Reconstruction of Black Hole Metric Perturbations from Weyl Curvature II: The Regge-Wheeler gaugeme

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Motivations

- The Teukolsky Equation gives us Ψ_4 and Ψ_0 only.
- Metric Reconstruction.
  1. Self-Force computations
  2. Horizon structure
  3. Second order perturbations
- Time domain
  1. Computational efficiency
  2. Easier interface to NR
- With B.Whiting we did the metric reconstruction in the *radiation* gauges, in vacuum, in the Schwarzschild background.
- Here we do the metric reconstruction in the Regge-Wheeler gauge, still in the Schwarzschild background, but with arbitrary source terms.

Weyl Scalars

$$\psi_4 = \frac{1}{2}\Big\{(\bar{\delta} + 3\alpha + \overline{\beta} - \overline{\tau})(\bar{\delta} + 2\alpha + 2\overline{\beta} - \overline{\tau})h_{\rm nn}$$
$$+ (\hat{\Delta} + \overline{\mu} + 3\gamma - \overline{\gamma})(\hat{\Delta} + \overline{\mu} + 2\gamma - 2\overline{\gamma})h_{\rm \overline{m}\overline{m}}$$
$$- \Big[2(\hat{\Delta} + \overline{\mu} + 3\gamma - \overline{\gamma})(\bar{\delta} - \overline{\tau} + 2\alpha)$$
$$+ 2\,(\bar{\delta} + 3\alpha + \overline{\beta} - \overline{\tau})(\overline{\mu})\Big]\; h_{\rm (n\overline{m})}\Big\}, \qquad (46)$$

and

$$\psi_0 = \frac{1}{2}\Big\{(\delta - \overline{\alpha} - 3\beta + \overline{\pi})(\delta - 2\overline{\alpha} - 2\beta + \overline{\pi})h_{\rm ll}$$
$$+ (\hat{D} - \overline{\rho} - 3\epsilon + \overline{\epsilon})(\hat{D} - \overline{\rho} - 2\epsilon + 2\overline{\epsilon})h_{\rm mm}$$
$$- \Big[2(\hat{D} - \overline{\rho} - 3\epsilon + \overline{\epsilon})(\delta + \overline{\pi} - 2\beta)$$
$$- 2\,(\delta - \overline{\alpha} - 3\beta + \overline{\pi})(\overline{\rho})\Big]\, h_{\rm (lm)}\Big\}. \qquad (47)$$

where $h_{\rm nn} = {\rm n}^\mu {\rm n}^\nu h_{\mu\nu}$, $h_{\rm lm} = {\rm l}^\mu {\rm m}^\nu h_{\mu\nu}$, etc.

Weyl Scalars (RW)

In the Regge-Wheeler gauge ($h_1^{\ell m} = h_0^{\ell m} = G^{\ell m} = 0 = {}^{(odd)}h_2^{\ell m}$), we get

$$\psi_4 \doteq \sum_{\ell m} \psi_4^{\ell m} \; {}_{-2}Y_{\ell m} = -\sum_{\ell m} \sqrt{\frac{(\ell-2)!}{(\ell+2)!}} \times \left\{ -\frac{f}{16 r^2} \left(H_0^{\ell m} - 2 H_1^{\ell m} + H_2^{\ell m} \right) - \frac{i}{8 r^2} \left[\partial_t - f \partial_r + f' \right] \left({}^{(odd)}h_0^{\ell m} - f \; {}^{(odd)}h_1^{\ell m} \right) \right\} \; {}_{-2}Y_{\ell m}, \tag{11}$$

and

$$\psi_0 \doteq \sum_{\ell m} \psi_0^{\ell m} \; {}_{+2}Y_{\ell m} = -\sum_{\ell m} \sqrt{\frac{(\ell-2)!}{(\ell+2)!}} \times \left\{ -\frac{1}{4 f r^2} \left(H_0^{\ell m} + 2 H_1^{\ell m} + H_2^{\ell m} \right) - \frac{i}{2 f^2 r^2} \left[\partial_t + f \partial_r - f' \right] \left({}^{(odd)}h_0^{\ell m} + f \; {}^{(odd)}h_1^{\ell m} \right) \right\} \; {}_{+2}Y_{\ell m}. \tag{12}$$

