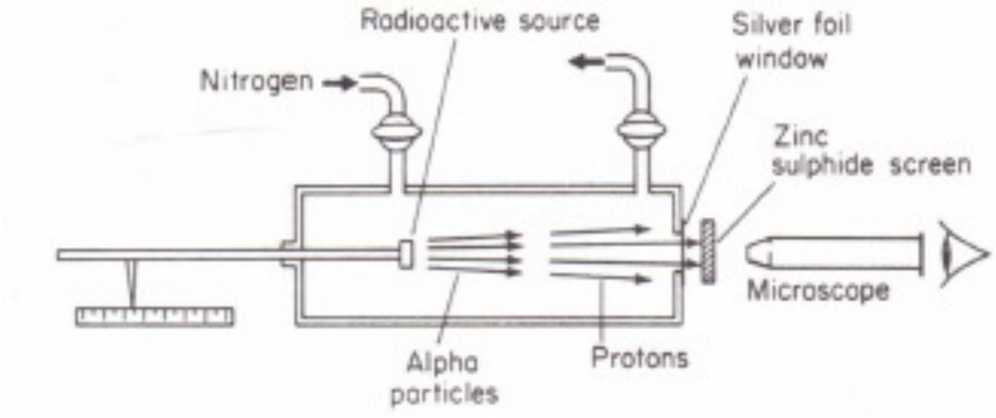
1908

Ernest Rutherford (1871-1937)
Father of Nuclear Physics and Atomic Physics

1911, discovery of nucleus
→ atom=nucleus+electron

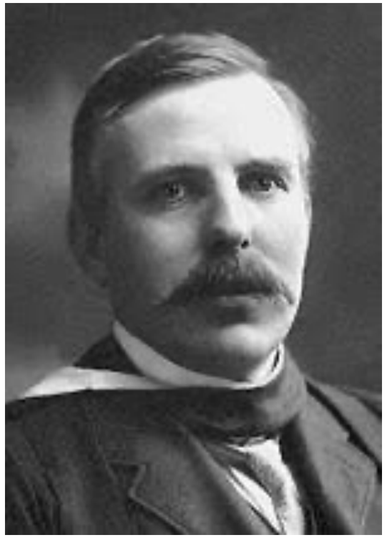


1919, discovery of proton
– 1^{st} hadron, lightest baryon



$$\alpha + {}^{14}N \rightarrow p + {}^{17}O$$

Rutherford proposed the neutron hypothesis



1908

In 1920, a public lecture

Bakerian Lecture: *Nuclear Constitution of Atoms*

by Sir E. Rutherford, F.R.S., Cavendish Professor of Experimental Physics, University of Cambridge.

hydrogen is ascribed to the movements of this distant electron. Under some conditions, however, it may be possible for an electron to combine much more closely with the H nucleus. forming a kind of neutral doublet. Such an atom would have very novel properties. Its external field would be practically zero, except very close to the nucleus, and in consequence it should be able to move freely through matter. Its presence would probably be difficult to detect by the spectroscope, and it may be impossible to contain it in a sealed vessel. On the other hand, it should enter readily the structure of atoms, and may either unite with the nucleus or be disintegrated by its intense field, resulting possibly in the escape of a charged H atom or an electron or both.

Discovery of Neutron – On the Importance of a Good Mentor

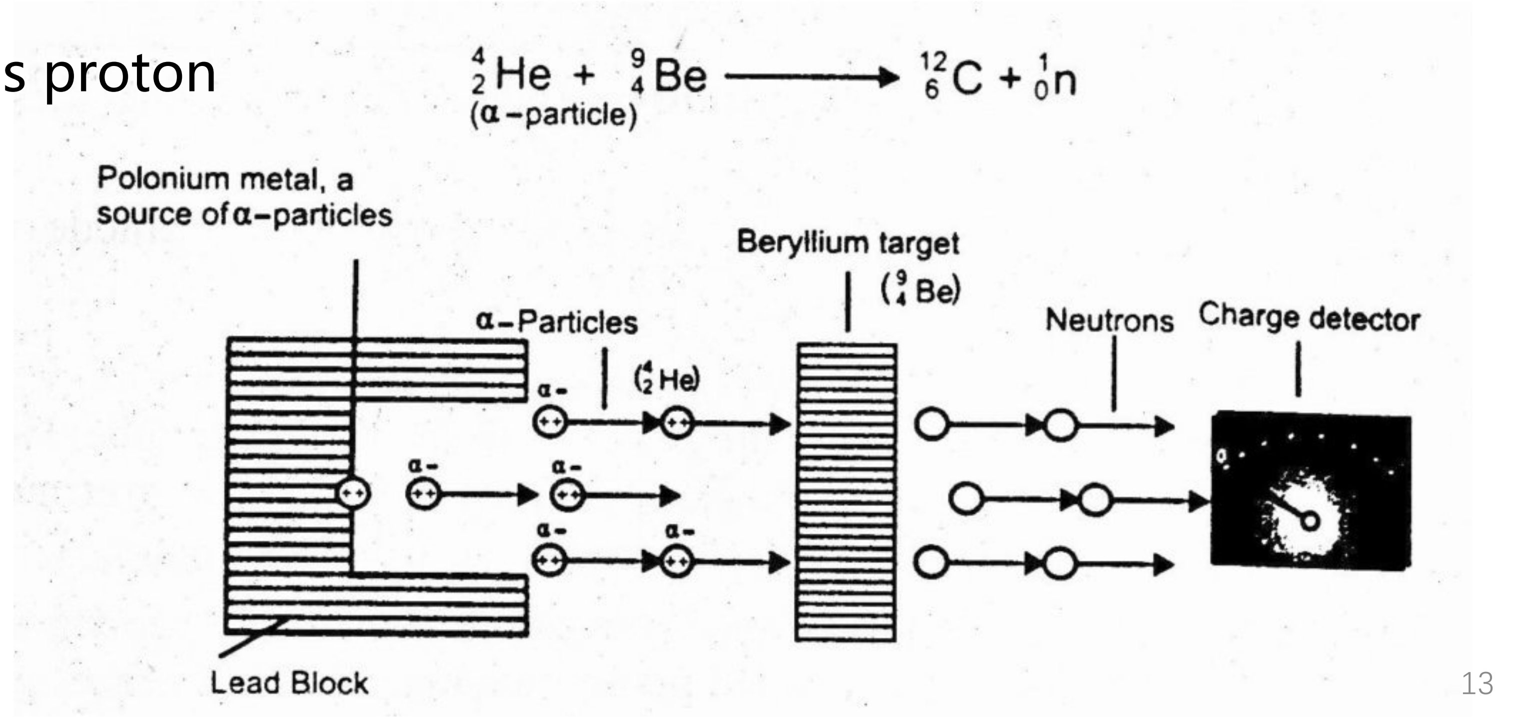
- In 1930, German scientist W. Bothe and his student H. Becker, bombarded beryllium (Be) with α rays, discovering a highly penetrating neutral radiation, which they believed to be γ rays
- 1932, the Curie couple repeated the experiment and found that neutral radiation could knock proton out of paraffin, with kinetic energy of few MeV. Based on conservation law, the energy of γ rays should be ~50 MeV. However, the E_b within the atomic nucleus is only a few MeV!
- 1932, Chadwick found that neutral radiation, when irradiated onto other substances, can cause nuclear recoil. Based on the energy of recoiled nitrogen nucleus, the energy of γ rays > 90 MeV.

Neutral radiation has a similar mass as proton

→ neutron

J. Chadwick

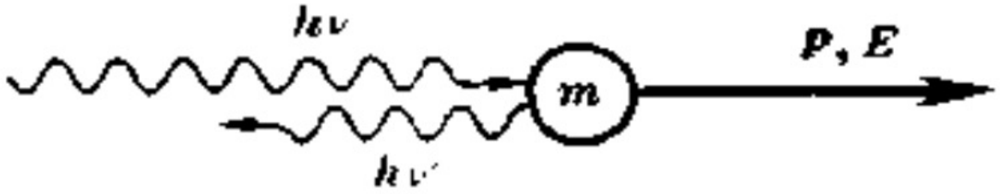
1935



Exercise: discovery of neutron

- The Curie couple bombarded beryllium (Be) and Boron (B) with α rays, discovering a highly penetrating neutral radiation. This radiation could knock proton out of paraffin, with a kinetic energy of 4.5 MeV for the Be experiment and 2 MeV for the B experiment, respectively.

  If the particle were a photon, please estimate the energy carried by the photon.



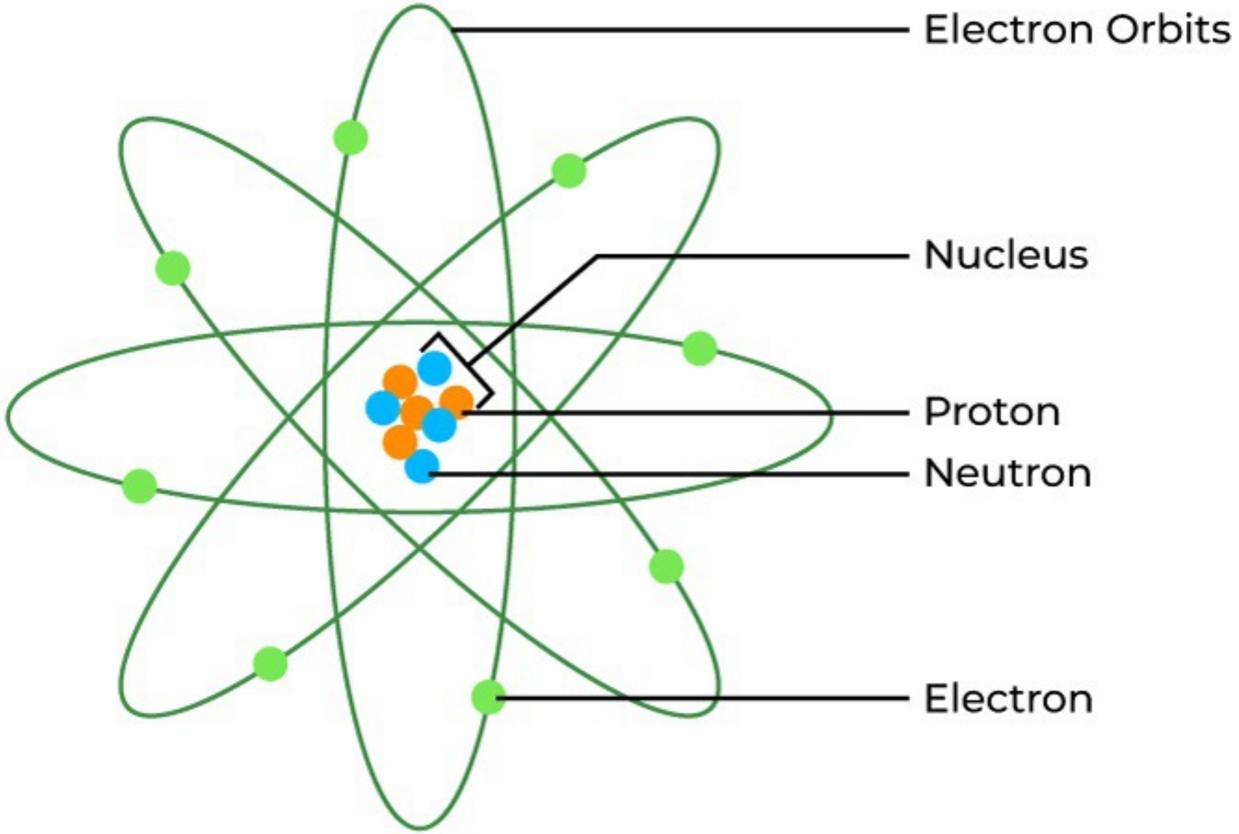
- Chadwick found that neutral radiation, when irradiated onto nitrogen (N) and hydrogen (H), the velocity of recoiled nitrogen and hydrogen is about 4.7×10^{6} m/s and 3.3×10^{7} m/s, respectively.

  Then what is the mass of the neutral particle?

Anecdotes in Science

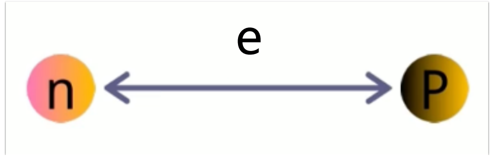
- Three Generations of Mentors and Disciples Discovering the Three Components of Atoms

J. Thomson → electron ; E. Rutherford → proton ; J. Chadwick →neutron



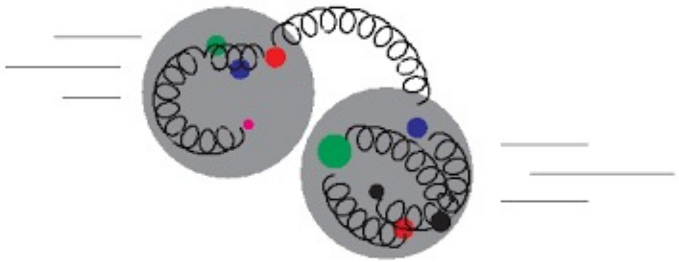
Heisenberg's Conjecture on Nuclear Forces

- Neutrons are electrically neutral and cannot neutralize the repulsion between protons, what purpose do they serve inside the atomic nucleus?
- What force is counteracting the electrostatic repulsion between protons to hold the nucleons together?



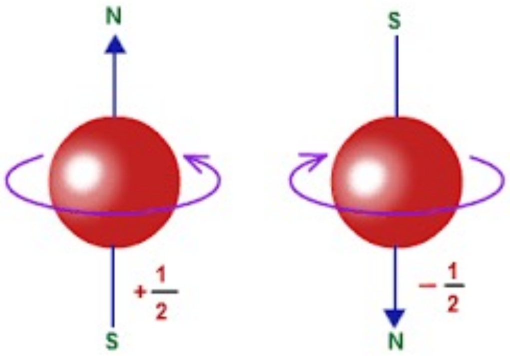
1936, M. Tuve and others discovered that the collision cross-section between nucleons is significantly larger than that of electromagnetic forces

W. Heisenberg (1901 - 1976)
A Founder of Quantum Mechanics
Representative Figure of Copenhagen School
Awarded the Nobel Prize in 1932



Introduction of Isospin

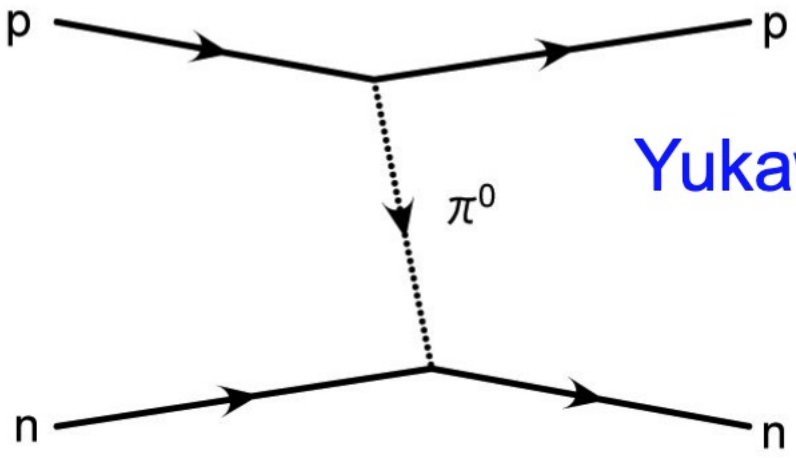
Analogous to Spin



Charge Space – Positive & Negative
Spin Space – Spin up & Spin down
Isospin Space – Isospin up & down

Theory of Pion Exchange by H. Yukawa

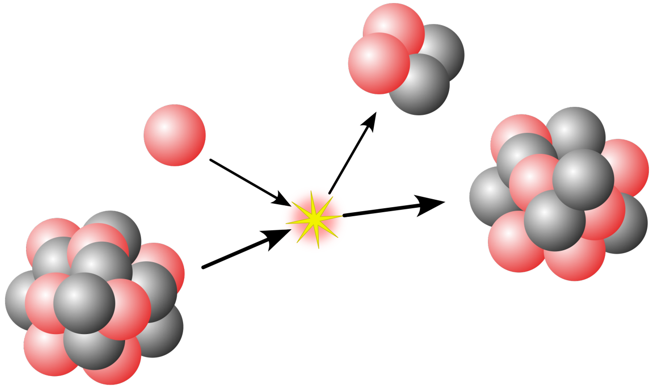
1949

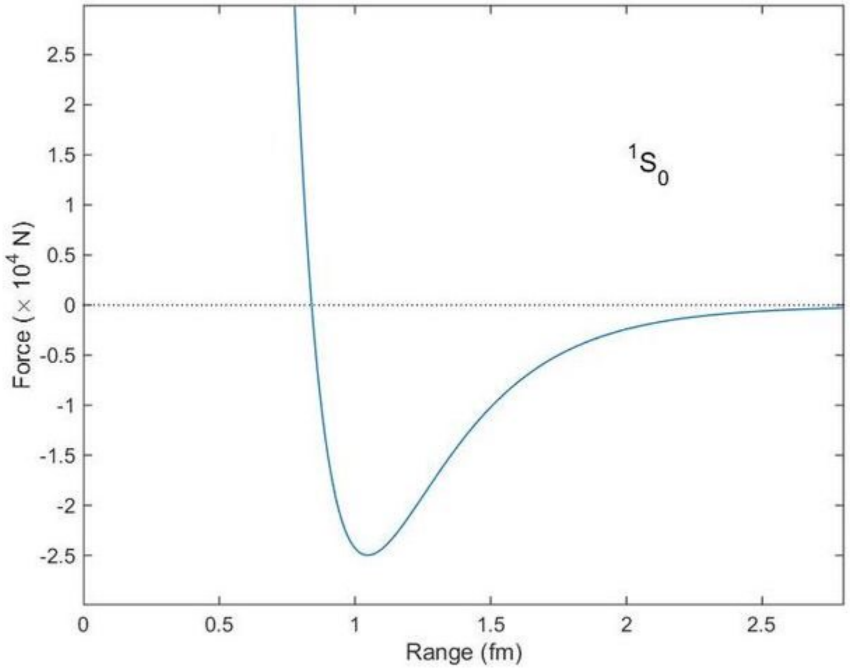


Yukawa potential

$$V(r) = \frac{g^2}{4\pi}\frac{e^{-m_\pi r}}{r}$$

In 1934, proposed the meson exchange theory of nuclear forces
→ predicted the existence of the π meson





The Search for Mesons

1949

1950

C. Powell

- In 1934, Yukawa predicted the meson mass
  - Strong nuclear force exists only within r_0 = 10^{-15} m
  - Using $E=mc^2$ and the principle of uncertainty

$$E = mc^2 \qquad \Delta t\, \Delta E \geq \hbar/2$$

$$r_0/c \qquad mc^2$$

$$m = \frac{\hbar}{2r_0 c} \sim 100\ \mathrm{MeV}$$

too heavy to be directly obtained through natural radioactivity experiments

- In 1936, Neddermeyer & Anderson observed cosmic rays from high mountains and discovered charged particles with mass of ~100 MeV, called μ meson (essentially not a meson)

- In 1947, Powell detected cosmic rays using a hot air balloon and discovered π meson with a mass of ~140 MeV

Fermi-Yang Model

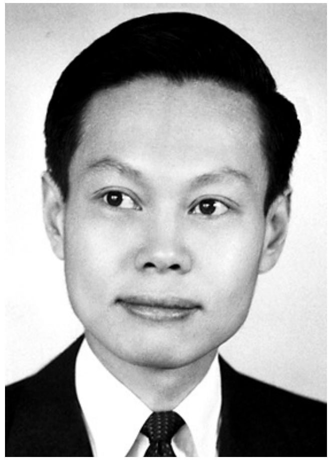
Are Mesons Elementary Particles?

E. FERMI AND C. N. YANG*
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois

(Received August 24, 1949)

The hypothesis that π-mesons may be composite particles formed by the association of a nucleon with an anti-nucleon is discussed. From an extremely crude discussion of the model it appears that such a meson would have in most respects properties similar to those of the meson of the Yukawa theory.

E. Fermi



C. N. Yang

- Five known particles: p、n、π^+、π^0、π^-

π^+ = p + anti-n
π^- = n + anti-p

Conservation of baryon number
Conservation of charge
Conservation of isospin

Development of accelerator and detector technologies

Cloud chambers, bubble chambers play important roles

D. Glaser

Invented the bubble chamber in 1952, making significant contributions to the study of hadron spectroscopy

Nobel prize in 1960

L. W. Alvarez

Developed hydrogen bubble chamber technology and data analysis methods in 1958, leading to the discovery of numerous hadrons

Nobel prize in 1968

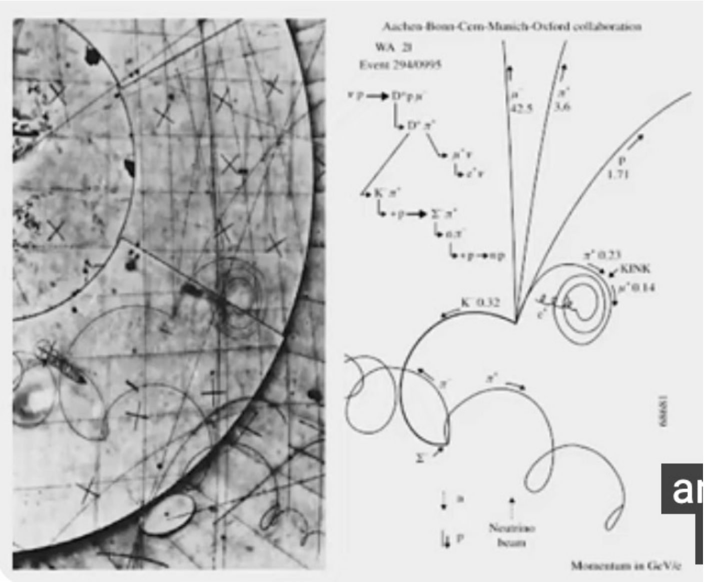
Discovering hadrons owes much to detector technology

Development of accelerator and detector technologies

In 1954, Bevatron at LBNL (a billion-electron-volt accelerator) began operating, leading to the discovery of a large number of hadrons, including strange mesons and hyperons.

Question 1: Are these particles fundamental particles?