Even and Odd parity decomposition

We define

$$\psi^{\pm} = \frac{1}{2}\left[\psi^{\ell,m} \pm (-)^m \overline{\psi}^{\ell,-m}\right] , \qquad (13)$$

where for notational simplicity we dropped the ℓ, m indexes. Thus, given the symmetric nature of the even parity metric perturbations, Eqs. (11) and (12) take the form

Even parity:

$$\psi_4^+ = \frac{f}{16r^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left(H_0^{\ell m} - 2H_1^{\ell m} + H_2^{\ell m}\right) , \qquad (14)$$

and

$$\psi_0^+ = \frac{1}{4fr^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left(H_0^{\ell m} + 2H_1^{\ell m} + H_2^{\ell m}\right) . \qquad (15)$$

From Eqs. (14) and (15) we can obtain the ℓm components of the metric perturbations as follows

$$H_1^{\ell m}(r,t) = -\frac{4r^2}{f}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\psi_4^+ - \frac{f^2}{4}\psi_0^+\right] \qquad (16)$$

and

$$H_0^{\ell m}(r,t) + H_2^{\ell m}(r,t) = \frac{8r^2}{f}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\psi_4^+ + \frac{f^2}{4}\psi_0^+\right] \qquad (17)$$

We now bring into the play the Hilbert-Einstein's equations in the Regge-Wheeler gauge, Eq. (A.14) give,

$$H_0^{\ell m}(r,t) - H_2^{\ell m}(r,t) = \frac{16\pi r^2}{\sqrt{2\lambda(\lambda+1)}}F^{\ell m} , \qquad (18)$$

Using Eq. (A.12) we can solve for $\partial_r K^{\ell m}$ and then replace it in Eq. (A.8) to find $K^{\ell m}$ in terms of the other even parity metric coefficients and source terms

$$
\begin{aligned}
&K(r,t)^{\ell m} = \\
&2\,\frac{(r-M)\,\frac{\partial}{\partial r}H_0^{\ell m}(r,t)}{\lambda} + \frac{(r-2\,M)\,r\,\frac{\partial^2}{\partial r^2}H_0^{\ell m}(r,t)}{\lambda} \\
&-\frac{r^2\,\frac{\partial^2}{\partial r\,\partial t}H_1^{\ell m}(r,t)}{\lambda} + \frac{M\,\frac{\partial}{\partial r}H_2^{\ell m}(r,t)}{\lambda} \\
&-\frac{(2\,r^2-8\,r\,M+9\,M^2)\,H_0^{\ell m}(r,t)}{r\lambda\,(r-2\,M)} \\
&+\frac{(-r^2\lambda+2\,r\,M\lambda+3\,M^2-2\,r\,M)\,H_2^{\ell m}(r,t)}{r\lambda\,(r-2\,M)} \\
&+\frac{r\,(-3\,r+7\,M)\,\frac{\partial}{\partial t}H_1(r,t)}{\lambda\,(r-2\,M)} + 8\,\frac{(r-2\,M)\,\pi\,r^2\,\frac{\partial}{\partial r}B^{\ell m}(r,t)}{\sqrt{\lambda+1}\lambda} \\
&-8\,\frac{(7\,M-4\,r)\,\pi\,r\,B^{\ell m}(r,t)}{\sqrt{\lambda+1}\lambda} - \frac{A_0^{\ell m}(r,t)r^3}{\lambda\,(r-2\,M)}.
\end{aligned} \tag{19}
$$