Question 2: Why do strange particles appear in pairs?



$$p^+ + \pi^- \longrightarrow K^0 + \Lambda^0$$
$$p^+ + \pi^- \longrightarrow \Sigma^- + K^+$$

Sakata Model

- Discovery of Strange Particles: $\boldsymbol{\Sigma}^+$、$\boldsymbol{\Sigma}^0$、$\boldsymbol{\Sigma}^-$、$\boldsymbol{\Lambda}$、Ξ^0、Ξ^-、K^0、K^+、K^-、anti-K^0

A new quantum number – Strangeness $\rightarrow$ p and n cannot form strange particles

In 1955, M. Sakata proposed considering p, n, and Λ particles as fundamental particles

Partially successful model

- Can explain the composition of K mesons

  K^+ = p + anti-$\boldsymbol{\Lambda}$
- Successfully predicted new particle η
- Unable to explain the composition of hyperons $\boldsymbol{\Sigma}^+$、$\boldsymbol{\Sigma}^0$、$\boldsymbol{\Sigma}^-$

Classification of Hadrons

M. Gell-Mann

In 1960, Gell-Mann (also Y. Ne'eman and others) classified hadrons by combining various 'charges'

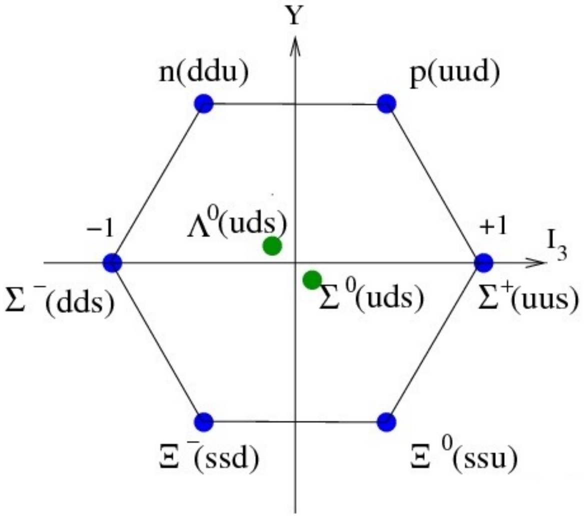
"*Hypercharge*": $Y = \underset{\text{baryon}}{B} + \underset{\text{strangeness}}{S}$

Electric charge: $Q = \underset{\text{isospin}}{I} + \underset{\text{hypercharge}}{Y}/2$

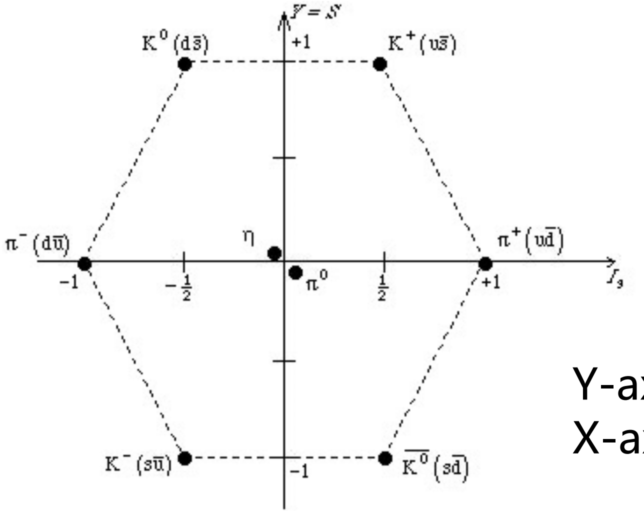
proton: $I = +\frac{1}{2}$, $Y = (1 + 0)$, so $Q = (\frac{1}{2} + \frac{1}{2}) = +1$

neutron: $I = -\frac{1}{2}$, $Y = (1 + 0)$, so $Q = (-\frac{1}{2} + \frac{1}{2}) = 0$

Octet of Baryons



Octet of Mesons



Y-axis: hypercharge
X-axis: isospin

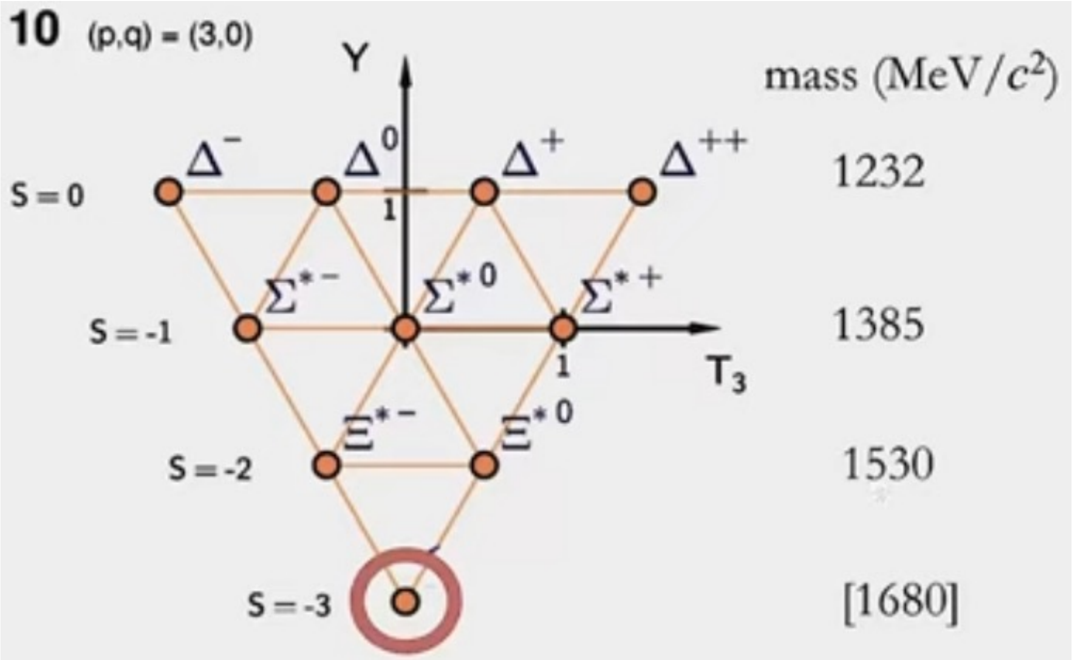
Create a periodic table for hadrons

Decuplet of Baryons: Spin=3/2, fully flavor symmetric

1969

M. Gell-Mann
Received Nobel Prize in 1969

For classification of fundamental particles, rather than for quark model



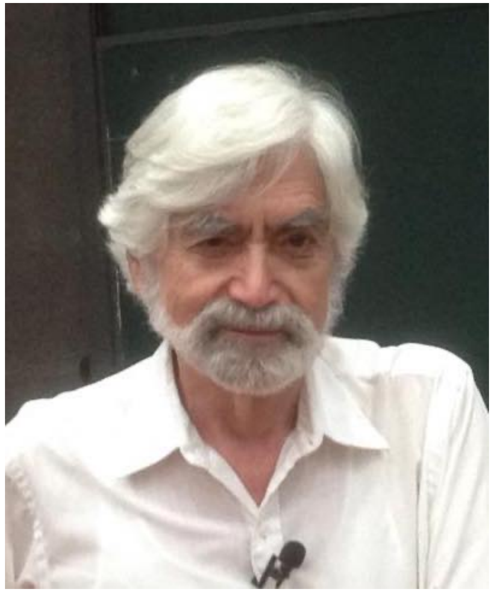
In 1962, predicted the existence of **Ω**$^-$(sss)

BROOKHAVEN NATIONAL LABORATORY

In 1964, BNL discovered it!

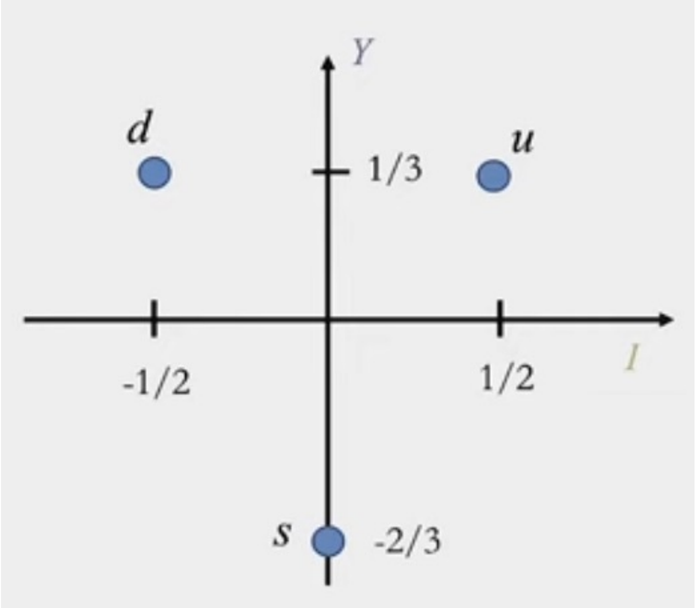
Quark Model

M. Gell-Mann

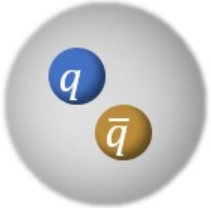


G. Zweig

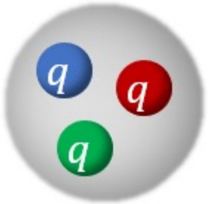
In 1964, Gell-Mann and Zweig, based on symmetry, proposed that hadrons are composed of three constituent particles. Gell-Mann called them "Quarks" while Zweig referred to them as "Aces"



- Three types of quarks: up (u), down (d), strange (s)
- Mesons are composed of a quark and an antiquark
- Baryons are composed of three quarks



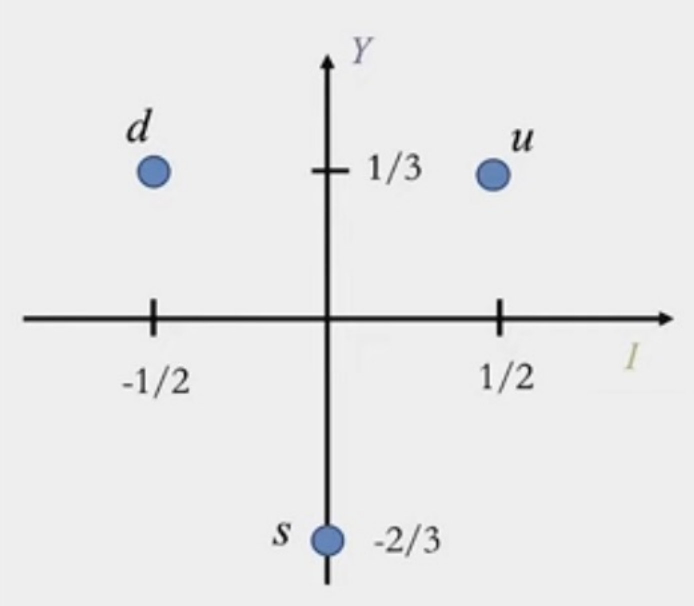
π^+ $(u\bar{d})$
K^+ $(u\bar{s})$



p (uud)
n (udd)

Challenges faced by the quark model

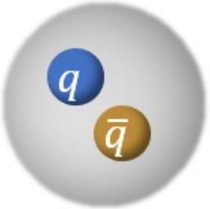
① Quarks carry fractional electric charge



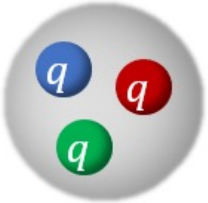
	B	S	Y	I	Q
u	1/3	0	1/3	1/2	+2/3
d	1/3	0	1/3	-1/2	-1/3
s	1/3	-1	-2/3	0	-1/3

"Hypercharge": $Y = B + S$

Electric charge: $Q = I + Y/2$



π^+ $(u\bar{d})$
K^+ $(u\bar{s})$



p (uud)
n (udd)

② Some particles violate Pauli exclusion principle

Charge State of Δ	I_3	*Final State*
Δ^{++}	$\frac{3}{2}$	$p\pi^+$
Δ^+	$\frac{1}{2}$	$p\pi^0$
		$n\pi^+$
Δ^0	$-\frac{1}{2}$	$p\pi^-$
		$n\pi^0$
Δ^-	$-\frac{3}{2}$	$n\pi^-$

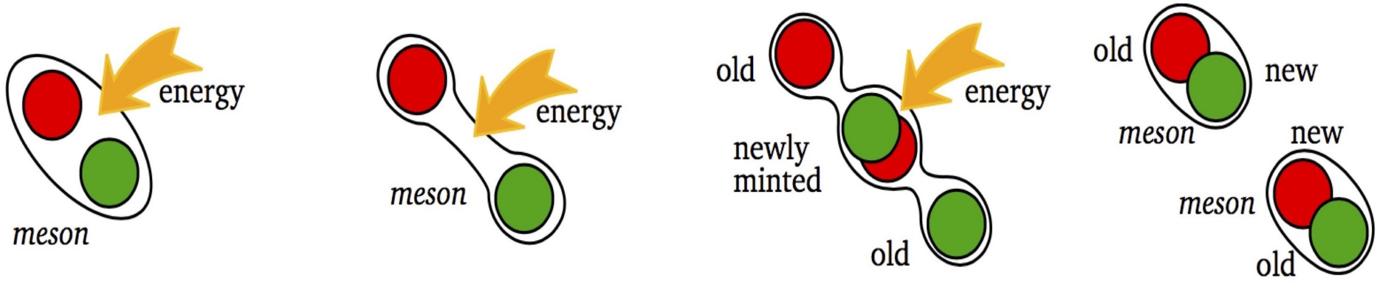
Δ^{++} : Spin=3/2 , Isopsin I_3=3/2
Composed of 3 u quarks
all spins aligned upwards?

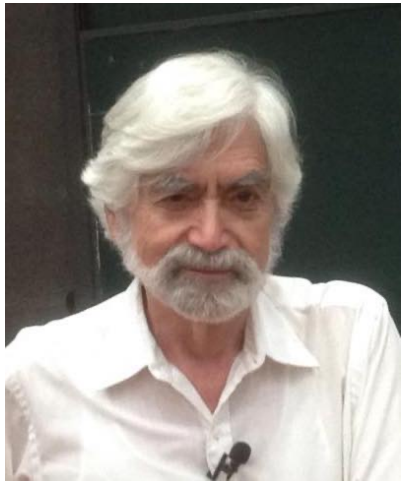
Ω^- also face the same issue…

Challenges faced by the quark model

③ Free quarks have not been observed

④ Dynamical mechanism by which quarks form hadrons?





G. Zweig

Zweig's paper was never accepted for publication!

M. Gell-Mann

It is fun to speculate about the way quarks would behave if they were physical particles of finite mass (instead of purely mathematical entities). … A search for stable quarks at the highest energy accelerators would help to reassure us of the non-existence of real quarks.

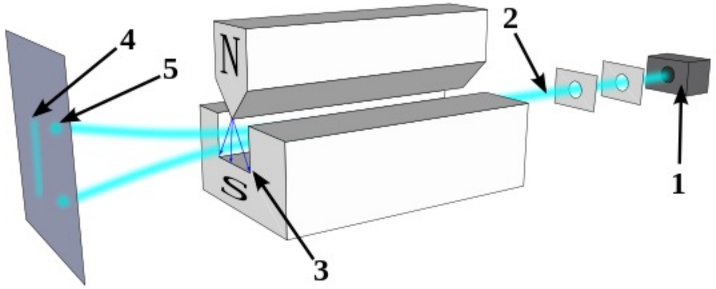
——A schematic model of baryons and mesons

Protons have internal structure

Otto Stern

1933, measured proton's magnetic moment, 2.5 times of the expected value

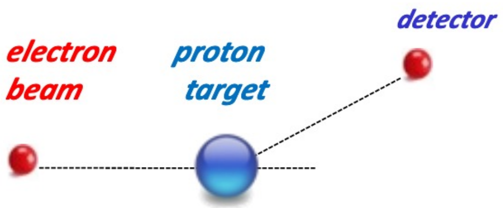
1943, Nobel prize in Physics

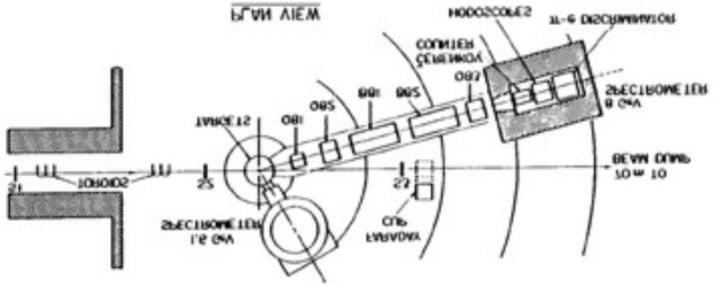


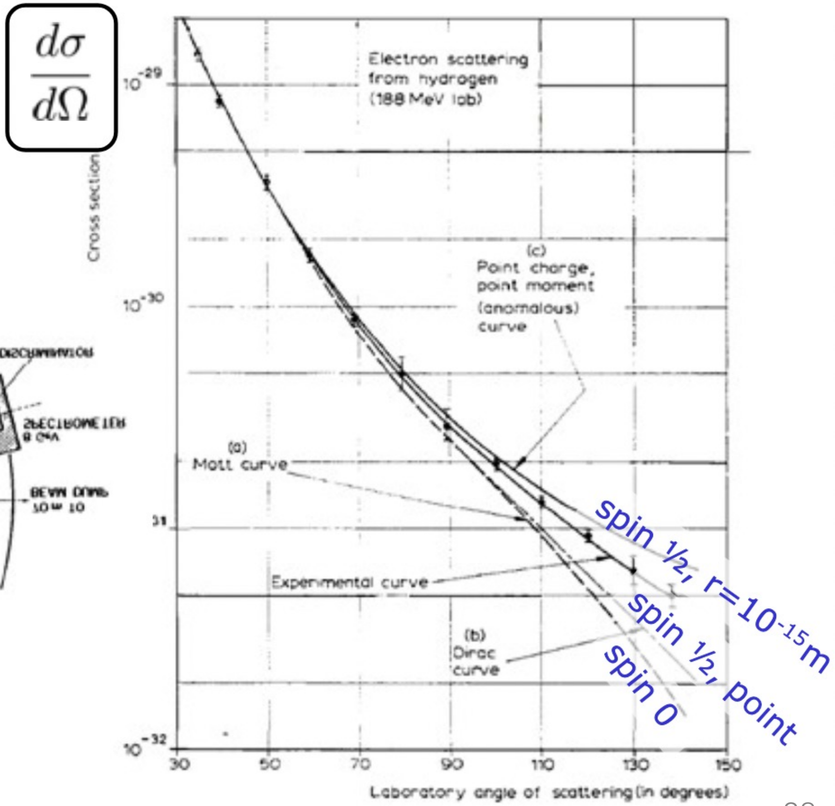
R. Hofstadter

1955, measured proton size through e-p scattering

1961, Nobel prize in Physics







Microscope of high-energy physicists

Optical microscope
Photon → Target material
Observation scale: 10^{-7} m

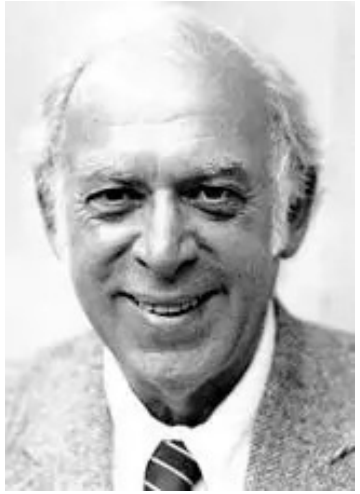
Electron microscope
Electron → Target material
Observation scale: 10^{-10} m

Accelerator
High-energy particle → Target material
Observation scale: 10^{-15} m

High-energy particles (photons, protons, electrons) serve as probes → Detectors act as the "eye"

Observing the microstructure inside hadrons becomes possible!

Nucleons have internal components

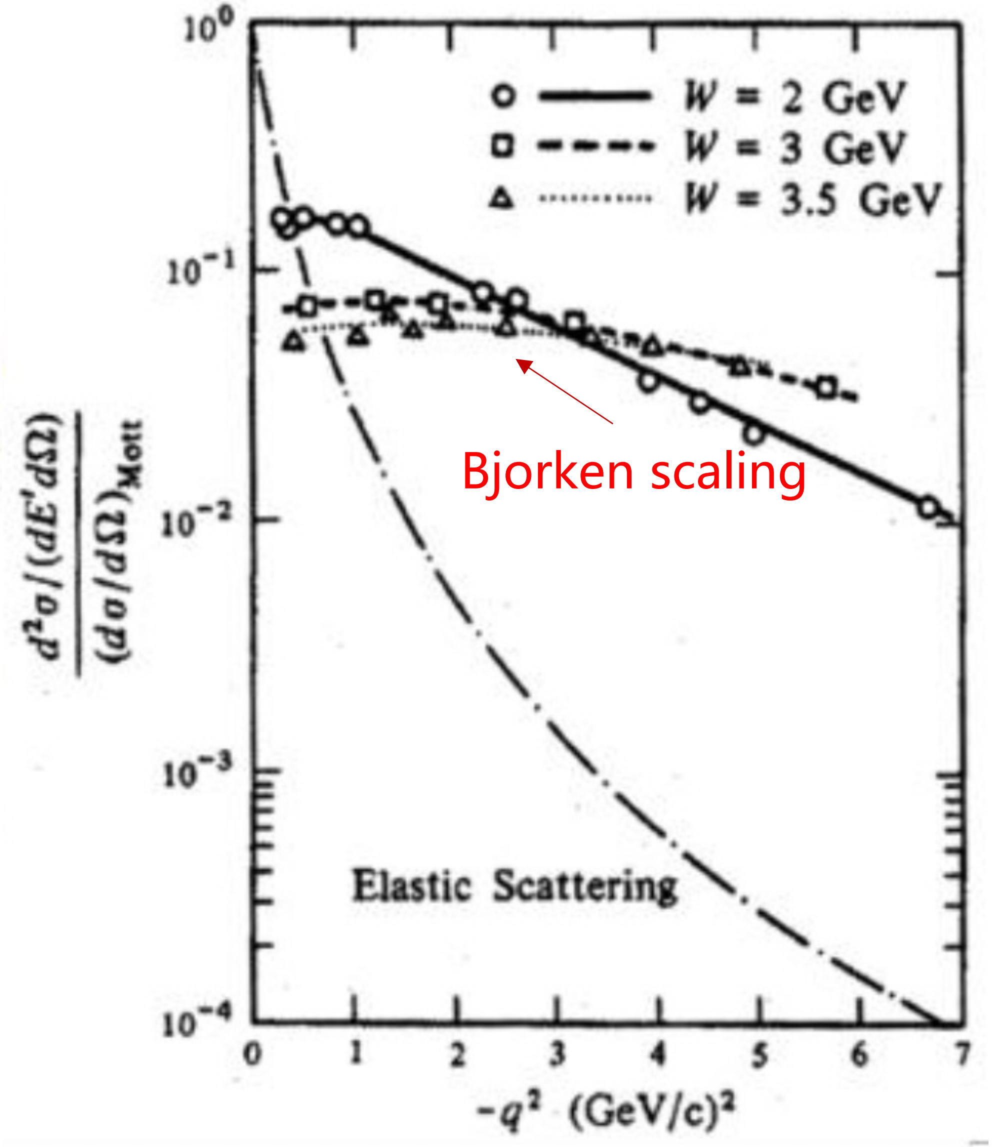

J. Friedman


H. Kendall


R. Taylor

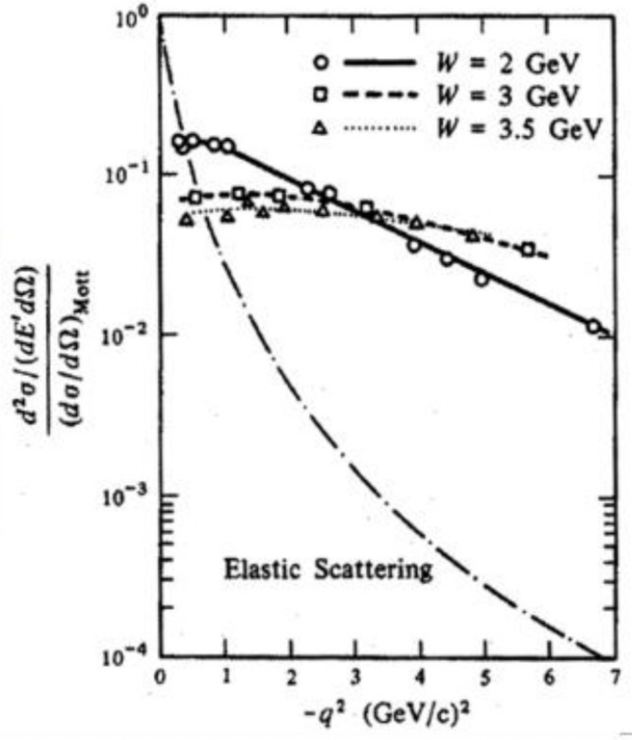
The 1969 deep inelastic e-p scattering experiments confirmed the existence of "point-like particle" inside nucleons

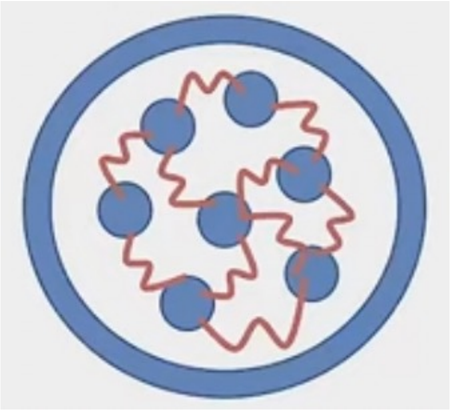
Nobel prize in 1990



Introduction of the concept of partons in nucleons

R. Feynman and J. Bjorken explained the SLAC scattering experiment data using the concept of partons





At low energies, the internal particles in nucleons interact very strongly and have very high energy and momentum

In DIS, from the electron's perspective, nucleon moves at near-light speed, the partons inside appear almost stationary due to the time dilation effect

Strictly speaking, Gell-Mann's quarks correspond to the strongly interacting components in low-energy bound states, while Feynman's partons correspond to the nearly free internal particles during high-energy scattering

Introduction of color

- Some particles Violate Pauli exclusion principle

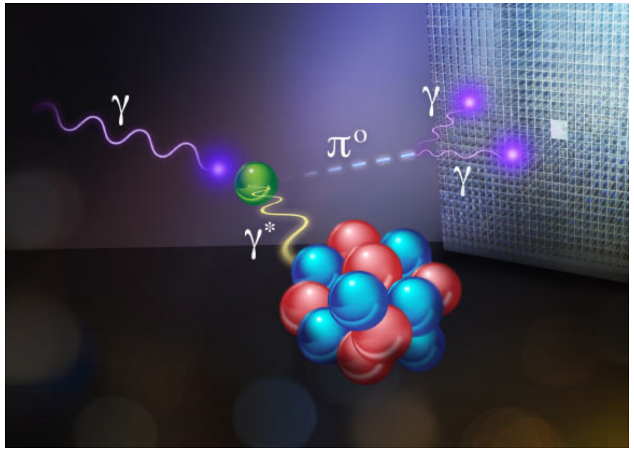
In 1964, Greenberg proposed a new quantum number – color

Composed of 3 identical quarks with all spins aligned upwards, but carrying different colors

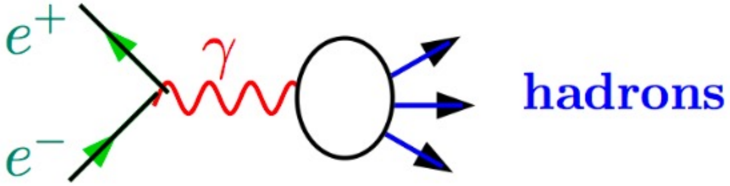
- Neutral Pion Decay

first measured in 1963 @CERN

Decay rate of $\pi^0 \to \gamma\gamma$ is $9=3^2$ times larger than expected



- Measurement of the R Value



$$R = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

$$R(s)^{\text{pert}} = N_c \sum_f Q_f^2 \frac{v_f}{2}(3 - v_f^2)\Theta(s - 4m_f^2) \times (1 + \alpha_s c_1 + \alpha_s^2 c_2 + \cdots)$$

$N_c = 3$, provided direct exp. evidence for 3 colors

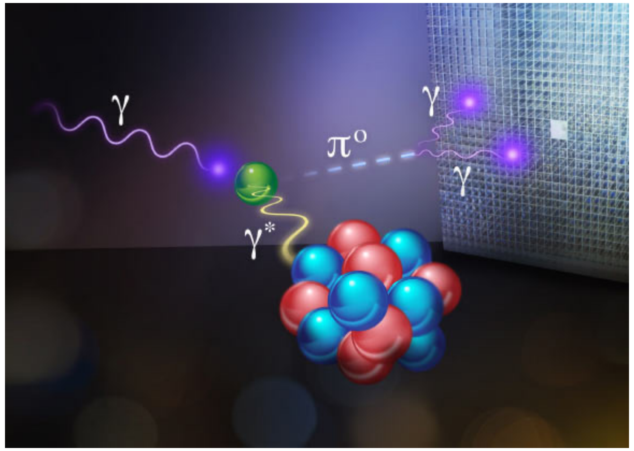
Excercise

Demonstrate that decay rate of $\pi^0 \to \gamma\gamma$ is proportional to N_c^2 while R ratio is proportional to N_c

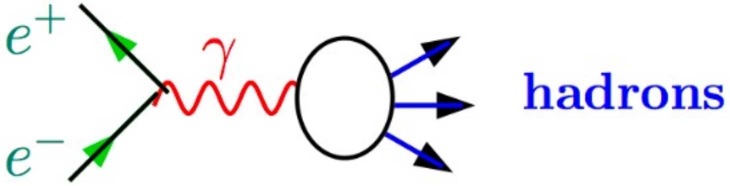
- Neutral Pion Decay

first measured in 1963 @CERN

Decay rate of $\pi^0 \to \gamma\gamma$ is $9 = 3^2$ times larger than expected



- Measurement of the R Value



$$R = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

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$N_c = 3$, provided direct exp. evidence for 3 colors

Vector Bosons Carrying Color - Gluons

Three Triplet Model with Double SU(3) Symmetry #3

M.Y. Han (Syracuse U.), Yoichiro Nambu (Chicago U., EFI) (1965)

Published in: *Phys.Rev.* 139 (1965) B1006-B1010

M. Y. Han

Y. Nambu

we introduce now eight gauge vector fields which behave as (1,8), namely as an octet in $SU(3)''$, but as singlets in $SU(3)'$. Since their coupling to the individual triplets is proportional to λ_i'' [the generators of $SU(3)''$], the interaction energy arising from the exchange of these vector fields will yield the first and second terms of Eq. (27). If these mesons obey again a similar type of

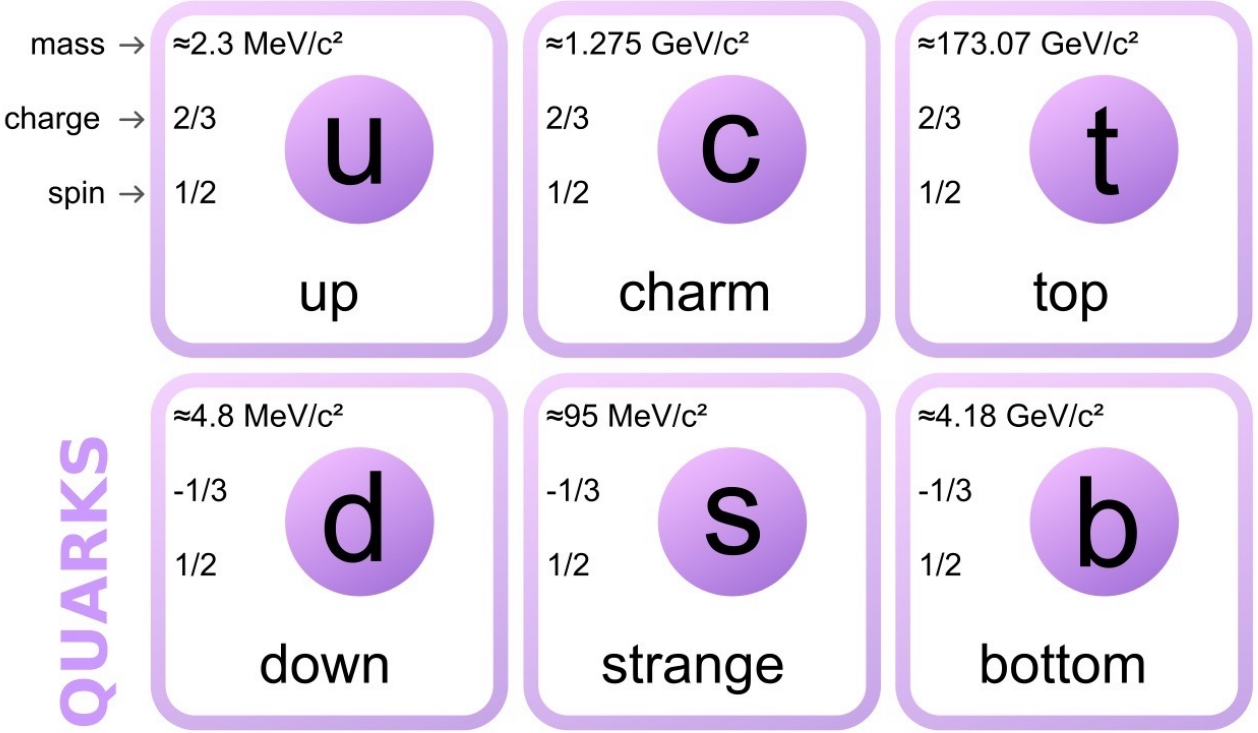
The discovery of heavy quarks

- In 1974, S. Ting and B. Richter discovered J/ψ, demonstrating the existence of the 4th quark - charm

S. Ting

B. Richter

1976



- In 1977, L. Lederman discovered **Υ**(9.5 GeV), composed of bottom & anti-bottom quarks
- In 1995, CDF and D0 experiment at Fermilab discovered the top quark, with m_t=173 GeV

Challenges faced by the quark model

① Quarks carry fractional electric charge

→ Change in mindset

② Some particles Violate Pauli exclusion principle

→ Introduction of color

③ Free quarks have not been observed

Assume:

Particles carrying color charge are confined within hadrons. All hadronic states are color singlets

→ Color confinement

Strong interaction is very strong !

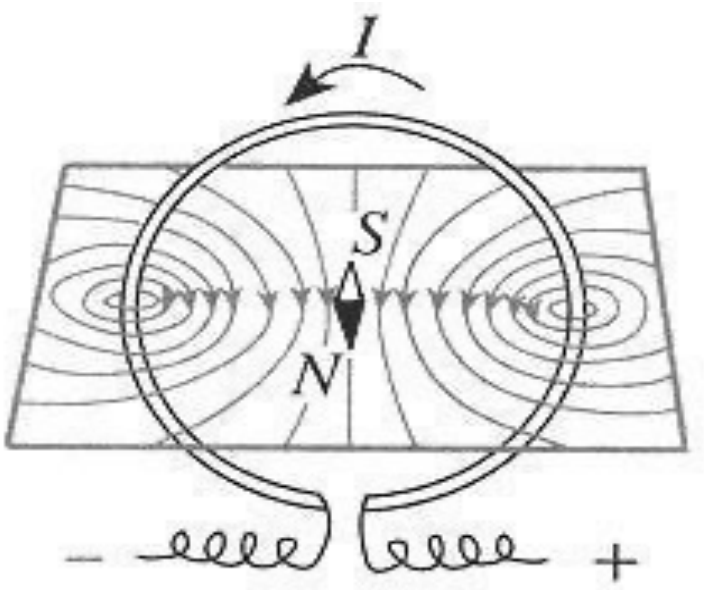
④ Dynamical mechanism by which quarks form hadrons?

Arise of QCD

Magnetic moment of a current-carrying coil

In 1820, Ampère's experiment

A. Ampère
1775-1836



- Size of the magnetic moment:
  current intensity × area of current circuit

$$I = e / T \qquad T = 2\pi R / v$$

$$S = \pi R^2$$

$$\mu = IS = \frac{1}{2} eRv$$

- Size of orbital angular momentum

$$L = mRv$$

- Relationship between magnetic moment and angular momentum:

$$\mu_L = g \frac{e}{2m} L \qquad g = 1$$ is Landé g-factor

Magnetic moment of a charged gyroscope

Imagine a charged gyroscope rotating to form a ring of circular electric current

- The faster the gyroscope spins
- The more charge it carries

→ The stronger the circular electric current

Magnetic moment of a rotating charged body is directly proportional to its charge and angular velocity

- angular velocity = rotation angular momentum ÷ mass

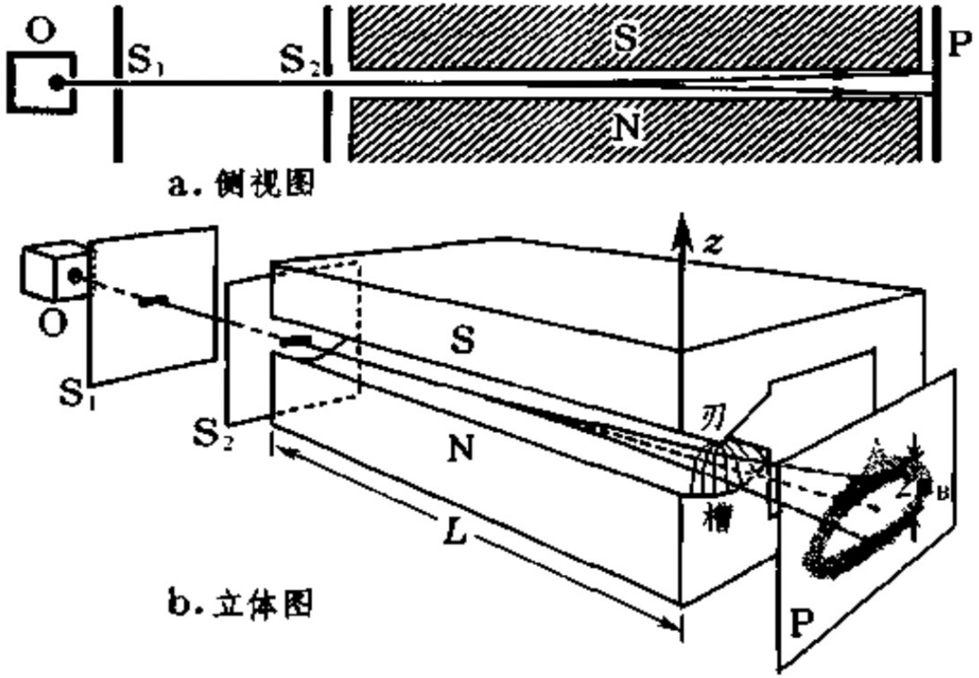
spin

finally **Magnetic moment** is proportional to **charge × spin ÷ mass**

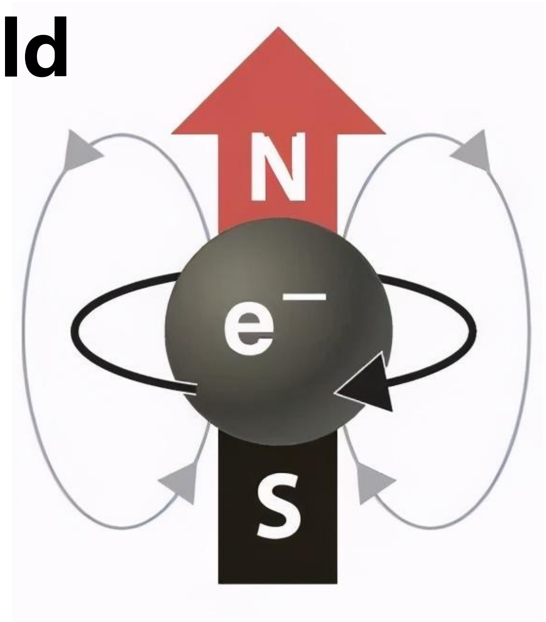
The spin magnetic moment in the quantum world

In addition to the classical magnetic moment produced by orbital motion, microscopic particles also carry an intrinsic magnetic moment

Stern-Gerlach experiment



Silver atoms deflect in a magnetic field, with magnetic moment mainly contributed by electron spin



Without any movement, an electron itself is also a small magnet

The proposal of spin

- Orbital angular momentum L should be an integer
- The silver atom beam should split into 2L+1 lines
- Splitting into two lines means L=1/2

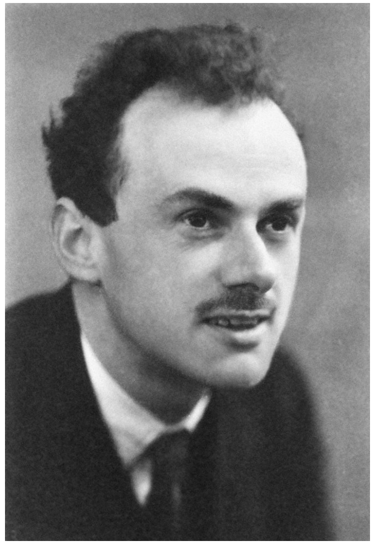
In 1925, Uhlenbeck & Goudsmit proposed the spin with a quantum number of 1/2

Dirac's theory predicts the spin magnetic moment

Current-carrying coil — Relationship between magnetic moment and angular momentum: $\mu_L = g\dfrac{e}{2m}L$ $\quad g = 1$

Electron spin — Relationship between magnetic moment and spin: $\mu_s = g\dfrac{e}{2m}s$ $\quad$?

In the classical world, there are no answers to this questions



1933

In 1928, Paul Dirac proposed the relativistic quantum mechanical equation describing the motion of electrons - the Dirac equation

$$g = 2$$

Landé g-factor of various particles

CODATA provides experimental measurements of g factors

Electron	$g = 2.002\ 319\ 304\ 362\ 56(35)$
Muon	$g = 2.002\ 331\ 8418(13)$
Proton	$g = 5.585\ 694\ 6893(16)$
Neutron	$g = -3.826\ 085\ 45(90)$

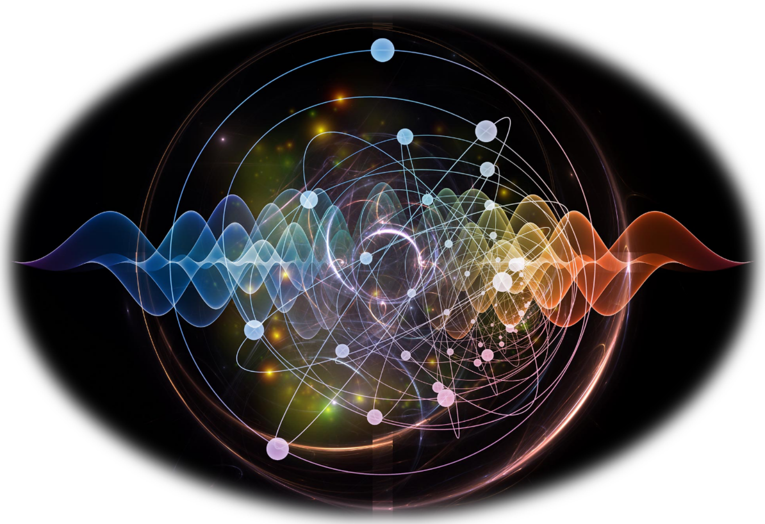
g factors of four spin-1/2 particles are not strictly equal to 2

$\Rightarrow$ Anomalous magnetic moment $a = (g-2)/2$

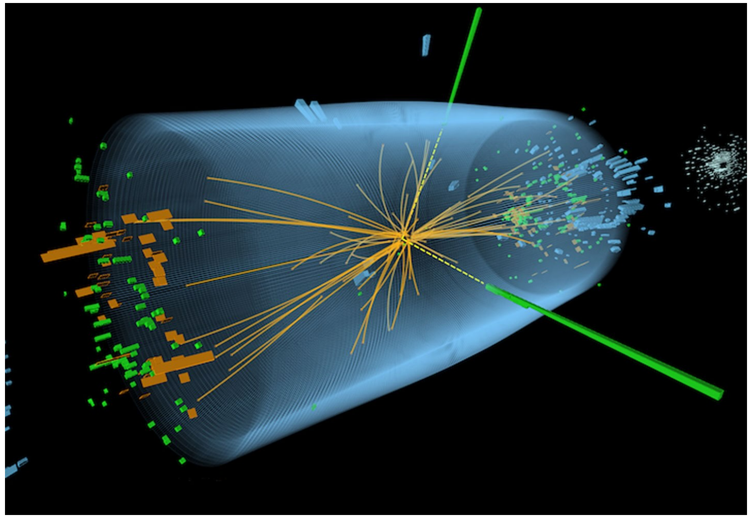
Why is there such a big difference in the anomalous magnetic moments (e, μ vs p, n)?

When relativity meets quantum mechanics

Approaching the speed of light
→ relativistic effects $E=mc^2$



Reaching atomic scales
→ quantum effects $E=\hbar\omega$



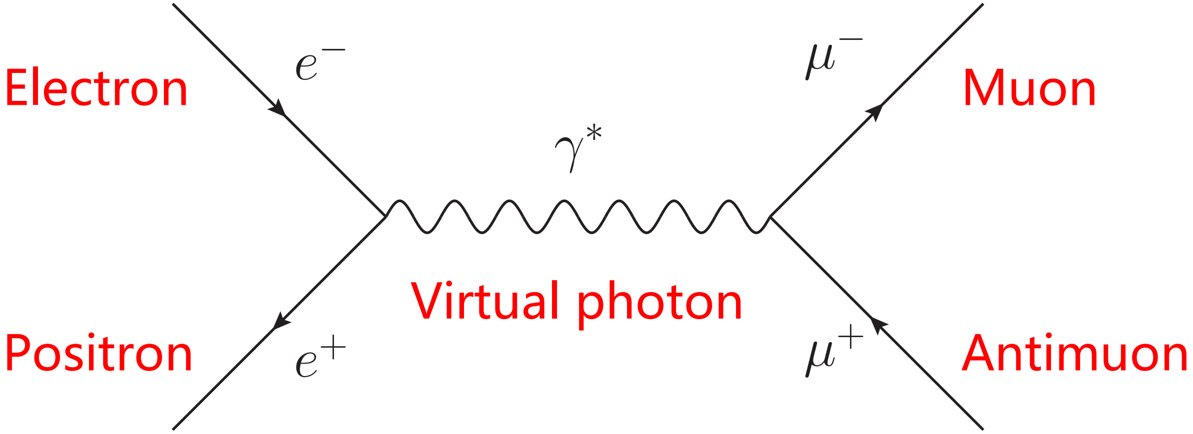
Characteristics of elementary particles:
- Simultaneously at the microscopic scale and moving at high speeds
- There will be particle creation and annihilation

→ Described using quantum field theory

Feynman diagrams - visual representation of QFT

R. Feynman

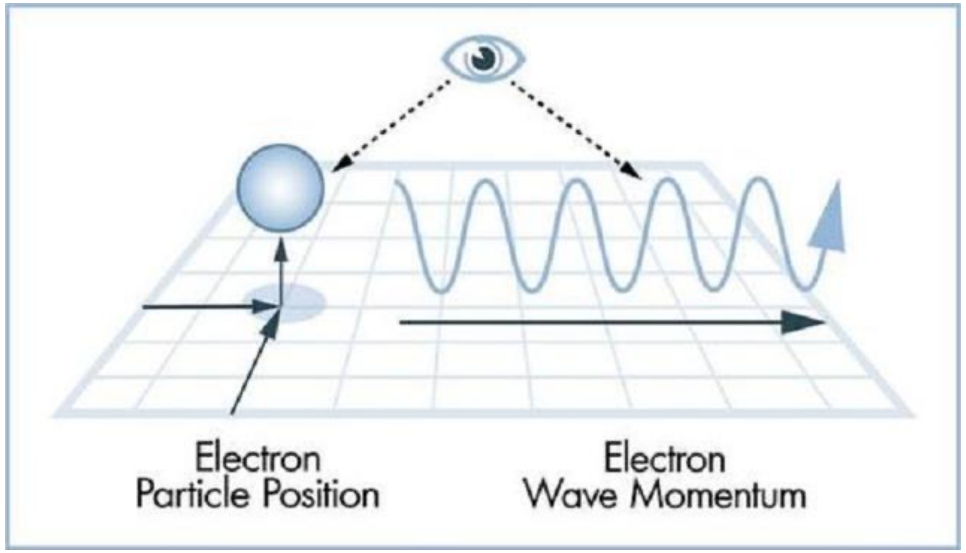
Electron-positron collision producing muon and antimuon pairs



The production rate of muon pairs is governed by the fine-structure constant, which describes the strength of electromagnetic interactions

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

Quantum fluctuations: the creation and annihilation of particles



- Heisenberg Uncertainty Principle

$$\Delta x \, \Delta p \geq \hbar/2$$

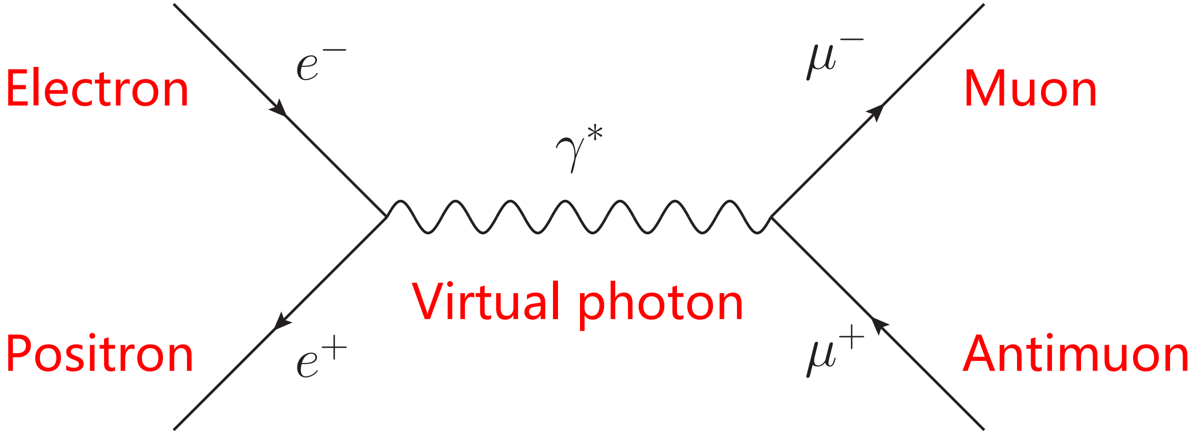
Energy can fluctuate significantly over short periods of time