This form of the metric coefficient $K^{\ell m}$, involves second derivatives of the Weyl scalars. One can consider an alternative integral form (on the hypersurface $t = constant$) derived from Eq. (A.12)

$$
\begin{aligned}
K^{\ell m} = H_0^{\ell m} + \int_{2M}^{r} \frac{dr}{1-\frac{2M}{r}} \Bigg[&-\frac{\partial H_1^{\ell m}}{\partial t} + \frac{2M}{r^2} H_0^{\ell m} \\
&-16\pi\,(r-2M)\,\frac{F^{\ell m}}{\sqrt{2\lambda(\lambda+1)}} - \frac{8\pi(r-2M)}{\sqrt{\lambda+1}} B^{\ell m} \Bigg]
\end{aligned} \tag{20}
$$

Odd Parity

From Eq. (11) and (12), given the antisymmetric behaviour of the odd parity metric coefficients, we get

$$\psi_4^- = \frac{i}{8r^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left[\partial_t - f\partial_r + f'\right]\left({}^{(odd)}h_0^{\ell m} - f\ {}^{(odd)}h_1^{\ell m}\right), \qquad (21)$$

and

$$\psi_0^- = \frac{i}{2f^2r^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left[\partial_t + f\partial_r - f'\right]\left({}^{(odd)}h_0^{\ell m} + f\ {}^{(odd)}h_1^{\ell m}\right). \qquad (22)$$

A linear combination of these previous equations produces

$$\psi_4^- + \frac{f^2}{4}\psi_0^- = \frac{i}{4r^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left[\partial_t {}^{(odd)}h_0^{\ell m} + (f\partial_r - f')\left(f\ {}^{(odd)}h_1^{\ell m}\right)\right], \qquad (23)$$

and

$$\psi_4^- - \frac{f^2}{4}\psi_0^- = \frac{-i}{4r^2}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left[f\partial_t {}^{(odd)}h_1^{\ell m} + (f\partial_r - f')\left({}^{(odd)}h_0^{\ell m}\right)\right]. \qquad (24)$$

Odd Parity (end)

which integrated produces

$$h_1^{\ell m} = \frac{1}{\sqrt{1-\frac{2M}{r}}} \left\{ \int_{2M}^{r} S_1^{\ell m}(r',t) \sqrt{1-\frac{2M}{r'}}\, dr' + C_1^{\ell m}(t) \right\}, \tag{26}$$

and

$$h_0^{\ell m} = \left(1-\frac{2M}{r}\right) \left\{ \int_{2M}^{r} \frac{S_0^{\ell m}(r',t)}{1-\frac{2M}{r'}}\, dr' + C_0^{\ell m}(t) \right\}. \tag{29}$$

where

$$S_1^{\ell m}(r,t) \doteq \frac{-2ir^2}{f^2} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \left(\psi_4^- + \frac{f^2}{4}\psi_0^-\right) - \frac{2\pi\, ir^2 D_{\ell m}}{f\sqrt{\lambda(\lambda+1)}}, \tag{25}$$

$$S_0^{\ell m}(r,t) \doteq \frac{4ir^2}{f} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \left(\psi_4^- - \frac{f^2}{4}\psi_0^-\right) - \partial_t\, h_1^{\ell m}, \tag{28}$$

Waveforms' metric reconstruction: Even Parity

$$K^{\ell m} = \frac{6M^2 + 3M\lambda r + \lambda(\lambda+1)r^2}{r^2(\lambda r + 3M)} \psi^{\ell m}_{\rm even} + \left(1 - \frac{2M}{r}\right) \partial_r \psi^{\ell m}_{\rm even} - \frac{8\pi r^3 A^{(0)}_{\ell m}}{(\lambda+1)(\lambda r + 3M)}, \tag{B.5}$$

and

$$H_2^{\ell m} = -\frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(\lambda+1)r^3}{r^2(\lambda r + 3M)^2} \psi^{\ell m}_{\rm even} + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)} \partial_r \psi^{\ell m}_{\rm even} + (r - 2M)\partial_r^2 \psi^{\ell m}_{\rm even} - \frac{8\pi r^4}{(\lambda+1)(\lambda r + 3M)} \partial_r A^{(0)}_{\ell m} + \frac{8\pi r^3(\lambda^2 r^2 - 2\,\lambda\, r^2 + 10\,\lambda\, r M - 9\, r M + 27\, M^2)}{(\lambda+1)(r - 2M)(\lambda r + 3M)^2} A^{(0)}_{\ell m}. \tag{B.6}$$

Even Parity (Contd.)