- Special Theory of Relativity

$$E = mc^2$$

Energy and mass can be converted into each other

- When large energy converts into mass, new particles are born



Quantum fluctuations: the creation and annihilation of particles

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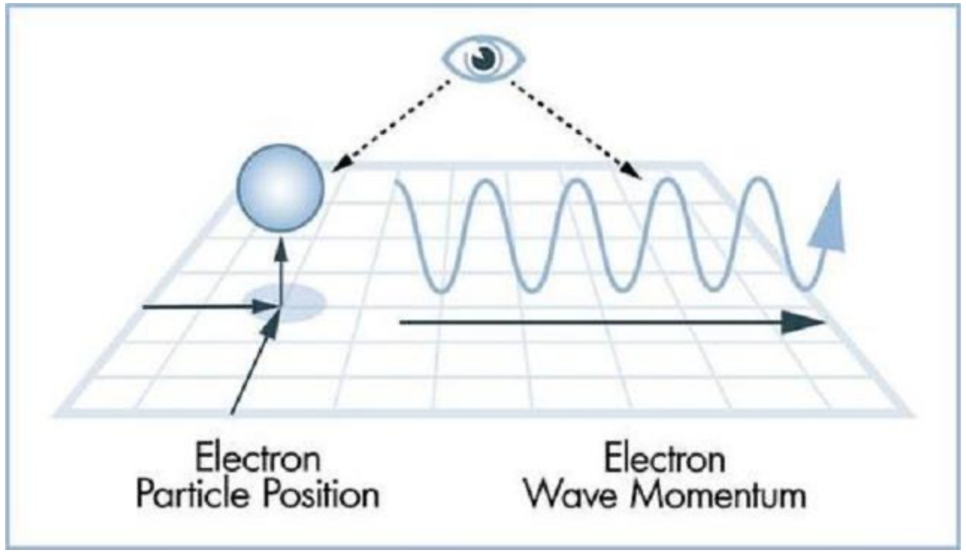
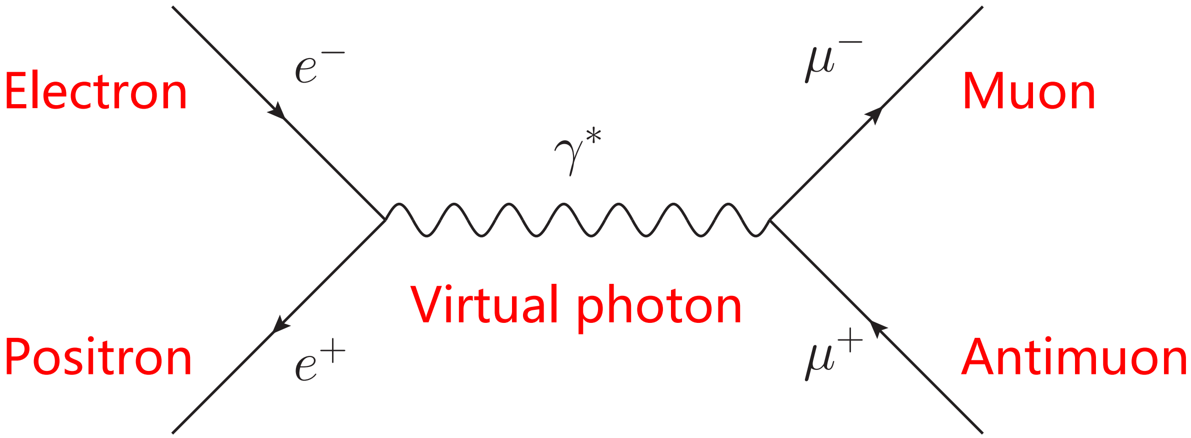


- Does quantum fluctuation violate conservation of energy?

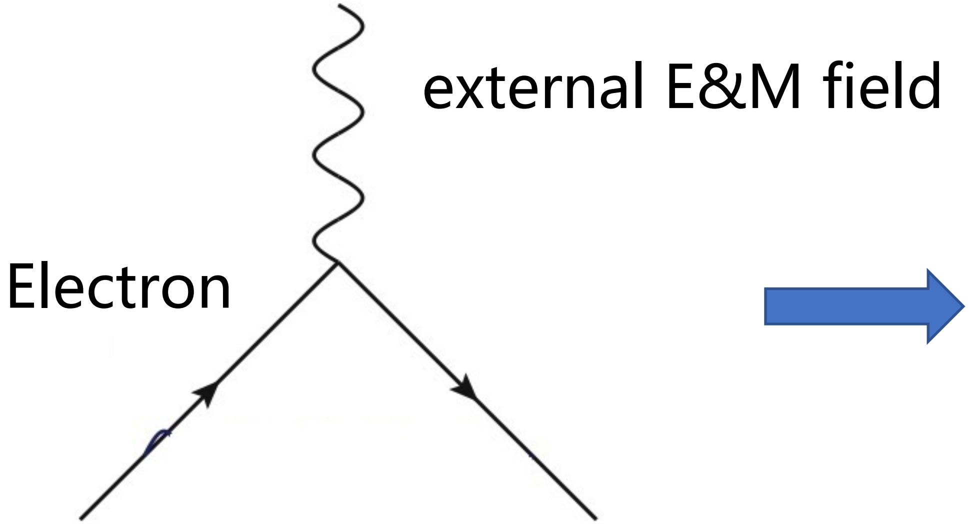
If new particles are created using high energy, but the energy is returned within the shortest time dictated by the uncertainty principle, there will be no violation of the conservation of energy law observed

Particles produced in this manner are called virtual particles



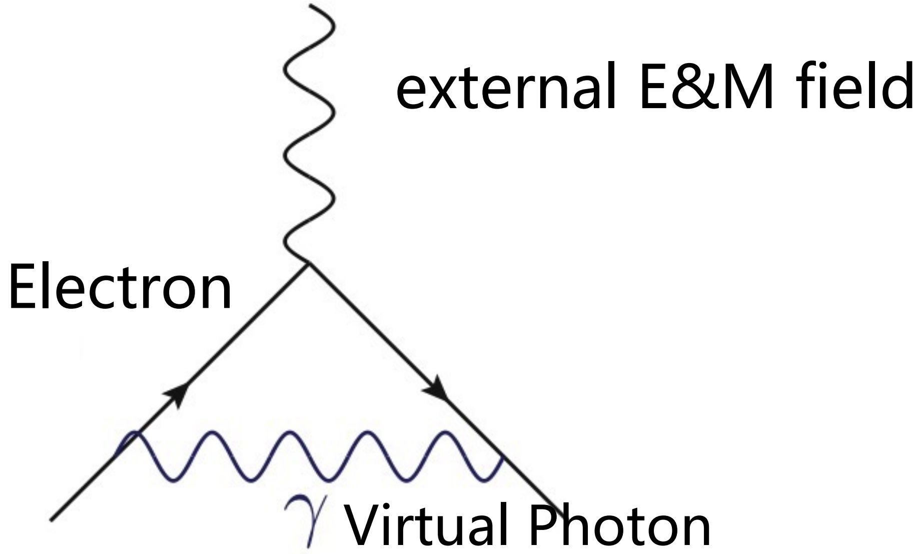
Quantum fluctuations can cause changes in the electron g-2

Electron in an external E&M field



Tree diagram, O(1)

$$g = 2$$



One-loop diagram, O(α)

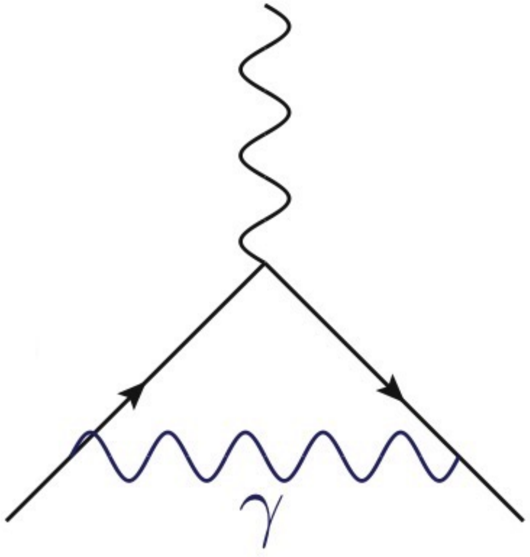
$$a = (g-2)/2$$

Schwinger's contribution

The leading-order calculation of the electron's anomalous magnetic moment was first provided by Julian Schwinger in 1948

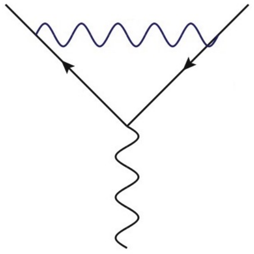
$$a_e \approx \frac{\alpha}{2\pi} = 0.001\ 161\ 4$$

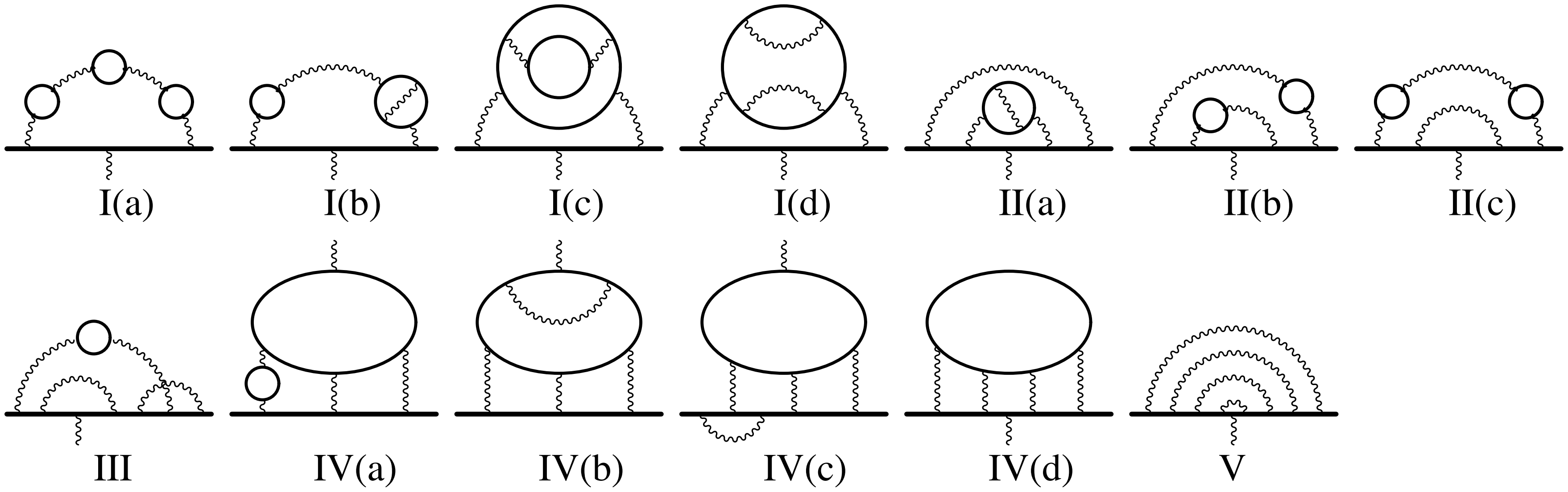
J. Schwinger



Quantum fluctuations can be very complex

- There can also be fluctuations within fluctuations
- Virtual particles themselves can generate other virtual particles





Theoretical calculation of lepton's anomalous magnetic moment has reached $O(\alpha^5)$

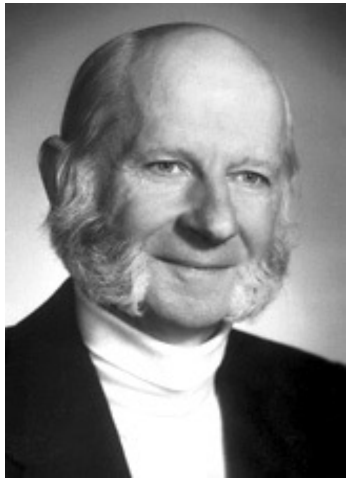
PRL **109,** 111808 (2012) PHYSICAL REVIEW LETTERS week ending 14 SEPTEMBER 2012

Complete Tenth-Order QED Contribution to the Muon $g-2$

Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]

Experimental measurement

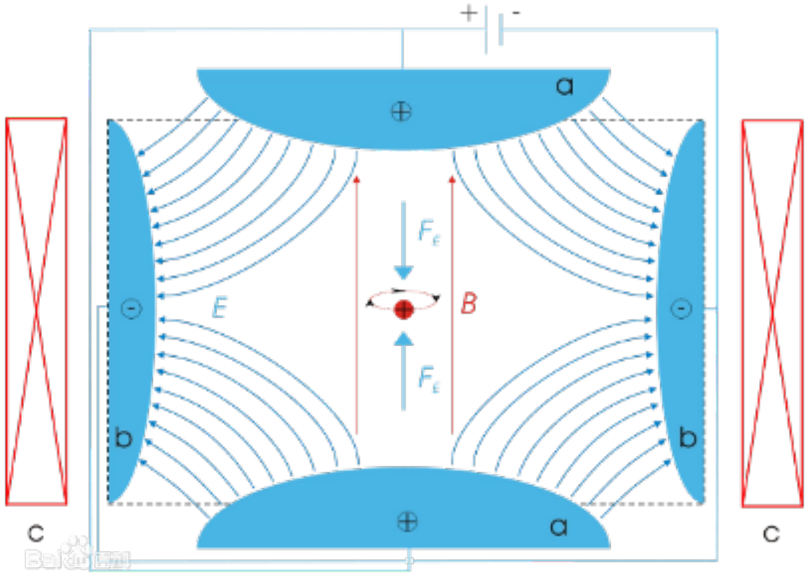
Dehmelt drew inspiration from Penning's designed magnetron and invented ion trap technology, naming it the Penning ion trap



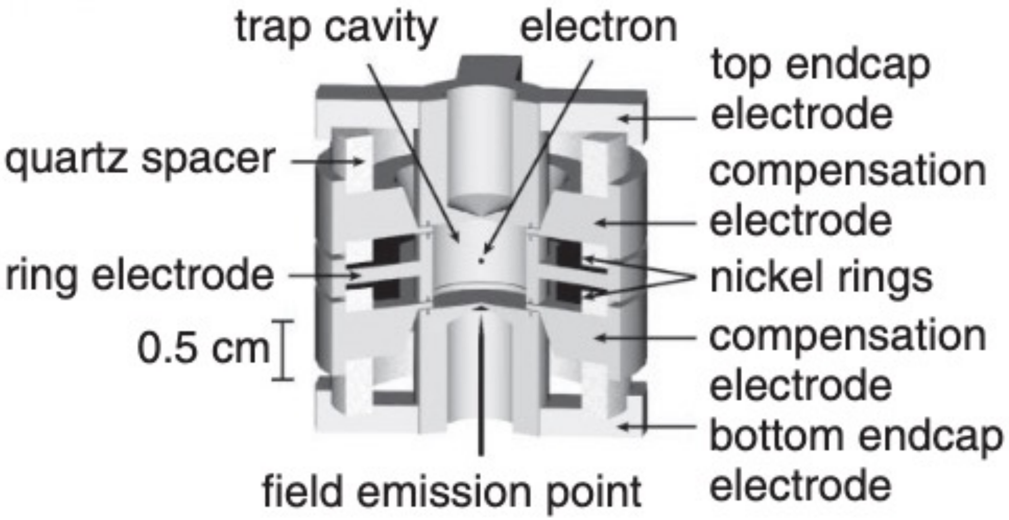
H. Dehmelt

1989

In 1987, using this technology, the electron's magnetic moment was precisely measured



In 2011, Harvard University utilized Penning ion trap technology to measure the electron g-2 with extremely high precision



The electron g-2 laid foundation for QED

Theory $$a_e = 1\ 159\ \ 652\ \ 181.61(23)\times 10^{-12}$$

Experiment $$a_e = 1\ 159\ \ 652\ \ 181.28(18)\times 10^{-12}$$

S. Tomonaga

J. Schwinger

R. Feynman

For establishing Quantum Electrodynamics, received the Nobel Prize in Physics in 1965

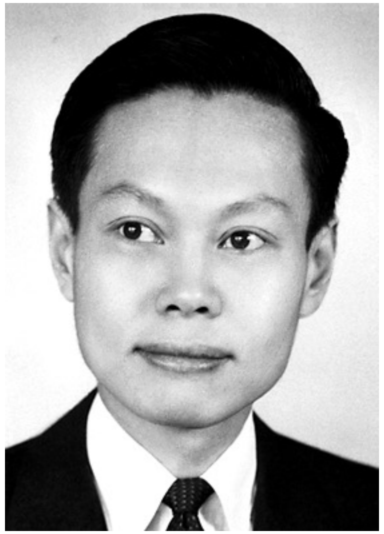
From U(1) to SU(N)

QED describes E&M interactions and exhibits U(1) gauge symmetry

E&M interactions → Strong interactions

One electric charge → Three colors

$U(1) \rightarrow SU(3)$



C. Yang

R. Mills

- Greenberg proposed the concept of color in 1964
- Yang & Mills proposed Yang-Mills theory in 1954
- Standard Model is constructed based on gauge theory

$$SU(3) \otimes SU(2) \otimes U(1)$$

The dilemmas of Yang-Mills theory

- Just as the photon in U(1) gauge theory is massless, Yang-Mills theory also predicts massless gauge bosons

Known: photon, graviton, neutrino

W. Pauli

Pauli was continuously pestering Yang, asking about the mass of the non-Abelian gauge bosons, knowing that they were massless and, therefore, a killer of the theory

- Physicists do not know how to calculate

Feynman accepted the challenge

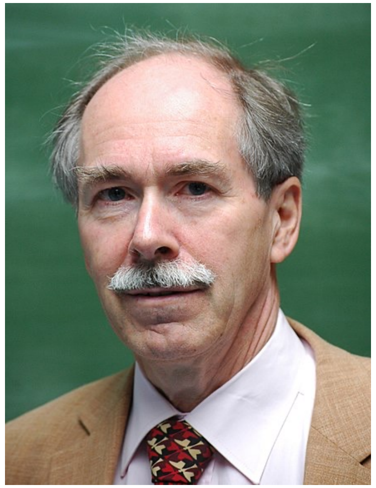
R. Feynman

To make the theory self-consistent, it is necessary to introduce additional ghost fields

Quantum Field Theory - Turning Point

- In 1967, Faddeev and Popov completed the quantization of the Yang-Mills theory

- In the same year, 't Hooft proved the renormalizability of Yang-Mills theory



1999

- Many loop diagrams yield results with infinite values - ultraviolet divergences
- If by introducing a finite number of counterterms into the Lagrangian, one can resolve the divergence issues in arbitrary loop diagrams, then the theory is said to be renormalizable

Quantum Field Theory - Breakthrough

- In 1971, K. G. Wilson proposed the theory of renormalization group

1982

- Concerned with the low-energy, long-range properties of physical systems
- Integrating out high-frequency degrees of freedom → renormalizes the parameters in the Lagrangian, causing the coupling parameters to run with the energy scale

- In 1973, H. Fritzsch, M. Gell-Mann, H. Leutywyler, reconsidered the color charge and the Yang-Mills theory based on experimental guidance

A theory describing strong interactions, established based on color gauge symmetry

Named as Quantum Chromodynamics (QCD)

QCD Lagrangian

- Maxwell's equations for E&M fields

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$	Gauss's law for electricity
$\vec{\nabla} \cdot \vec{B} = 0$	Gauss's law for magnetism
$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law of induction
$\vec{\nabla} \times \vec{B} = 4\pi\vec{j} + \frac{\partial \vec{E}}{\partial t}$	Maxwell-Ampère law

$\Rightarrow$ 4D covariant form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial^\mu F_{\mu\nu} = 0$$

- Starting from equations of motion for E&M field, deduce the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu)$$

Easy to verify that the Euler-Lagrange equations are equivalent to the Maxwell's equations

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \qquad \Rightarrow \qquad \partial_\nu F^{\mu\nu} = 0$$

QCD Lagrangian

- Equations for non-Abelian gauge fields

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{j} + \frac{\partial \vec{E}}{\partial t}$$

color a=1,2,3 … 8

Adjoint representation of SU(3)

All color electric field E^a, color magnetic field B^a, color charge density ρ^a, color current j^a carry color charge

- Quark field ψ

$$(iD_\mu\gamma^\mu - m)\psi = 0, \quad \rho^a = \psi^\dagger t^a \psi, \quad \vec{J}^a = \bar{\psi}\vec{\gamma} t^a \psi$$

- The equations of motion are a rephrasing of Maxwell's equations, but none of them are laws derived from experiments

QCD Lagrangian

The Symmetry group of vector and axial vector currents #1

Murray Gell-Mann (Caltech) (May, 1964)

Published in: *Physics Physique Fizika* 1 (1964) 63-75

pdf DOI cite claim reference search 342 citations

We start with the simple Lagrangian model of quarks discussed in ref. 11. There is a triplet t of fermion fields corresponding to three spin 1/2 quarks: the isotopic doublet u and d, with charges 2/3 and $-1/3$ respectively, and the isotopic singlet s, with charge $-1/3$. A neutral vector meson field B_α is introduced, too. The Lagrangian is simply

$$-\bar{t}\,\gamma_\alpha\,\partial_\alpha\,t - \mathscr{L}_B - i\,F\,B_\alpha\,\bar{t}\,\gamma_\alpha\,t$$

QCD Lagrangian

Current algebra: Quarks and what else? #6

Harald Fritzsch (CERN), Murray Gell-Mann (CERN) (1972)

Published in: *eConf* C720906V2 (1972) 135-165 • Contribution to: ICHEP 72 • e-Print: hep-ph/0208010 [hep-ph]

pdf links cite claim reference search 230 citations

Now the interesting question has been raised lately whether we should regard the gluons as well as the quarks as being non–singlets with respect to color[5]. For example, they could form a color octet of neutral vector fields obeying the Yang–Mills equations. (We

Advantages of the Color Octet Gluon Picture #4

H. Fritzsch (Caltech), Murray Gell-Mann (Caltech), H. Leutwyler (Caltech) (1973)

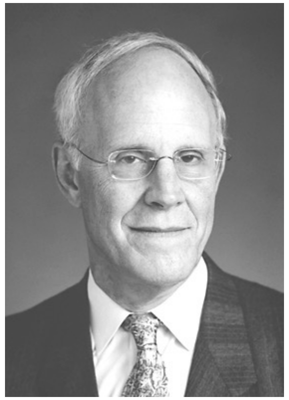
Published in: *Phys.Lett.B* 47 (1973) 365-368

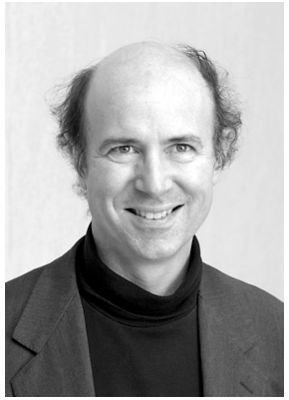
pdf links DOI cite claim reference search 2,564 citations

Last year was the 50th anniversary of QCD

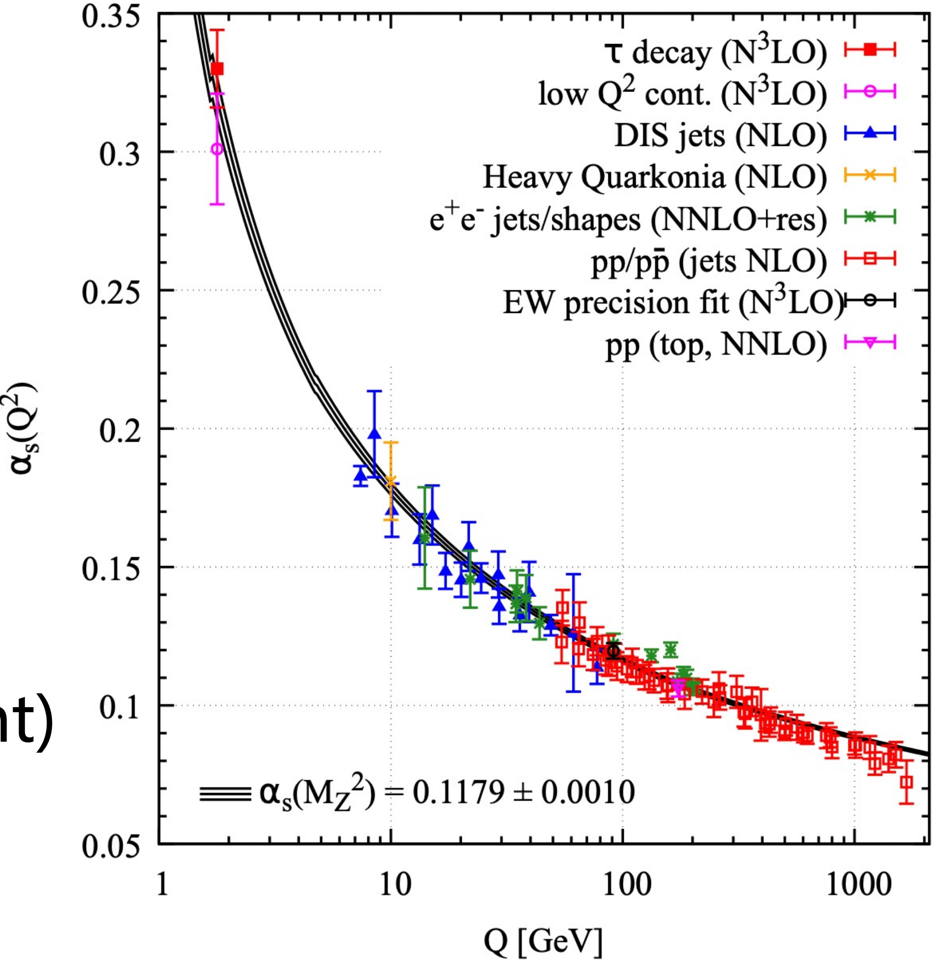
QCD makes a triumphant return

In 1973, Gross, Politzer, and Wilczek discovered that non-Abelian gauge theory exhibits asymptotic freedom





Gross (left), Politzer (Middle), Wilczek (right) awarded Nobel Prize in Physics in 2004

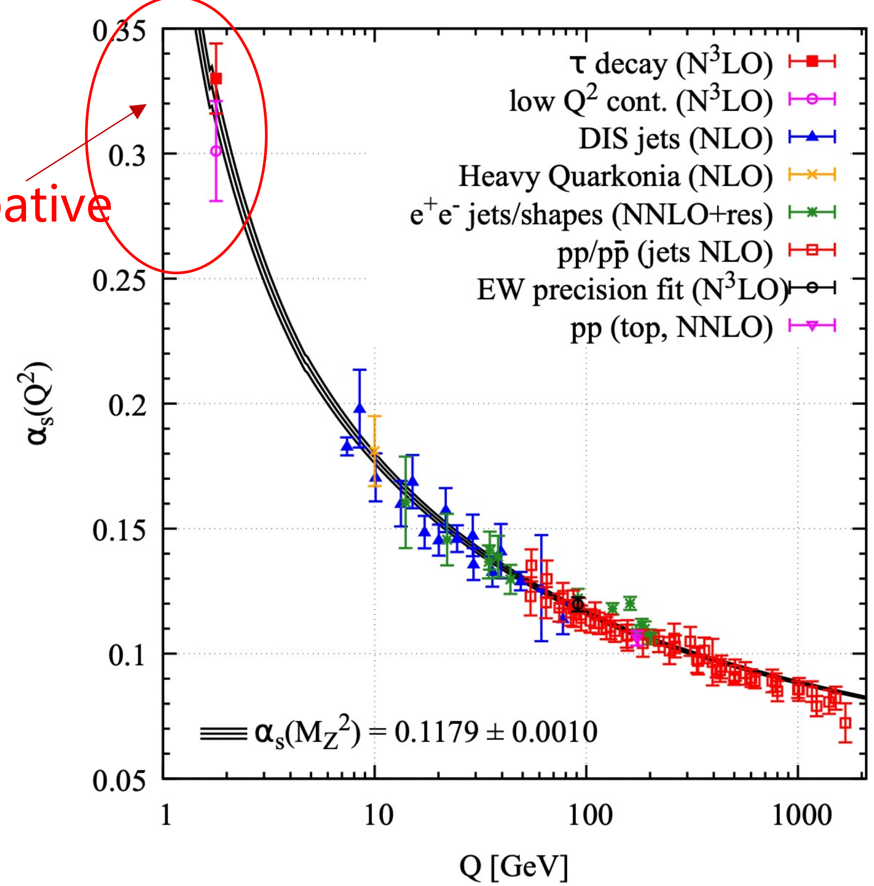


Establishment of Lattice QCD

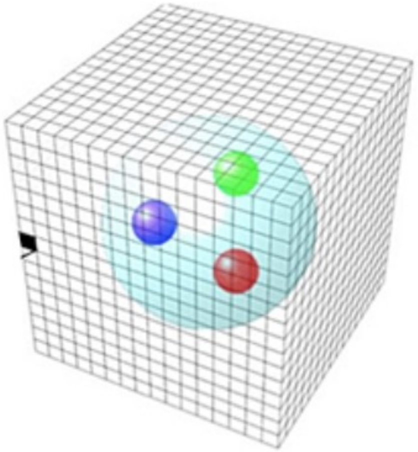
In 1974, K. G. Wilson established lattice gauge theory to study quark confinement

- ➢ Allow precise calculations in the non-perturbative regime from first principles

non-perturbative regime



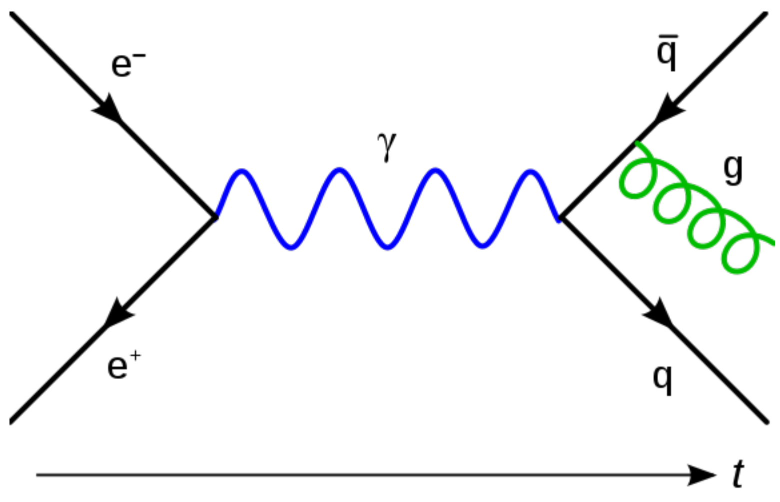
K. G. Wilson

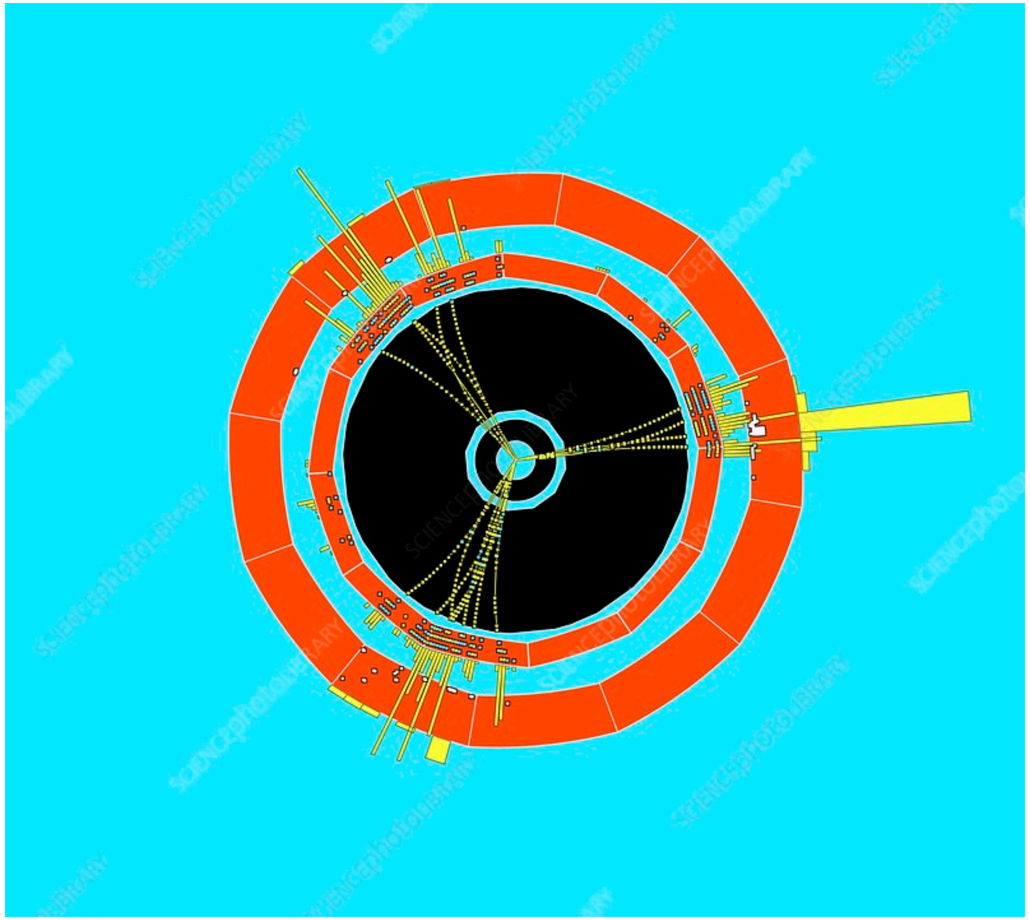


Lattice QCD

Supercomputers enable precise calculations

Three-jet events - "seeing" gluons





In 1979, three-jet events were discovered at the PETRA electron-positron collider at DESY

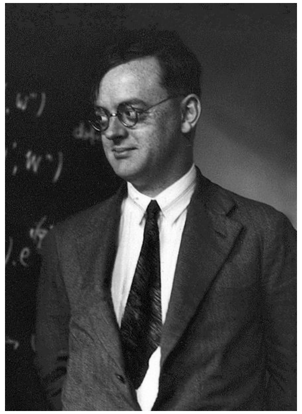
Running coupling constant

Fields and harmonic oscillators

- Using harmonic oscillators to describe fields can be traced back to 1925-1926

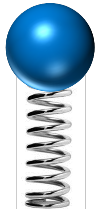

M. Born


W. Heisenberg


P. Jordan

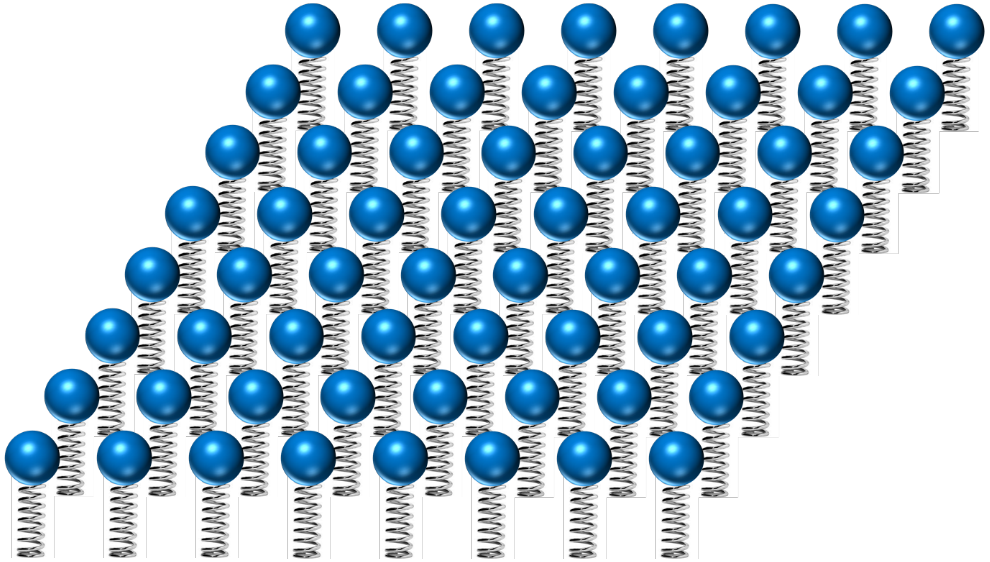
Using quantum harmonic oscillators to handle the canonical quantization of E&M field

- Considering a harmonic oscillator that moves only in the vertical direction



$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

- Field with infinite degrees of freedom → Harmonic oscillator at every point in space



Each ball's height q_i is represented by a single number → scalar field

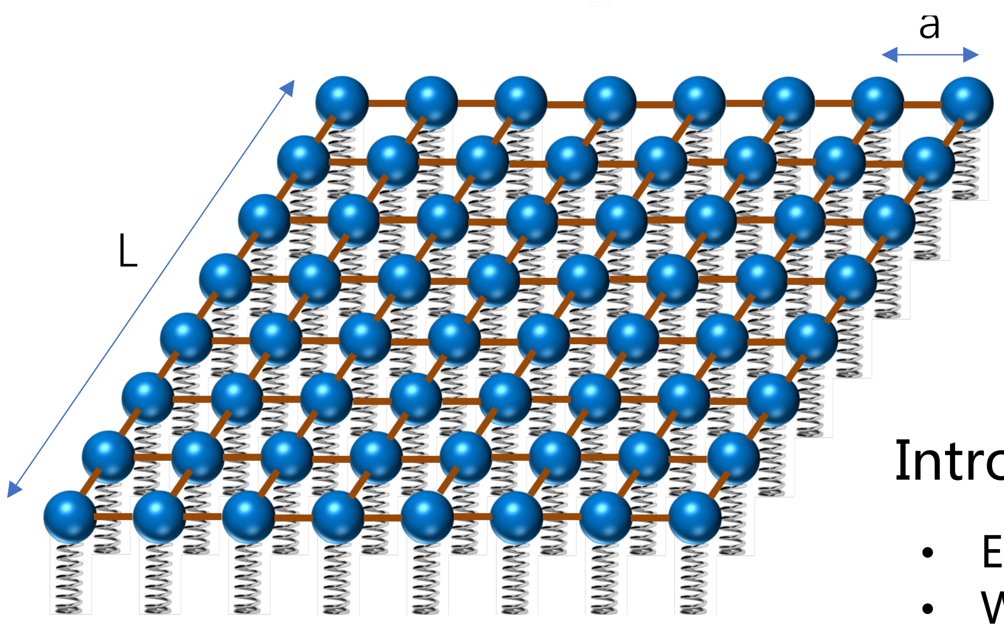
$$L = \frac{1}{2}\sum_i m\dot{q}_i^2 - \frac{1}{2}\sum_i kq_i^2$$

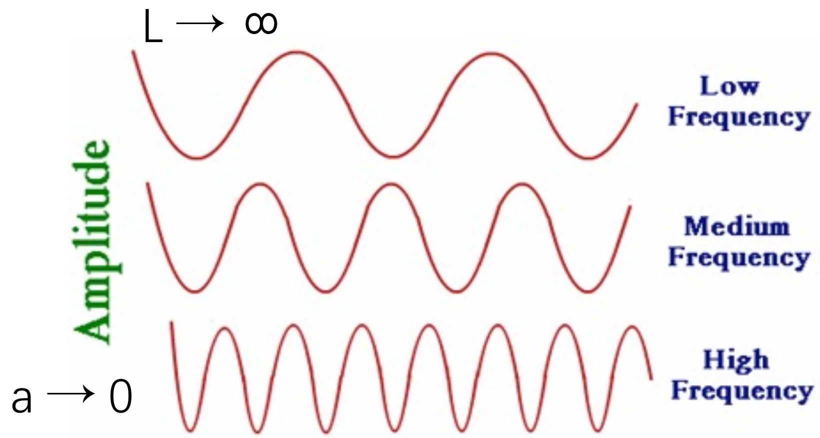
Fields and harmonic oscillators

- Considering that oscillators are not independent of each other — mattress configuration

Introducing interactions between oscillators

$$V(q_1, q_2, \cdots, q_N) = \frac{1}{2}\sum_{ij} k_{ij}(q_i - q_j)^2 + \cdots$$



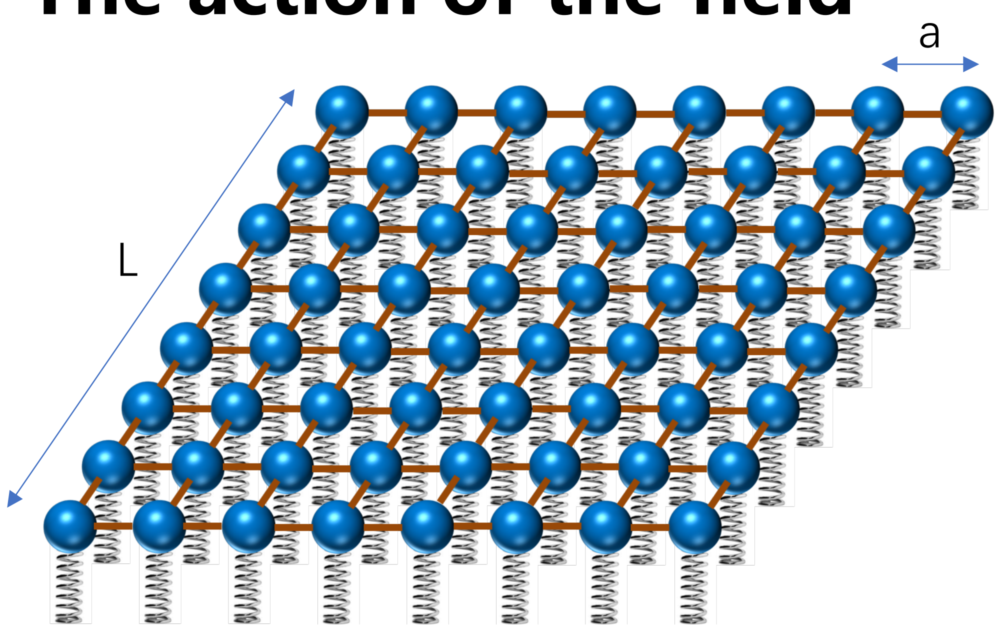


Giving the field a disturbance
→ Pressing down on the mattress releases a wave
→ When quantized, this is understood as a particle propagating

Introducing higher-order terms of q_i

- Eigenmodes can couple; a wavepacket can potentially split into two
- Wave packets meeting can scatter, potentially creating additional wave packets
- Consistent with particle creation and annihilation behavior

The action of the field



$$V(q_1, q_2, \cdots, q_N) = \frac{1}{2}\sum_{ij} k_{ij}(q_i - q_j)^2 + \cdots$$

- Assuming interest only in phenomena where the scale is much larger than the lattice spacing a

Take continuum limit: a → 0

$$i \to \vec{x}$$

$$q_i(t) \to q(t, \vec{x}) \to \phi(t, \vec{x}) \to \phi(x)$$

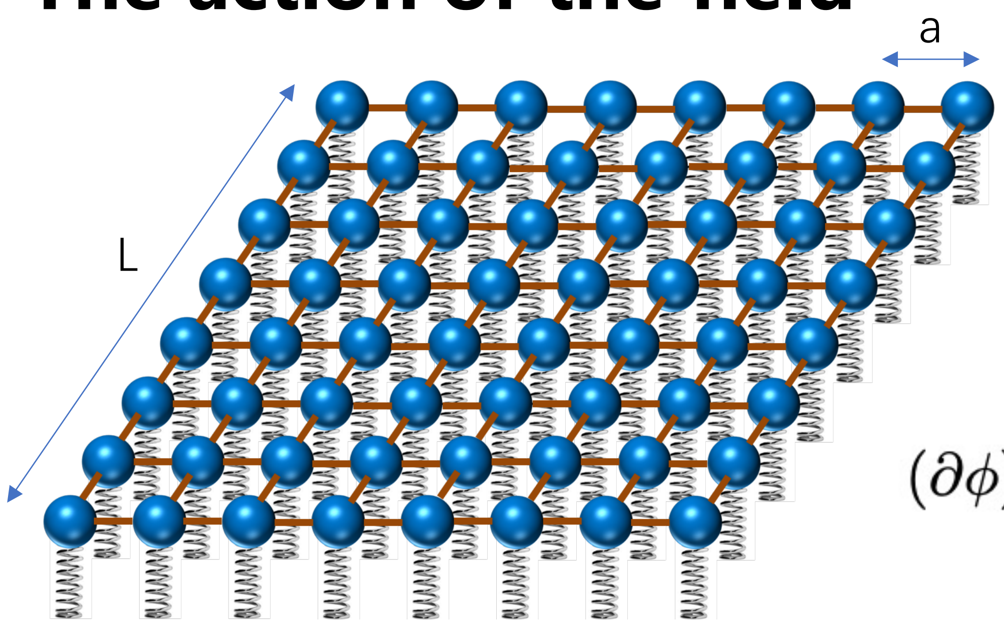
- Kinetic term $\sum_i \frac{1}{2} m\dot{q}_i^2$
- Potential term $V(q_1, q_2, \cdots, q_N) = \frac{1}{2}\sum_{ij} k_{ij}(q_i - q_j)^2 + \cdots$

$$\sum_i \to \frac{1}{a^2}\int d^2\vec{x}$$

$$\sum_i \frac{1}{2} m\dot{q}_i^2 \to \int d^2\vec{x}\, \frac{1}{2}\sigma \left(\frac{\partial \phi}{\partial t}\right)^2$$

$$(q_i - q_j)^2 = a^2 \left(\frac{\partial \phi}{\partial x}\right)^2 + \cdots$$

The action of the field



Consider the symmetry

$$S = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \cdots\right]$$

$$(\partial\phi)^2 = \partial_\mu\phi\,\partial^\mu\phi = \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2$$

$$i \to \vec{x}$$

$$q_i(t) \to q(t,\vec{x}) \to \phi(t,\vec{x}) \to \phi(x)$$

Quantum mechanics is equivalent to (0+1)-dimensional quantum field theory

The simplest scalar field

- Free scalar field (very similar to a harmonic oscillator)

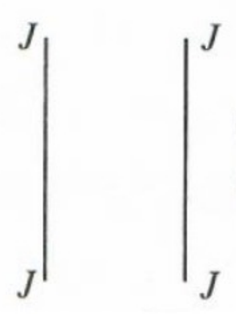
$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

- Introducing interaction terms

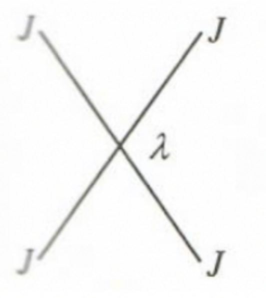
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad \rightarrow \quad \phi^4 \text{ Theory: } \mathcal{L}_1(\phi) = -\frac{\lambda}{4!}\phi^4$$

- 2→2 scalar particle scattering

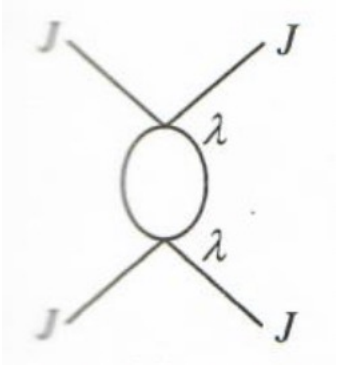
$O(\lambda^0)$



$O(\lambda^1)$

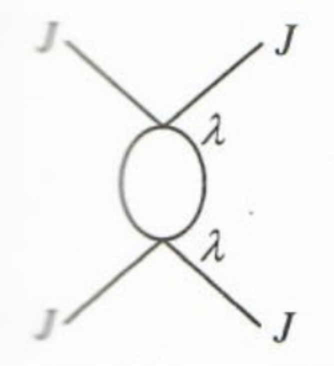


$O(\lambda^2)$



Ultraviolet divergence and regularization

O(λ^2)



➢ The origin of divergence

2→2 scalar particle scattering, O(λ^2) amplitude

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P-k)^2 - m^2 + i\epsilon}$$

Logarithmic divergent! Divergence arises from high-momentum region, named UV divergence

➢ The nature of divergence

- Fourier transform tells us that information in the high-energy momentum range corresponds to short-range, small-scale regions in position space
- Heisenberg Uncertainty Principle tells us that extremely high energies correspond to very small resolutions. Low-energy, long waves typically pass around objects without directly probing their microscopic structures.
- Difficulty of UV divergence actually arises because we assume that QFT system is continuous in spacetime. When we switch to momentum-energy space, momentum and energy can tend towards infinity.