From Eq. (A.9) and the expressions for $\partial_t K^{\ell m}$ and $\partial_t H_2^{\ell m}$ in terms of $\partial_t \psi_{\text{even}}^{\ell m}$, we find the $H_1^{\ell m}$ metric coefficient in the Regge-Wheeler gauge

$$H_1^{\ell m} = r\partial_r(\partial_t \psi_{\text{even}}^{\ell m}) + \frac{\lambda r^2 - 3M\lambda r - 3M^2}{(r-2M)(\lambda r + 3M)}\partial_t \psi_{\text{even}}^{\ell m} - \frac{8\pi r^5}{(\lambda+1)(r-2M)(\lambda r+3M)}\partial_t A_{\ell m}^{(0)} + \frac{4\sqrt{2} i \pi r^2}{(\lambda+1)} A_{\ell m}^{(1)}. \tag{B.7}$$

These equations together with

$$H_0^{\ell m} = H_2^{\ell m} + \frac{16\pi r^2 F_{\ell m}}{\sqrt{2\lambda(\lambda+1)}}, \tag{B.8}$$

Completes the four even parity metric coefficients.

Odd parity

One can use the field equations to write the metric perturbation in the Regge–Wheeler gauge

$$h_0^{\ell m}(r,t) = \frac{1}{2}(1-\frac{2M}{r})\partial_r\left(r\psi_{\rm odd}^{\ell m}\right) + \frac{4\pi r^3 Q_{\ell m}^{(0)}}{\lambda\sqrt{(\lambda+1)}} \qquad \text{(B.21)}$$

$$h_1^{\ell m}(r,t) = \frac{1}{2}\frac{r}{(1-\frac{2M}{r})}\partial_t\psi_{\rm odd}^{\ell m} + \frac{4\pi i r^3 Q_{\ell m}}{\lambda\sqrt{(\lambda+1)}}$$

And this completes the whole metric reconstruction in the RW gauge in terms of Moncrief waveforms.

Chandrasekhar transformations

$$\psi_4^+ = \frac{1}{16r}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left\{2\psi^{\rm even}_{,r*r*} - 2\psi^{\rm even}_{,tr*} + W^+(\psi^{\rm even}_{,r*} - \psi^{\rm even}_{,t}) - V^+\psi^{\rm even}\right.$$
$$+\frac{16\pi r^3}{(\lambda r+3M)(\lambda+1)}\left(\partial_t A^{(0)}_{\ell m} - \partial_{r*}A^{(0)}_{\ell m}\right) - \frac{8i\,(r-2\,M)\,\sqrt{2}\pi\,A^{(1)}_{\ell m}(r,t)}{\lambda+1}$$
$$+16\,\frac{\pi\,r\,(\lambda^2r^2 - 2\,\lambda\,r^2 + 10\,\lambda\,r\,M - 9\,r\,M + 27\,M^2)\;A^{(0)}_{\ell m}(r,t)}{(\lambda+1)\,(\lambda\,r+3\,M)^2}$$
$$\left.-8\,\frac{F_{\ell m}(r,t)\,\sqrt{2}\pi\;(r-2\,M)}{\sqrt{\lambda\,(\lambda+1)}}\right\}, \qquad \text{(C.1)}$$