Is the world truly infinitely divisible?

- Ancient Eastern civilizations

A rod one foot long, cut in half each day, will never be exhausted even after ten thousand generations

- Ancient Western civilizations

Democritus' "Atomism":
Atom is derived from the Greek word "Atomos", meaning uncuttable

- The perspective of modern experiments:

- Microscopes have a certain resolution; it is impossible to create a microscope that can resolve infinitely small scales
- Colliders also have an upper energy limit; it is impossible to build a collider with infinitely high energy.

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Atom is derived from the Greek word "Atomos," meaning uncuttable

- We must acknowledge that we know little about the ultraviolet region

- Assuming that unknown regions are described by continuous spacetime quantum field theory is not necessarily correct
- Our current theory should be an effective theory only in the low-energy, experimentally detectable regime

How to deal with the ultraviolet region?

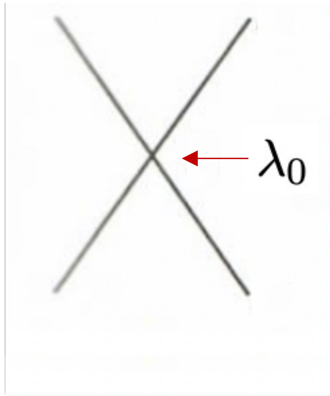
- Introducing UV cutoff Λ, remove our ignorance
  - Momentum cutoff: ensuring that all momentum values are smaller than a certain value
  - Lattice regularization: introducing a non-zero lattice spacing to discretize spacetime for the fields
  - Pauli-Villars method: $\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\epsilon)^2} - \frac{1}{(k^2-\Lambda^2+i\epsilon)^2}$
  - Dimensional regularization: $\int d^D k \sim \int k^{D-1} dk$

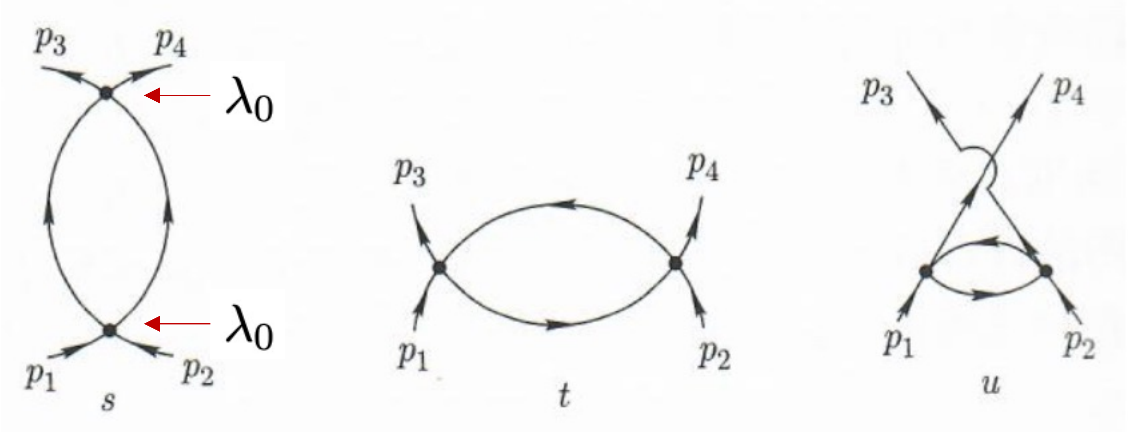
    If $D=4$ integral is divergent , then let $D=4-\epsilon$, the integral become convergent

    Explanation: $\int d^D k \sim \sum_{\vec{k}}$ As the dimension increases, more high-momentum modes accumulate; $D=4 \to 4-\epsilon$ suppresses contributions from high momentum
- Once regularization is introduced, the theory becomes an effective theory
  - Physics above the cutoff energy scale cannot be directly described by the existing theory
  - But UV region should have an impact on our physical observations, how is it manifested?

$\Longrightarrow$
- In an effective theory, all parameters (including coupling constants, masses, etc.) should depend on the cutoff energy scale
- The contributions cut off from the ultraviolet region should be compensated for by bare parameters

Returning to scalar particle 2→2 scattering





- Introducing kinematic variables (Mandelstam variables) based on different diagrams

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

s, t, u are functions of the scattering center-of-mass energy and scattering angles

- Introducing Pauli-Villars cutoff, the loop integrals are rendered finite

$$\mathcal{M} = -i\lambda_0 - iC\lambda_0^2 \left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u} \right] + \mathcal{O}(\lambda_0^3)$$

Obviously, λ_0 depends on the cutoff energy scale

Question: What is the use of this scattering amplitude expression?

Theory meets experiments

- Find an experimental physicist to conduct 2→2 scattering experiments
  - In experiments, specific scattering energies and scattering angles are chosen, corresponding to Mandelstam variable
    $$\{s, t, u\} = \{s_0, t_0, u_0\}$$
  - Experimentally measured differential scattering cross-sections can be used to infer the scattering amplitude $\mathcal{M}_{\text{exp}}$
  - It is equivalent to experimentally measuring the physical coupling constants for the 2→2 process $\mathcal{M}_{\text{exp}} = -i\lambda_P$

- Theoretical and experimental agreement yields

$$-i\lambda_P = -i\lambda_0 - iC\lambda_0^2 \left[\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{t_0} + \ln\frac{\Lambda^2}{u_0} \right] + \mathcal{O}(\lambda_0^3)$$

- λ_0 is the bare, unphysical quantity; expressing the scattering amplitude in terms of physical coupling constants would be perfect!

Theory meets experiments

- Expressing the scattering amplitude in terms of physical coupling constants

$$-i\lambda_P = -i\lambda_0 - iC\lambda_0^2\left[\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{t_0} + \ln\frac{\Lambda^2}{u_0}\right] + \mathcal{O}(\lambda_0^3)$$

$$\Downarrow$$

$$\lambda_0 = \lambda_P - C\lambda_0^2\left[\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{t_0} + \ln\frac{\Lambda^2}{u_0}\right] + \mathcal{O}(\lambda_0^3) = \lambda_P - C\lambda_P^2\left[\ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{t_0} + \ln\frac{\Lambda^2}{u_0}\right] + \mathcal{O}(\lambda_P^3)$$

Substitute the expression for the scattering amplitude

$$\begin{aligned}\mathcal{M} &= -i\lambda_0 - iC\lambda_0^2\left[\ln\frac{\Lambda^2}{s} + \ln\frac{\Lambda^2}{t} + \ln\frac{\Lambda^2}{u}\right] + \mathcal{O}(\lambda_0^3) \\ &= -i\lambda_P - iC\lambda_P^2\left[\ln\frac{s_0}{s} + \ln\frac{t_0}{t} + \ln\frac{u_0}{u}\right] + \mathcal{O}(\lambda_P^3)\end{aligned}$$

- The dependence of physical quantities $\mathcal{M}$ on the cutoff Λ disappears
- Physical coupling constants λ_P depend on energy scale s_0, t_0, u_0 .
  For simplicity, let $s_0, t_0, u_0 = \mu^2$

$$\mu\frac{d\lambda_P}{d\mu} = -6C\lambda_P^2 + \mathcal{O}(\lambda_P^3)$$

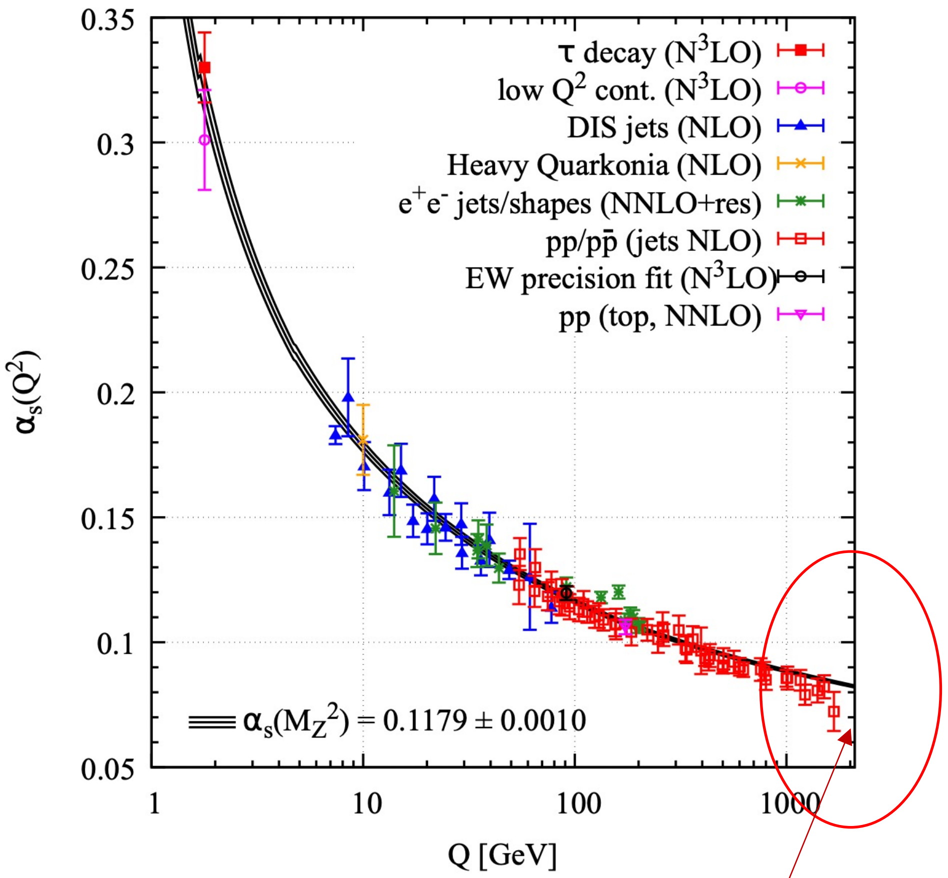
QCD coupling constant running

Near α_s=0 ultraviolet fixed point

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2_{\rm QCD}}$$

$$\beta_0 = 11 - \frac{2N_f}{3} > 0$$

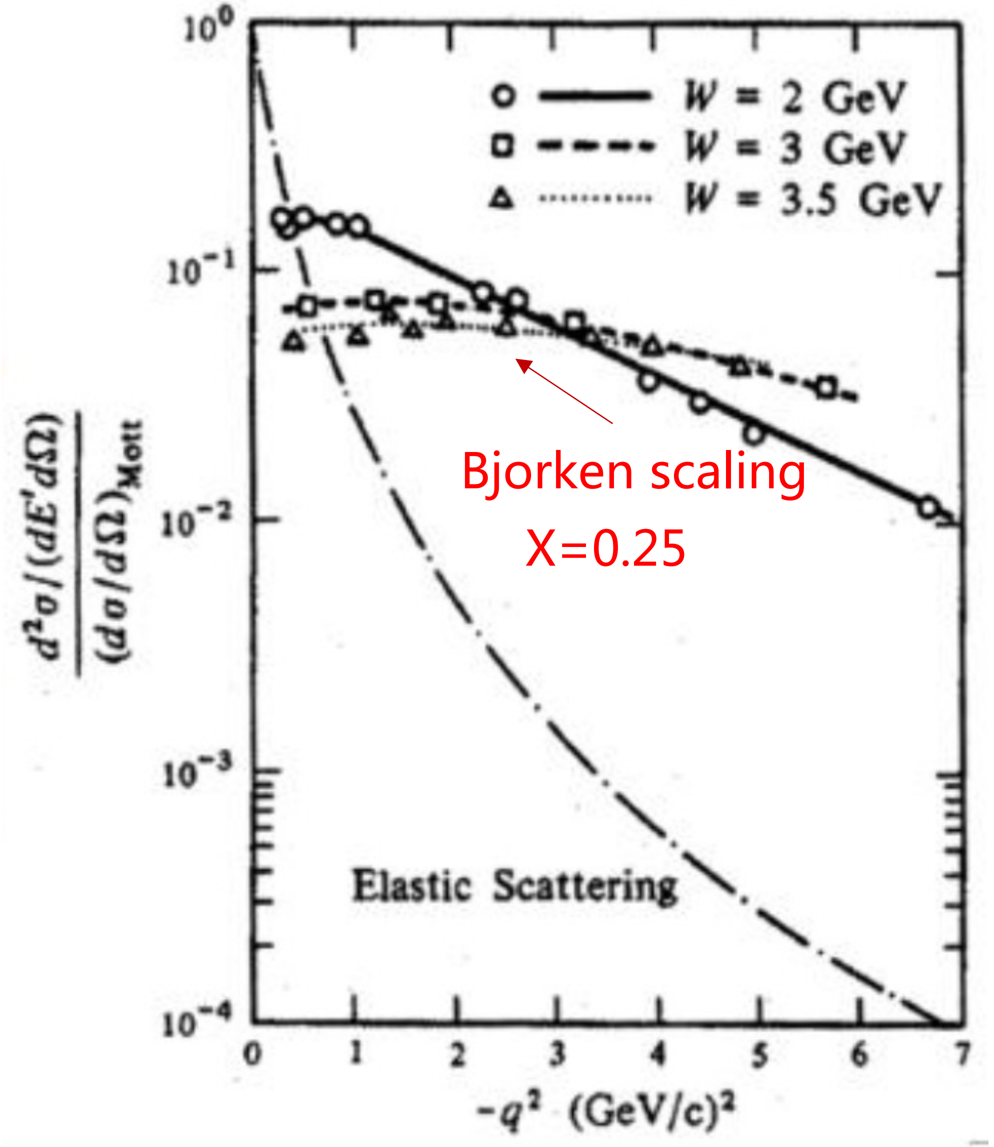
$\Lambda_{\rm QCD} \approx 300\ {\rm MeV}$ is the non-perturbative hadronic scale

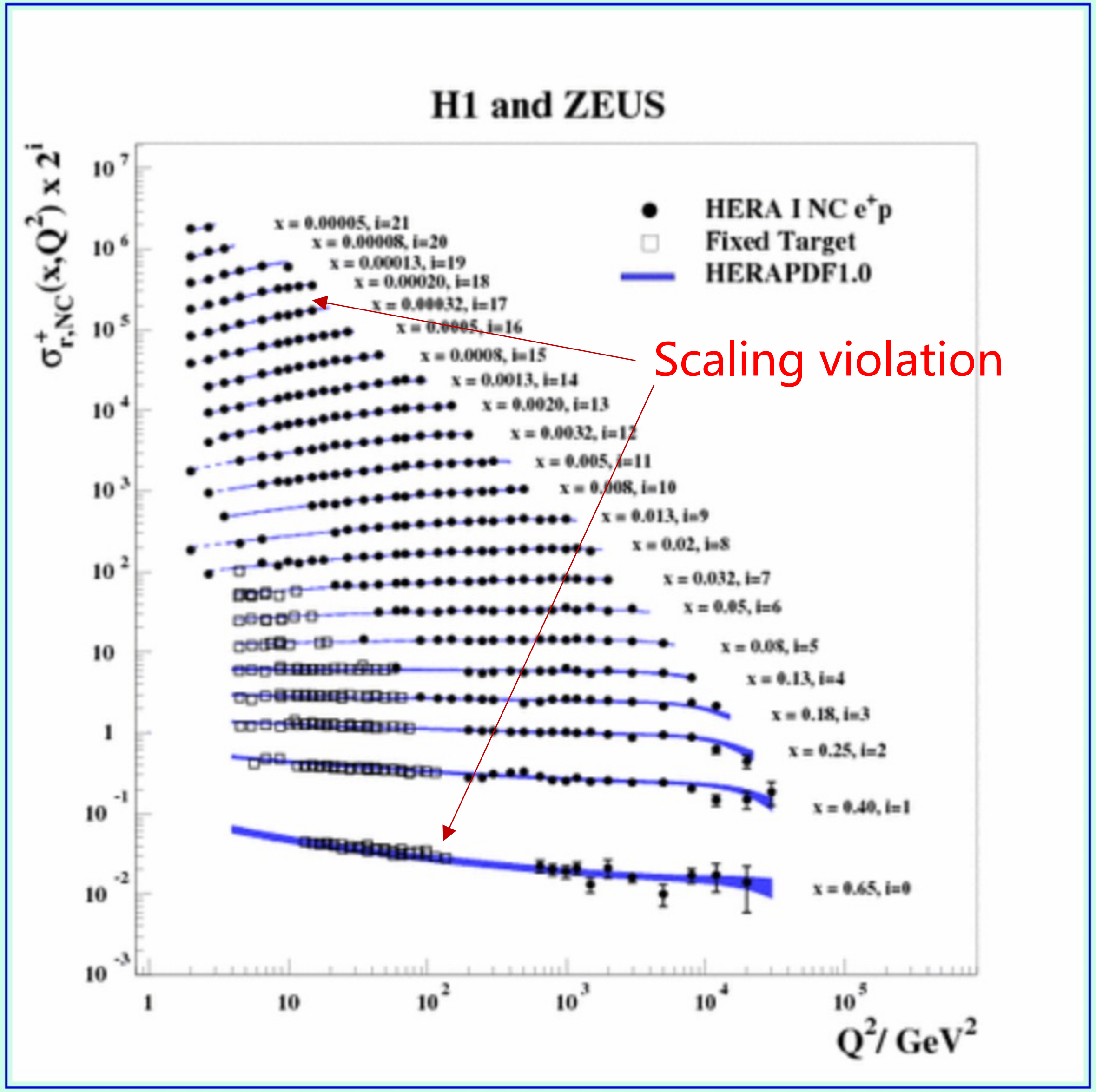


Ultraviolet regime

Success of pQCD

Perturbative tests of QCD



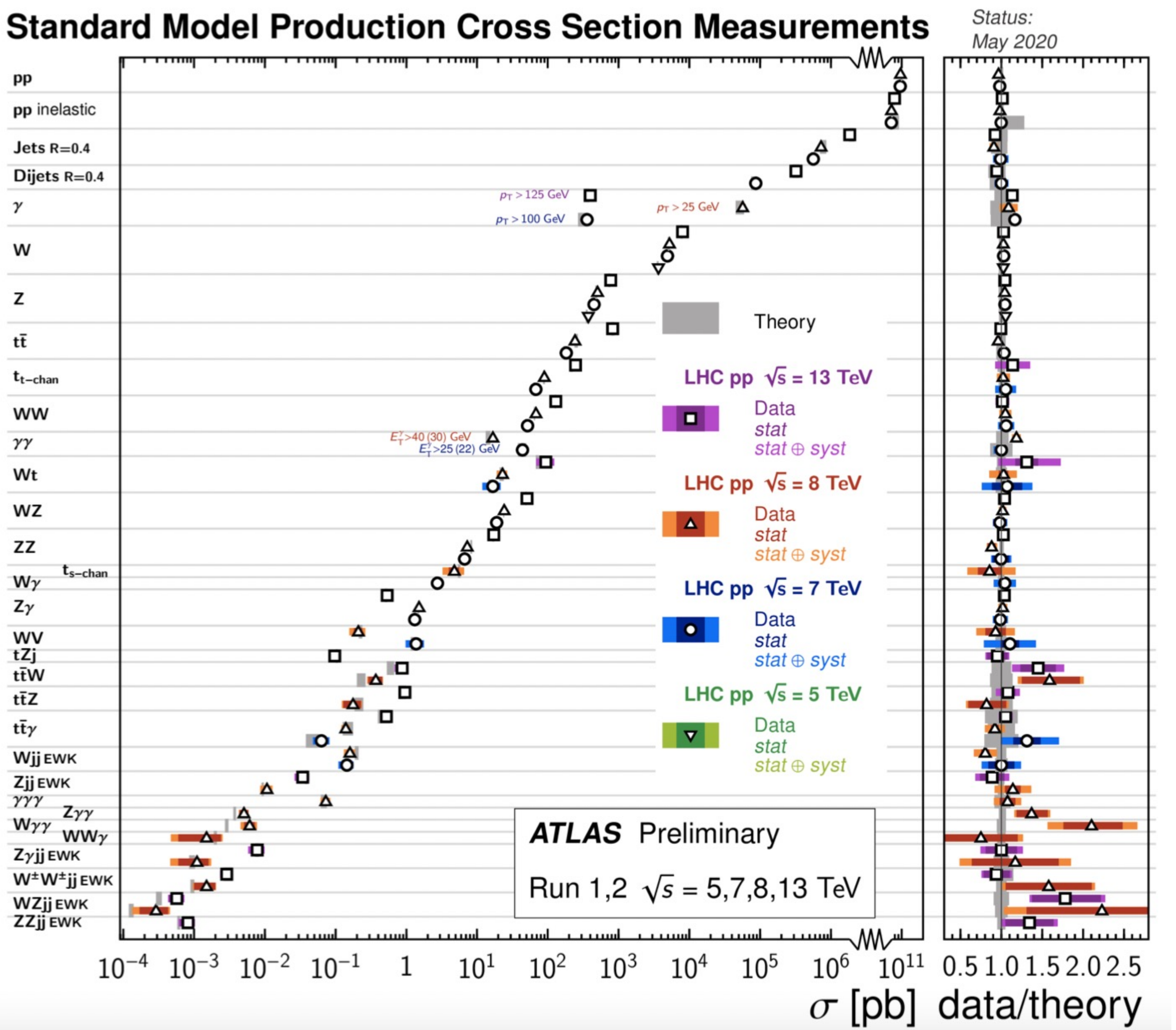


HERA combined NC e+p reduced cross section and fixed-target data as a function of Q2

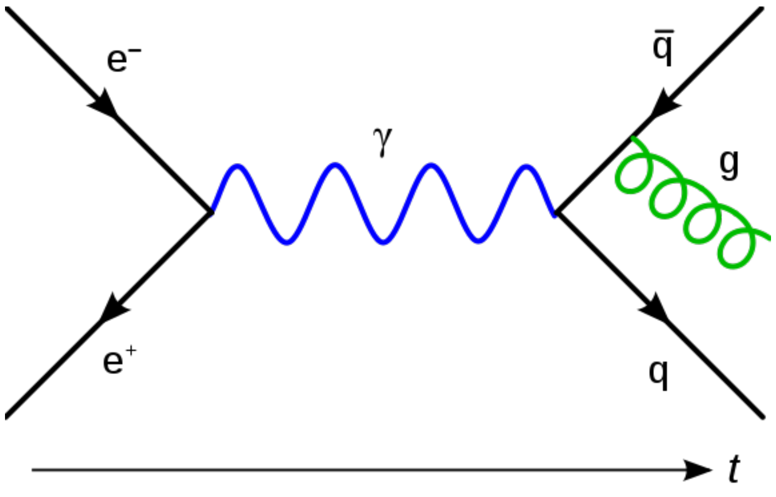
Comparison of Standard Model with experiments at the LHC

very abundant processes

very rare processes



No strong interaction process is perturbative throughout



- Gluons can be produced in high-energy processes through quark radiation
- Production happens at very short distance; no detector can detect gluons at such distance
- Experiment observed three-jets rather than three gluons

- Gluons undergo hadronization, where non-perturbative processes come into play

Why we need non-perturbative QCD

- In many cases, pQCD and LQCD work together
  - pQCD used for short distance: Wilson coefficients
  - LQCD used for long distance: hadronic matrix elements

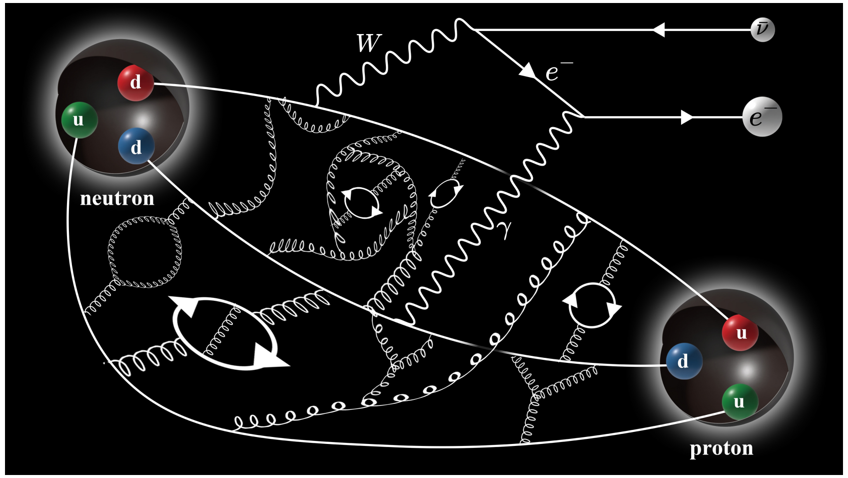
- The role played by lattice QCD is irreplaceable
  - High-order pQCD calculation is challenging: QED up to 5 loop (e.g. g-2); QCD up to NNNLO
  - More is different——P. W. Anderson

    Perturbative and nonperturbative regimes are intrinsically different

    For example, QCD vacuum is nonperturbative and has chiral symmetry spontaneously breaking

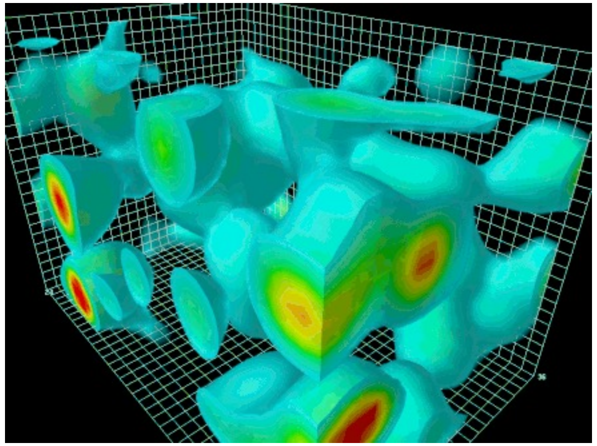
  - Lattice QCD simulates QCD vacuum structure

    ——nontrivial topological charge density fluctuation



$$\underbrace{0+0+0+0+\cdots+0+0+0}\neq 0$$

An infinite sum of zeros can be nonzero



Lattice QCD

Starting from Euclidean spacetime, continuous field theory

- Quark and gluon field

flavor

Quark field $\psi^{(f)}(x)_{\substack{\alpha\\c}}$, $\overline{\psi}^{(f)}(x)_{\substack{\alpha\\c}}$ spin

color

Gluon field

$A_\mu(x)_{cd}$ Hermitian, traceless

- Action for quark field

Defined in Euclidean spacetime

$$S_F[\psi,\overline{\psi},A] = \sum_{f=1}^{N_f} \int \mathrm{d}^4x\, \overline{\psi}^{(f)}(x) \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A_\mu(x)\right) + m^{(f)}\right) \psi^{(f)}(x)$$

- Gauge transformation $\psi(x) \to \psi'(x) = \Omega(x)\psi(x)$, $\overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\Omega(x)^\dagger$

here $\Omega(x)$ is a SU(3) matrix

Under a gauge transformation, the action must remain invariant, so the gluon field must follow the certain transformation $A_\mu(x) \to A'_\mu(x)$

Starting from Euclidean spacetime, continuous field theory

- Under a gauge transformation, the action becomes

$$S_F[\psi', \overline{\psi}', A'] = \int \mathrm{d}^4 x\, \overline{\psi}(x) \Omega(x)^\dagger \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A'_\mu(x)\right) + m\right) \Omega(x) \psi(x)$$

- The action is gauge invariant, requiring

$$\begin{aligned} \partial_\mu + \mathrm{i}\, A_\mu(x) &= \Omega(x)^\dagger \left(\partial_\mu + \mathrm{i}\, A'_\mu(x)\right) \Omega(x) \\ &= \partial_\mu + \Omega(x)^\dagger \left(\partial_\mu \Omega(x)\right) + \mathrm{i} \Omega(x)^\dagger A'_\mu(x) \Omega(x) \end{aligned}$$

$$\Longrightarrow \quad A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega(x)^\dagger + \mathrm{i} \left(\partial_\mu \Omega(x)\right) \Omega(x)^\dagger$$

- Define covariant derivative $D_\mu(x) = \partial_\mu + \mathrm{i}\, A_\mu(x)$

  satisfying transformation $D_\mu(x) \to D'_\mu(x) = \partial_\mu + \mathrm{i}\, A'_\mu(x) = \Omega(x) D_\mu(x) \Omega(x)^\dagger$

Starting from Euclidean spacetime, continuous field theory

- Benefits of defining a covariant derivative

$$F_{\mu\nu}(x) = -\mathrm{i}[D_\mu(x), D_\nu(x)] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \mathrm{i}[A_\mu(x), A_\nu(x)]$$

satisfying transformation $F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu}(x) \Omega(x)^\dagger$

- Therefore, one defines the action of gauge fields

$$S_G[A] = \frac{1}{2\,g^2} \int \mathrm{d}^4x \ \mathrm{tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right]$$

If letting $\frac{1}{g} A_\mu(x) \ \to \ A_\mu(x)$

then the coefficient $1/g^2$ disappears, and the covariant derivative will change to $D_\mu(x) \ \to \ \partial_\mu + \mathrm{i}\, g A_\mu(x)$

Starting from Euclidean spacetime, continuous field theory

$$S_G[A] = \frac{1}{2\,g^2}\int \mathrm{d}^4x\ \mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right]$$

- If we expand $A_\mu(x)$ in terms of generators

$$A_\mu(x) = \sum_{i=1}^{8} A_\mu^{(i)}(x)\,T_i \quad \longrightarrow \quad F_{\mu\nu}(x) = \sum_{i=1}^{8} F_{\mu\nu}^{(i)}(x)\,T_i\ ,$$

$$F_{\mu\nu}^{(i)}(x) = \partial_\mu A_\nu^{(i)}(x) - \partial_\nu A_\mu^{(i)}(x) - f_{ijk}A_\mu^{(j)}(x)A_\nu^{(k)}(x)$$

- Finally, the gauge field action can be expressed as

$$S_G[A] = \frac{1}{4\,g^2}\sum_{i=1}^{8}\int \mathrm{d}^4x\, F_{\mu\nu}^{(i)}(x)F_{\mu\nu}^{(i)}(x)$$

In Euclidean spacetime, the action is positive definite!

Discretization of spacetime

$$\Lambda = \{n = (n_1, n_2, n_3, n_4) \mid n_1, n_2, n_3 = 0, 1, \ldots, N-1 \,;\, n_4 = 0, 1, \ldots, N_T - 1\}$$

$$\psi(n)\,,\, \overline{\psi}(n)\,,\quad n \in \Lambda \quad \Rightarrow \quad \partial_\mu \psi(x) \;\rightarrow\; \frac{1}{2a}\left(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})\right)$$

- The action for free fermion

$$S_F^0[\psi, \overline{\psi}] = a^4 \sum_{n \in \Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \, \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} \;+\; m\,\psi(n) \right)$$

$\overline{\psi}(n)\psi(n+\hat{\mu})$ is not gauge invariant

- Introducing gauge fields using gauge transformations $U_\mu(n)$

$$\psi(n) \rightarrow \psi'(n) = \Omega(n)\,\psi(n)\,,\; \overline{\psi}(n) \rightarrow \overline{\psi}'(n) = \overline{\psi}(n)\,\Omega(n)^\dagger$$

$$\overline{\psi}'(n)\,U'_\mu(n)\,\psi'(n+\hat{\mu}) \;=\; \overline{\psi}(n)\,\Omega(n)^\dagger\,U'_\mu(n)\,\Omega(n+\hat{\mu})\,\psi(n+\hat{\mu})$$

The combination can be gauge invariant as long as the gauge field satisfies the transformations

$$U_\mu(n) \;\rightarrow\; U'_\mu(n) = \Omega(n)\,U_\mu(n)\,\Omega(n+\hat{\mu})^\dagger$$

Discretization of spacetime

- The action of fermions coupled to gauge fields

$$S_F[\psi,\overline{\psi},U] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n)\left(\sum_{\mu=1}^{4}\gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} + m\,\psi(n)\right)$$

- The relationship between two gauge fields

$$U_\mu(n) = \exp\left(\mathrm{i}\, a A_\mu(n)\right)$$

group elements Lie algebra

$$U_\mu(n) = \mathbb{1} + \mathrm{i}\, a A_\mu(n) + \mathcal{O}(a^2)$$

$$\Downarrow$$

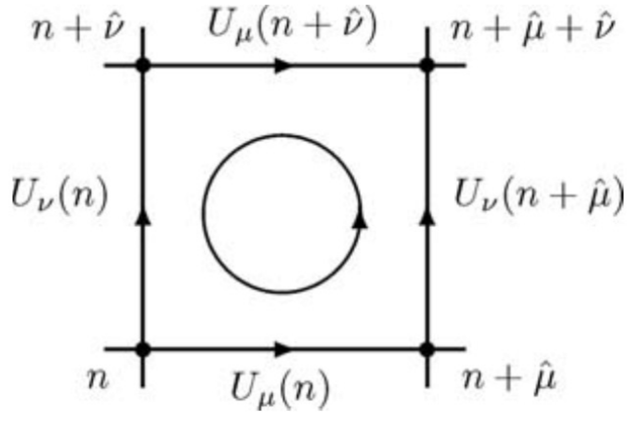
In the continuum limit, this returns to the action of a continuous field theory

- Gauge-invariant quantities

$$\overline{\psi}(n_0)\, P[U]\, \psi(n_1) \quad \text{or} \quad L[U] = \mathrm{tr}\left[\prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\right]$$

Discretization of spacetime

➢ Gauge field action



$$
\begin{aligned}
U_{\mu\nu}(n) &= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_{-\mu}(n+\hat{\mu}+\hat{\nu})\, U_{-\nu}(n+\hat{\nu}) \\
&= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu(n+\hat{\nu})^\dagger\, U_\nu(n)^\dagger\ .
\end{aligned}
$$

$$
S_G[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\ \mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right]
$$

In the continuum limit

$$
S_G[U] = \frac{a^4}{2\,g^2} \sum_{n\in\Lambda} \sum_{\mu,\nu} \mathrm{tr}[F_{\mu\nu}(n)^2] + \mathcal{O}(a^2)
$$

Exercise 1: Demonstrate that the the action has the correct continuum limit

Exercise 2: How to design a gauge-invariant CP-violating term?

How about in discrete spacetime?

Path integral

- Consider a 1d quantum mechanical system $\hat{H} = \hat{H}(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x})$

- A fundamental concept in quantum mechanics is the transition amplitude from an initial state to a final state

$$\langle x_f | e^{-i\hat{H}T} | x_i \rangle$$

- The time evolution of any quantum state can be written as

$$\psi(x,t) = \int dx' \langle x | e^{-i\hat{H}t} | x' \rangle \psi(x', 0)$$

wave function

- How to compute the transition amplitude from an initial state $|x_i\rangle$ at time t_i to the final state $|x_f\rangle$?

- Sum over all possible paths $\{x(t) \quad \text{for} \quad t = t_i \to t_f\}$

- Given initial and final times, fix

$$x(t_i) = x_i, \quad x(t_f) = x_f$$

- If choosing $x_i = x_f \equiv x, \quad t_f - t_i \equiv T$, the amplitude can be written as

$$C(T) = \langle x | e^{-i\hat{H}T} | x \rangle = \sum_n \langle x | E_n \rangle e^{-iE_n T} \langle E_n | x \rangle$$

Path integral

- Discretize the time direction

$$t_j = t_i + ja, \quad \text{for } j = 0, 1, 2, \cdots, N \qquad a = (t_f - t_i)/N$$

- The configuration we desire is given by a $N+1$-dimensional vector

$$x = \{x(t_0), x(t_1), \cdots, x(t_N)\} = \{x_0, x_1, \cdots, x_N\}$$

- Let's first consider a simpler free Hamiltonian $\hat{H}_0 = \dfrac{\hat{p}^2}{2m}$

$$\langle x_N | e^{-i\hat{H}_0(t_f - t_i)} | x_0 \rangle = \int dx_1\, dx_2, \cdots dx_{N-1} \langle x_N | e^{-i\hat{H}_0 a} | x_{N-1} \rangle \langle x_{N-1} | e^{-i\hat{H}_0 a} | x_{N-2} \rangle \cdots \langle x_1 | e^{-i\hat{H}_0 a} | x_0 \rangle$$

Here, the basic unit is $\langle x_{j+1} | e^{-i\hat{H}_0 a} | x_j \rangle = \int dp\, \langle x_{j+1} | e^{-i\hat{H}_0 a} | p \rangle \langle p | x_j \rangle$

Using the definition of plane waves $\langle x | p \rangle = \dfrac{1}{\sqrt{2\pi}} e^{ipx}$

$$\int dp\, \langle x_{j+1} | p \rangle e^{-i\frac{a\hat{p}^2}{2m}} \langle p | x_j \rangle = \int \frac{dp}{2\pi}\, e^{ip(x_{j+1} - x_j)} e^{-i\frac{ap^2}{2m}} = \sqrt{\frac{m}{i2\pi a}} e^{i\frac{m(x_{j+1} - x_j)^2}{2a}}$$

Path integral

$$\langle x_N|e^{-i\hat{H}_0(t_f-t_i)}|x_0\rangle = \left(\frac{m}{i2\pi a}\right)^{\frac{N}{2}} \int dx_1\, dx_2, \cdots\, dx_{N-1}\, e^{i\sum_{j=0}^{N-1}\frac{m(x_{j+1}-x_j)^2}{2a}}$$

Rewrite the result

$$\langle x_f|e^{-i\hat{H}(t_f-t_i)}|x_i\rangle = \int \mathcal{D}[x(t)]\, e^{iS[x]}$$

$$S = \sum_{j=0}^{N-1}\frac{m(x_{j+1}-x_j)^2}{2a} \xrightarrow{a\to 0} S = \int dt\, \frac{m\dot{x}^2}{2}$$

$$\int \mathcal{D}[x(t)] \to C_N \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_{N-1}$$

- A Hamiltonian with interaction $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

Using Zassenhaus formula $e^{\lambda(\hat{A}+\hat{B})} = e^{\lambda\hat{A}}e^{\lambda\hat{B}}(1+\mathcal{O}(\lambda^2))$

$$e^{-i\hat{H}a} = e^{-i\frac{V(\hat{x})}{2}a}e^{-i\frac{\hat{p}^2}{2m}a}e^{-i\frac{V(\hat{x})}{2}a}(1+\mathcal{O}(a^2))$$

Path integral

$$\langle x_{j+1}|e^{-i\hat{H}a}|x_j\rangle = e^{-i\frac{V(x_{j+1})}{2}a}\langle x_{j+1}|e^{-i\hat{H}_0 a}|x_j\rangle e^{-i\frac{V(x_j)}{2}a} \approx \langle x_{j+1}|e^{-i\hat{H}_0 a}|x_j\rangle e^{-iV\left(\frac{x_{j+1}+x_j}{2}\right)a}$$

- $t_i \to t_f$ transition matrix element

$$\langle x_N|e^{-i\hat{H}(t_f-t_i)}|x_0\rangle \approx \left(\frac{m}{i2\pi a}\right)^{\frac{N}{2}} \int dx_1\, dx_2, \cdots dx_{N-1}\, e^{i\sum_{j=0}^{N-1}\left[\frac{m(x_{j+1}-x_j)^2}{2a} - V\left(\frac{x_{j+1}+x_j}{2}\right)a\right]}$$

- The transition amplitude in discretized time

$$\langle x|e^{-i\hat{H}T}|x\rangle \approx C_N \int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1} e^{iS[x]}$$

- The action on the lattice is given by

$$S[x] \equiv \sum_{j=0}^{N-1}\left[\frac{m}{2a}(x_{j+1}-x_j)^2 - V\left(\frac{x_{j+1}+x_j}{2}\right)a\right]$$

Path integral

➢ Path integral in Euclidean spacetime

Let us verify $t_M \to -it_E, \quad iS_M \to -S_E$

- Under Wick rotation, transition matrix changes to $\langle x_N | e^{-\hat{H}T_E} | x_i \rangle$

The key point here is that the Hamiltonian remains unchanged

- Lattice discretization

$$\langle x_N | e^{-\hat{H}T} | x_0 \rangle \quad \to \quad \langle x_{j+1} | e^{-\hat{H}a} | x_j \rangle \quad \to \quad e^{-V(\frac{x_{j+1}+x_j}{2})a} \langle x_{j+1} | e^{-\frac{\hat{p}^2}{2m}a} | x_j \rangle$$

$$\langle x_{j+1} | e^{-\frac{\hat{p}^2}{2m}a} | x_j \rangle = \int \frac{dp}{2\pi} e^{-\frac{p^2}{2m}a} e^{ip(x_{j+1}-x_j)} = \sqrt{\frac{m}{2\pi a}} e^{-\frac{m(x_{j+1}-x_j)^2}{2a}}$$

- Transition amplitude

$$\langle x | e^{-\hat{H}T} | x \rangle \approx C_N \int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1} e^{-S_E[x]}$$

- Lattice action

$$S_E[x] \equiv \sum_{j=0}^{N-1} \left[\frac{m}{2a}(x_{j+1}-x_j)^2 + V\left(\frac{x_{j+1}+x_j}{2}\right) a \right]$$

Ensure positivity!

Path integral

$$\langle x|e^{-HT}|x\rangle = \sum_n \langle x|E_n\rangle e^{-E_nT}\langle E_n|x\rangle$$

T is large, only the ground state energy contributes

$$\langle x|e^{-HT}|x\rangle \xrightarrow{T\to\infty} e^{-E_0T}|\langle x|E_0\rangle|^2$$

wave function for ground state

eigen-energy for ground state

Exercise:

Take a=1/2, N=8. V(x) takes the form of harmonics oscillator V(x)=x^2/2. The mass is m=1. Please compute the amplitude at several different positions x using a 7-dimensional integral, for instance, from x=0 to 2, and then verify

$$\langle x|e^{-\hat{H}T}|x\rangle \approx |\langle x|E_0\rangle|^2 e^{-E_0T}$$

where $E_0 = 1/2$, $\langle x|E_0\rangle = \frac{e^{-x^2/2}}{\pi^{1/4}}$

Path integral

- The transition amplitude can be written as $\langle x|e^{-\hat{H}T}|x\rangle$
- If perform integral $\int dx\,\langle x|e^{-\hat{H}T}|x\rangle \Rightarrow \mathrm{Tr}\left[e^{-\hat{H}T}\right]$

  Incorporate $x_i = x_f \equiv x$ into path integral

  T is very large

$$\mathrm{Tr}\left[e^{-\hat{H}T}\right] = \sum_n \langle n|e^{-\hat{H}T}|n\rangle = \sum_n e^{-E_nT} \xrightarrow{T\to\infty} e^{-E_0T}$$

- If using path integrals to compute correlation functions

$$\langle \hat{x}(t_2)\hat{x}(t_1)\rangle \equiv \frac{\int \mathcal{D}x\, x(t_2)x(t_1)\, e^{-S[x]}}{\int \mathcal{D}x\, e^{-S[x]}}$$

  is equivalent to computing

$$\int dx\,\langle x|e^{-\hat{H}(t_f-t_2)}\hat{x}(0)e^{-\hat{H}(t_2-t_1)}\hat{x}e^{-\hat{H}(t_1-t_i)}|x\rangle$$

$$\langle \hat{x}(t_2)\hat{x}(t_1)\rangle = \frac{\sum e^{-E_nT}\langle E_n|\hat{x}(0)e^{-(\hat{H}-E_n)t}\hat{x}|E_n\rangle}{\sum e^{-E_nT}} = \langle 0|\hat{x}(0)e^{-(\hat{H}-E_0)t}\hat{x}(0)|0\rangle$$

A pure gauge field system

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U] \, \mathrm{e}^{-S_G[U]} \, O[U]$$

- Partition function $$Z = \int \mathcal{D}[U] \, \mathrm{e}^{-S_G[U]}$$
- Measure of path integral $$\int \mathcal{D}[U] = \prod_{n \in \Lambda} \prod_{\mu=1}^{4} \int \mathrm{d}U_\mu(n)$$
- Action $$S_G[U] = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathrm{Re} \; \mathrm{tr} \left[\mathbb{1} - U_{\mu\nu}(n) \right]$$

  $\beta = \dfrac{6}{g^2}$ called inverse coupling
- Integral is invariant under variable transformations; similarly, path integrals are invariant under gauge transformations

$$Z = \int \mathcal{D}[U] \, \mathrm{e}^{-S_G[U]} = \int \mathcal{D}[U'] \, \mathrm{e}^{-S_G[U']} = \int \mathcal{D}[U'] \, \mathrm{e}^{-S_G[U]} \quad \Longrightarrow \quad \mathcal{D}[U] = \mathcal{D}[U']$$

A pure gauge field system

$$\mathrm{d}U_\mu(n) = \mathrm{d}U_\mu(n)' = \mathrm{d}\left(\Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger\right)$$

- Due to the arbitrariness of the gauge transformation matrix, the integral measure must satisfy

$$\mathrm{d}U = \mathrm{d}(U\,V) = \mathrm{d}(V\,U)$$

Mathematically known as the Haar measure

- Introduce a normalization condition $\int \mathrm{d}U\ 1 = 1$

- A very useful integral expression $\int_{\mathrm{SU}(3)} \mathrm{d}U\, f(U) = \int_{\mathrm{SU}(3)} \mathrm{d}U\, f(V\,U) = \int_{\mathrm{SU}(3)} \mathrm{d}U\, f(U\,W)$

It yields a consequence

$$\int_{\mathrm{SU}(3)} \mathrm{d}U\, U_{ab} = 0 \quad \Longrightarrow \quad \int_{\mathrm{SU}(3)} \mathrm{d}U\, U_{ab} = \int_{\mathrm{SU}(3)} \mathrm{d}U\, (V\,U)_{ab} = V_{ac}\int_{\mathrm{SU}(3)} \mathrm{d}U\, U_{cb}$$

A pure gauge field system

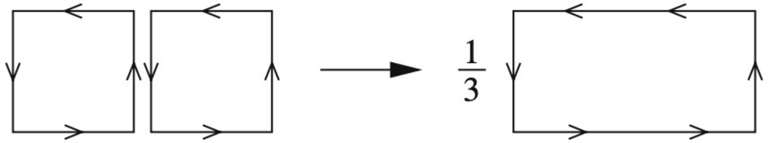
$$\int_{\mathrm{SU}(3)} \mathrm{d}U \, U_{ab} U_{cd} = 0 \, ,$$

$$\int_{\mathrm{SU}(3)} \mathrm{d}U \, U_{ab} (U^\dagger)_{cd} = \frac{1}{3} \, \delta_{ad} \delta_{bc} \, ,$$

$$\int_{\mathrm{SU}(3)} \mathrm{d}U \, U_{ab} U_{cd} U_{ef} = \frac{1}{6} \, \epsilon_{ace} \epsilon_{bdf} \, .$$

- If the integrand is not gauge invariant, then the integral result must be 0
- Question: Why can the results of the integrals be non-zero for the latter two expressions?
- A useful result

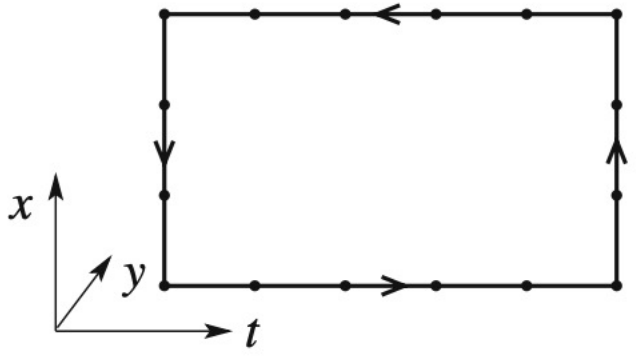
$$\int_{\mathrm{SU}(3)} \mathrm{d}U \, U_{ab} (U^\dagger)_{cd} = \frac{1}{3} \, \delta_{ad} \delta_{bc} \quad \Longrightarrow \quad \int \mathrm{d}U \;\; \mathrm{tr}[V \, U] \; \mathrm{tr}[U^\dagger \, W] = \frac{1}{3} \, \mathrm{tr}[V \, W]$$



integrate out the common links

Calculation of Wilson loops

$$L[U] = \operatorname{tr}\left[\prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\right]$$



- Choose a specific gauge

$$T(\boldsymbol{n}, n_t) = \prod_{j=0}^{n_t-1} U_4(\boldsymbol{n}, j) = \mathbb{1}$$

- Wilson loops are given by links in two spatial directions

$$W_{\mathcal{L}}[U] = \operatorname{tr}\left[S(\boldsymbol{m}, \boldsymbol{n}, n_t)\, S(\boldsymbol{m}, \boldsymbol{n}, 0)^\dagger\right]$$

$$\lim_{T\to\infty} \langle O_2(t)\, O_1(0)\rangle_T = \sum_n \langle 0|\widehat{O}_2|n\rangle\langle n|\widehat{O}_1|0\rangle\, \mathrm{e}^{-t\,E_n} \Longrightarrow \sum_k \langle 0|\widehat{S}(\boldsymbol{m},\boldsymbol{n})_{ab}|k\rangle\langle k|\widehat{S}(\boldsymbol{m},\boldsymbol{n})^\dagger_{ba}|0\rangle\, \mathrm{e}^{-tE_k}$$

Calculation of Wilson loops

- In the strong coupling limit

$$\langle W_{\mathcal{C}} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \, \exp\left(-\frac{\beta}{3} \sum_{P} \mathrm{Re} \, \mathrm{tr}[\mathbb{1} - U_P] \right) \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right]$$

$$\begin{aligned} \langle W_{\mathcal{C}} \rangle &= \frac{1}{Z'} \int \mathcal{D}[U] \, \exp\left(\frac{\beta}{3} \sum_{P} \mathrm{Re} \, \mathrm{tr}[U_P] \right) \, \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right] \\ &= \frac{1}{Z'} \int \mathcal{D}[U] \, \exp\left(\frac{\beta}{6} \sum_{P} \left(\mathrm{tr}[U_P] + \mathrm{tr}[U_P^\dagger] \right) \right) \, \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right] \end{aligned}$$

$$\exp\left(\frac{\beta}{6} \sum_{P} \left(\mathrm{tr}[U_P] + \mathrm{tr}[U_P^\dagger] \right) \right) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} \left(\frac{\beta}{6} \right)^{i+j} \left(\sum_{P} \mathrm{tr}[U_P] \right)^{i} \left(\sum_{P} \mathrm{tr}[U_P^\dagger] \right)^{j}$$

Calculation of Wilson loops

- The denominator part

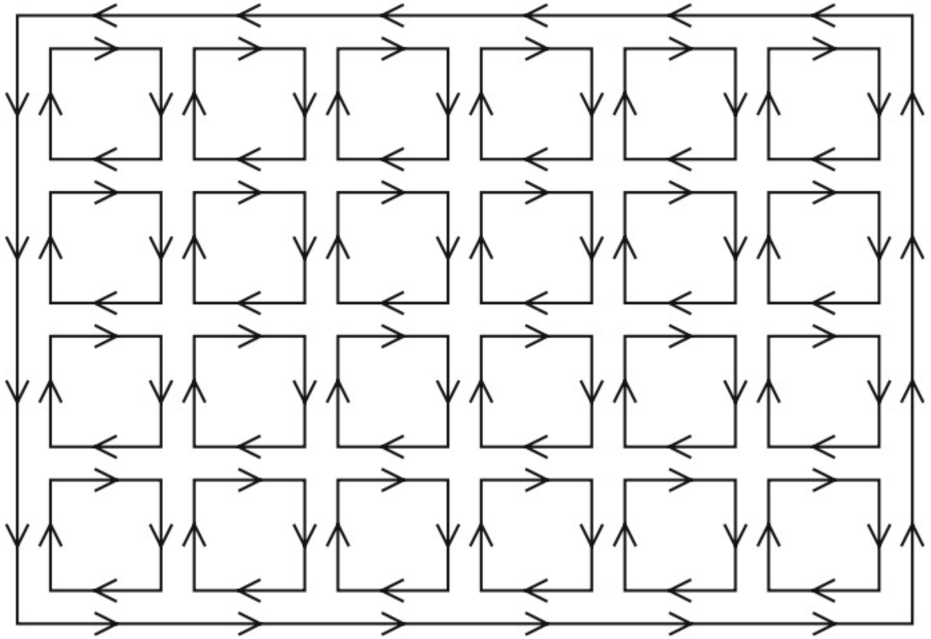
$$Z' = \int \mathcal{D}[U] \exp\left(\frac{\beta}{6}\sum_P \left(\mathrm{tr}[U_P]+\mathrm{tr}[U_P^\dagger]\right)\right) = \int \mathcal{D}[U]\left(1+\mathcal{O}(\beta)\right)$$

- The numerator part

$$\int \mathcal{D}[U]\frac{1}{n_A!}\left(\frac{\beta}{6}\right)^{n_A}\left(\sum_P \mathrm{tr}[U_P^\dagger]\right)^{n_A} \mathrm{tr}\left[\prod_{l\in\mathcal{C}} U_l\right]$$

$$= \left(\frac{\beta}{6}\right)^{n_A}\int \mathcal{D}[U]\prod_{P\in\mathcal{A}_\mathcal{C}} \mathrm{tr}[U_P^\dagger]\ \mathrm{tr}\left[\prod_{l\in\mathcal{C}} U_l\right]$$

$$= \mathrm{tr}[\mathbb{1}]\left(\frac{\beta}{6}\right)^{n_A}\left(\frac{1}{3}\right)^{n_A} = 3\exp\left(n_A \ln\left(\frac{\beta}{18}\right)\right)$$



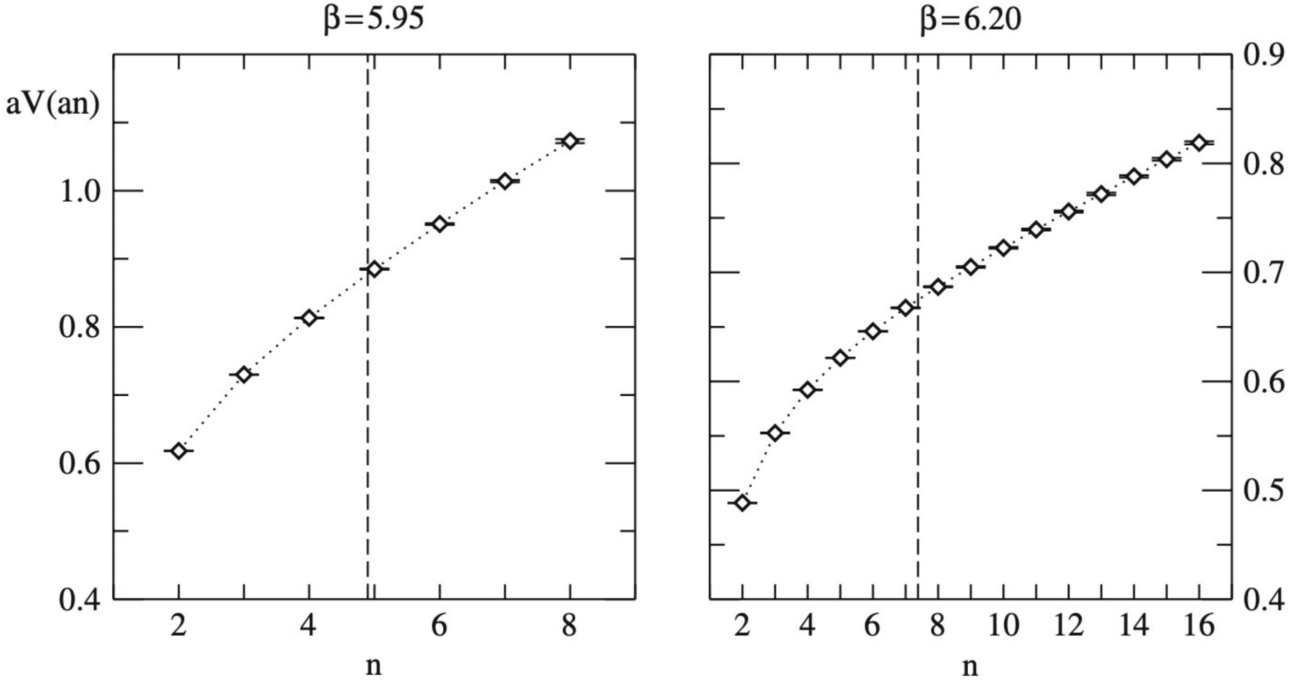
n_A describes the area of the Wilson loop—the area law

Calculation of Wilson loops

$$\langle W_{\mathcal{C}}\rangle = 3\exp\left(n_A\ \ln\left(\frac{\beta}{18}\right)\right)\ (1+\mathcal{O}(\beta)) = 3\exp\left(n_r\, n_t\ \ln\left(\frac{\beta}{18}\right)\right)\ (1+\mathcal{O}(\beta))$$

$$\langle W_{\mathcal{C}}\rangle \propto \exp\left(-a\, n_t\, V(r)\right)$$

$$V(r) = \sigma\, r \qquad \sigma = -\frac{1}{a^2}\ \ln\left(\frac{\beta}{18}\right)\ (1+\mathcal{O}(\beta))$$



Interpretation of the Wilson loop

- Fermion action

$$S_F[\psi,\overline{\psi},U]=a^4\sum_{n\in\Lambda}\overline{\psi}(n)\left(\sum_{\mu=1}^{4}\gamma_\mu\frac{U_\mu(n)\psi(n+\hat{\mu})-U_{-\mu}(n)\psi(n-\hat{\mu})}{2a}+m\,\psi(n)\right)$$

$$S_F[\psi,\overline{\psi},U] \;=\; a^4\sum_{n,m\in\Lambda}\;\sum_{a,b,\alpha,\beta}\overline{\psi}(n)_{\substack{\alpha\\a}}\,D(n|m)_{\substack{\alpha\,\beta\\a\,b}}\,\psi(m)_{\substack{\beta\\b}}$$

$$D(n|m)_{\substack{\alpha\,\beta\\a\,b}}=\sum_{\mu=1}^{4}(\gamma_\mu)_{\alpha\beta}\,\frac{U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}-U_{-\mu}(n)_{ab}\,\delta_{n-\hat{\mu},m}}{2a}+m\,\delta_{\alpha\beta}\,\delta_{ab}\,\delta_{n,m}$$

- Wilson Fermion

$$\widetilde{D}(p)\;=\;m\mathbb{1}\;+\;\frac{\mathrm{i}}{a}\sum_{\mu=1}^{4}\gamma_\mu\,\sin(p_\mu a)\;+\;\mathbb{1}\frac{1}{a}\sum_{\mu=1}^{4}(1-\cos(p_\mu a))$$

$$D^{(f)}(n|m)_{\substack{\alpha\,\beta\\a\,b}}=\left(m^{(f)}+\frac{4}{a}\right)\delta_{\alpha\beta}\,\delta_{ab}\,\delta_{n,m}-\frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1}-\gamma_\mu)_{\alpha\beta}\,U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}$$

Interpretation of the Wilson loop

- Dirac matrix

$$D = C\,(\mathbb{1} - \kappa H) \quad \text{with} \quad \kappa = \frac{1}{2\,(a\,m+4)}\,, \quad C = m + \frac{4}{a}$$

$$H(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \sum_{\mu=\pm 1}^{\pm 4} (\mathbb{1} - \gamma_\mu)_{\alpha\beta}\, U_\mu(n)_{ab}\, \delta_{n+\hat{\mu},m}\,.$$

- Quark propagator

$$\left\langle \psi(n)_{\substack{\alpha\\ a}} \overline{\psi}(m)_{\substack{\beta\\ b}} \right\rangle_F = a^{-4}\, D^{-1}(n|m)_{\substack{\alpha\,\beta\\ a\,b}}$$

$$D^{-1} = (\mathbb{1} - \kappa H)^{-1} = \sum_{j=0}^{\infty} \kappa^j\, H^j \qquad D^{-1}(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \sum_{j=0}^{\infty} \kappa^j\, H^j(n|m)_{\substack{\alpha\,\beta\\ a\,b}}$$

Interpretation of the Wilson loop

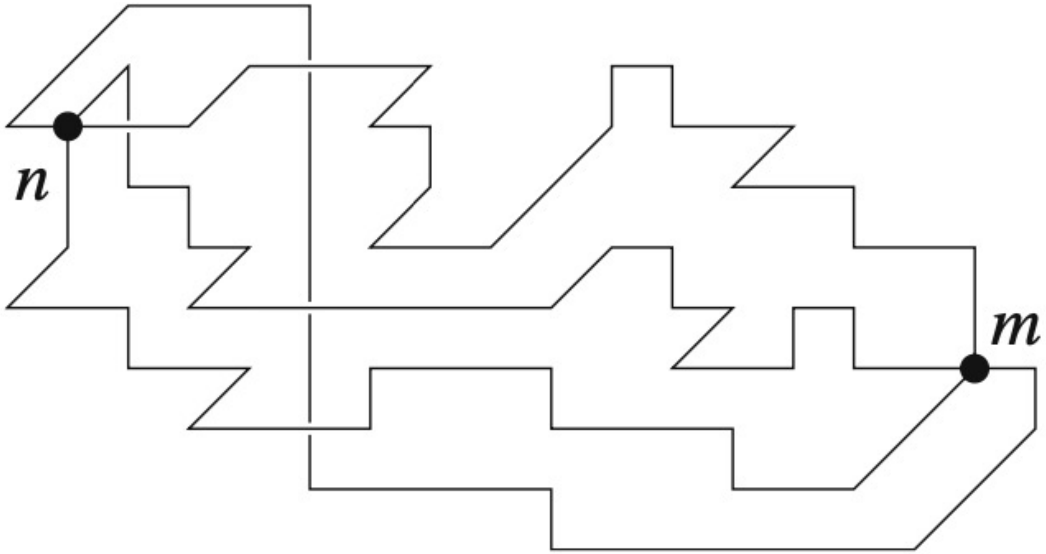
$$H^0(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \delta_{\alpha\beta}\,\delta_{ab}\,\delta_{n,m}\ ,$$

$$H^1(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \sum_{\mu=\pm1}^{\pm4}(\mathbb{1}-\gamma_\mu)_{\alpha\beta}\,U_\mu(n)_{ab}\,\delta_{n+\hat{\mu},m}\ ,$$

$$H^2(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \sum_{l,\rho,c} H(n,l)_{\substack{\alpha\,\rho\\ a\,c}}\,H(l,m)_{\substack{\rho\,\beta\\ c\,b}}$$

$$= \sum_{\mu,\nu=\pm1}^{\pm4}\big((\mathbb{1}-\gamma_\mu)(\mathbb{1}-\gamma_\nu)\big)_{\alpha\beta}\,\big(U_\mu(n)U_\nu(n+\hat{\mu})\big)_{ab}\,\delta_{n+\hat{\mu}+\hat{\nu},m}\ ,$$

$$H^j(n|m)_{\substack{\alpha\,\beta\\ a\,b}} = \sum_{\mu_i=\pm1}^{\pm4}\left(\prod_{i=1}^{j}(\mathbb{1}-\gamma_{\mu_i})\right)_{\alpha\beta} P_{\mu_1\ldots\mu_j}(n)_{ab}\,\delta_{n+\hat{\mu}_1+\ldots+\hat{\mu}_j,m}\ .$$



- In the static limit, leading-order effects are provided by the shortest Wilson line
- The Wilson loop provides the static quark potential for a pair of infinitely heavy quarks

Establishment of lattice QCD

Confinement of quarks, K. G. Wilson PRD 10 (1974) 2445

K. G. Wilson

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

As the strong coupling constant approaches infinity, the potential energy between static quarks, calculated analytically, increases linearly with distance

Numerical computation

- Leaving strong coupling limit, numerical computation is required

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]}\, O[U]$$

- Monte Carlo importance sampling: generating gauge configuration with probability $\mathrm{e}^{-S_G[U]}$

$$\langle O \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} O[U_n] \qquad \mathrm{d}P(U) = \frac{\mathrm{e}^{-S[U]}\, \mathcal{D}[U]}{\int \mathcal{D}[U]\, \mathrm{e}^{-S[U]}}$$

- Analogous to weighted integral

$$\langle f \rangle_\rho = \frac{\int_a^b \mathrm{d}x\, \rho(x)\, f(x)}{\int_a^b \mathrm{d}x\, \rho(x)} \Rightarrow \langle f \rangle_\rho = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n)$$

$x_n \in (a, b)$, according to probability density $\mathrm{d}P(x) = \dfrac{\rho(x)\mathrm{d}x}{\int_a^b \mathrm{d}x\, \rho(x)}$ to generate a series of x_n

Numerical computation

$$\langle O\rangle \sim \int [d\psi][d\bar{\psi}][dU]\, O\, e^{-S_f[\psi,\bar{\psi},U]-S_g[U]}$$

- Integrate out fermionic fields using Grassmann algebra

$$\langle O\rangle \sim \int [dU] O[U] \det(\not{D}+m) e^{-S_g[U]}$$

- Importance sampling: generating gauge configurations according to probability distribution

$$p[U] \propto \det(\not{D}+m) e^{-S_g[U]}$$

- The integral over gauge fields is obtained by approximating it with average over gauge configurations

$$\int [dU] \det(\not{D}+m) e^{-S_g[U]} \quad \rightarrow \quad \frac{1}{N}\sum_{\{U\}}$$

- The more configurations there are, the smaller the statistical error. Error is reduced by $1/\sqrt{N}$

Quenched approximation

$$\langle O \rangle \sim \int [dU] O[U] \det(\not{D} + m) e^{-S_g[U]}$$

- Quenched approximation

$$\det(\not{D} + m) \quad \rightarrow \quad 1$$

- Quenched approximation is equivalent to the heavy-quark limit approximation

$$\det(\not{D} + m) = \det[1 - \kappa H] = \exp\left(\mathrm{tr}\left[\ln(1 - \kappa H)\right]\right) = \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j} \kappa^j \,\mathrm{tr}\left[H^j\right]\right)$$

- Under the quenched approximation, the effects of sea quarks are completely removed, resulting in uncontrollable errors

Development of Lattice QCD

- The development of lattice QCD mainly relies on

  ① the advancement of supercomputing ② the improvement of algorithms

  ③ the breakthroughs in lattice methodology

- After entering the year 2000, there has been rapid development in the field of lattice QCD

  Quenched approximation (no sea quark)

  → Full QCD simulation (N_f=2, 2+1, 2+1+1)

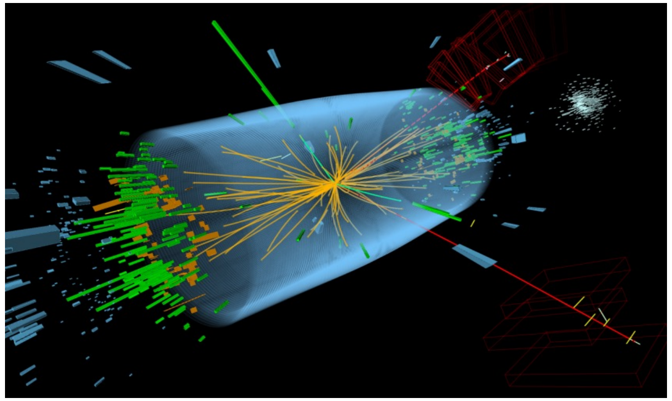
  → Physical point simulation (physical quark mass)

  → Inclusion of isopsin breaking effects (both from quark mass and QED effects)

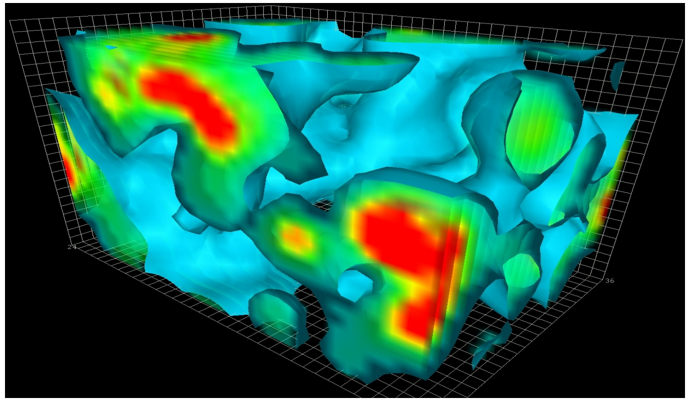
Experiment vs Lattice QCD

Collider (Energy, Luminosity)

Supercomputer (Performance, Memory)



Collision, Events



Simulation, QCD vacuum

Detector, measurement

Lattice QCD calculation

Correlation function

- Operator construction

x is the coordinate of spacetime

$$O_{\pi^+}(x) = \bar{d}(x)\gamma_5 u(x), \quad O_{\pi^-} = \bar{u}(x)\gamma_5 d(x)$$

$$O_p(x) = \epsilon_{abc} u(x)_a \left(u(x)_b^T C\gamma_5 d(x)_c \right)$$

Diquark is designed to have isospin I=0 and spin J=0

- Momentum projection

$$O_{\pi^+}(t,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} O_{\pi^+}(t,\vec{x})$$

- In Euclidean spacetime, the time dependence of operator is given by

$$O(t) = e^{Ht} O(0) e^{-Ht}$$

- Take the pion operator as an example, one can construct correlation function

$$\langle 0|O_\pi(t) O_\pi^\dagger(0)|0\rangle = \sum_n \langle 0|e^{Ht} O_\pi e^{-Ht}|n\rangle\langle n|O_\pi^\dagger|0\rangle$$

$$= \sum_n |\langle 0|O_\pi|n\rangle|^2 e^{-E_n t}$$

Lowest energy is given by pion mass

Realization of correlation function

- For any product of operators, denoted by A, one has to evaluate

$$\langle A \rangle = \langle \langle A \rangle_F \rangle_G = \frac{1}{Z} \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]}\, \mathcal{D}[\psi, \overline{\psi}]\, \mathrm{e}^{-S_F[\psi, \overline{\psi}, U]}\, A[\psi, \overline{\psi}, U]$$

$$\langle O_T(n)\, \overline{O}_T(m) \rangle = -\frac{1}{Z} \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]}\, \det[D_u] \det[D_d]$$

$$\times \operatorname{tr}\left[\Gamma D_u^{-1}(n|m)\, \Gamma D_d^{-1}(m|n) \right],$$

$$Z = \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]}\, \det[D_u]\, \det[D_d]\,.$$

- After integrating out the fermionic fields, ψ in S_F results in derterminant; ψ in the observable results in quark propagator

- Then the integrand becomes a quantity purely depending on gauge field

➡ Average over configurations

Lattice QCD~50 years

- First proposed by K. G. Wilson in 1974
- First numerical simulation on a computer was implemented by M. Creutz in 1979
- First computation of the light hadron spectrum in lattice QCD was completed by H. Hamber and G. Parisi in 1981

VOLUME 47, NUMBER 25 PHYSICAL REVIEW LETTERS 21 DECEMBER 1981

Numerical Estimates of Hadronic Masses in a Pure SU(3) Gauge Theory

H. Hamber
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

and

G. Parisi
Istituto Nazionale di Fisica Nucleare, Frascati, Italy, and Istituto di Fisica della Facoltà di Ingegneria, Rome, Italy

(Received 2 October 1981)

In lattice quantum chromodynamics, the hadronic mass spectrum is evaluated by computer simulations in the approximation where closed quark loops are neglected. Chiral symmetry is shown to be spontaneously broken and an estimate of the pion decay constant is given.

PACS numbers: 12.70.+q, 11.10.Np, 11.30.Jw, 12.40.Cc