$$\psi_4^- = \frac{-i}{16r}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}\left\{2\psi^{\rm odd}_{,r*r*} - 2\psi^{\rm odd}_{,tr*} + W^-(\psi^{\rm odd}_{,r*} - \psi^{\rm odd}_{,t}) - V^-\psi^{\rm odd}\right.$$
$$-\frac{16\pi r^2}{\lambda(\lambda+1)}\left(\partial_t Q^{(0)}_{\ell m} - \partial_{r*}Q^{(0)}_{\ell m}\right) + \frac{16i\pi r(r-2M)}{\lambda(\lambda+1)}\left(\partial_t Q_{\ell m} - \partial_{r*}Q_{\ell m}\right)$$
$$\left.-\frac{48\,i\pi\;(r-2\,M)^2\,Q_{\ell m}}{\lambda\,\sqrt{\lambda+1}\,r} + \frac{16\pi\;(3\,r-8\,M)\,Q^{(0)}_{\ell m}}{\lambda\,\sqrt{\lambda+1}} - S^-\right\}, \qquad \text{(C.2)}$$

And similar expressions for Ψ_0

Inverse Chandrasekhar transformations

$$\psi^{\text{even}} = \frac{r^2 (r - 2M) \frac{\partial^2}{\partial r^2} H_0(r,t)}{\lambda(\lambda+1)} - \frac{r^3 \frac{\partial^2}{\partial r \partial t} H_1(r,t)}{\lambda(\lambda+1)}$$
$$+ \frac{r(rM\lambda - 3M^2 + r^2\lambda + 6rM) \frac{\partial}{\partial r} H_0(r,t)}{(\lambda+1)\lambda(\lambda r + 3M)}$$
$$+ \frac{(2r^2\lambda - 5rM\lambda - 21M^2 + 9rM) r^2 \frac{\partial}{\partial t} H_1(r,t)}{(\lambda+1)(\lambda r + 3M)(-r + 2M)\lambda}$$
$$- \frac{(M^2\lambda r + 2r^2M\lambda^2 - 12r^2M - r^2M\lambda - 2r^3\lambda + 42rM^2 - r^3\lambda^2 - 63M^3) H_0(r,t)}{2(\lambda+1)(\lambda r + 3M)(-r+2M)\lambda}$$
$$- 4\frac{r^2\pi\sqrt{2\lambda+2}(-5r^2\lambda - 12rM + 9rM\lambda + 21M^2)\sqrt{2}B(r,t)}{(\lambda+1)^2\lambda(\lambda r + 3M)}$$
$$+ \frac{r^4 A^{(0)}(r,t)}{\lambda(\lambda+1)(-r+2M)}$$
$$+ 4\frac{\sqrt{2}\sqrt{\lambda(\lambda+1)}r^2\pi(2rM - 11M^2 - r^2\lambda + 2rM\lambda) F(r,t)}{(-r+2M)(\lambda+1)^2\lambda^2}$$
$$- 8\frac{r^3(-r+2M)\pi \frac{\partial}{\partial r} B(r,t)}{(\lambda+1)^{3/2}\lambda} - 8\frac{r^3 M\pi\sqrt{2}\frac{\partial}{\partial r} F(r,t)}{(\lambda+1)\sqrt{\lambda(\lambda+1)}\lambda}, \qquad \text{(C.10)}$$

Inverse Chandrasekhar transformations

So, finally

$$\psi^{\rm odd} = \frac{r}{\lambda}\left\{-\frac{2}{r}(1-\frac{M}{r})\int_{2M}^{r}\frac{S_0(r',t)}{1-\frac{2M}{r'}}\,dr' + \frac{4ir^2}{f}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\psi_4^- - \frac{f^2}{4}\psi_0^-\right)\right.$$
$$\left. - \frac{2}{\sqrt{1-\frac{2M}{r}}}\left[\int_{2M}^{r}\partial_t S_1(r',t)\sqrt{1-\frac{2M}{r'}}\,dr'\right]\right\}. \qquad \text{(C.12)}$$

Gives the odd parity waveforms.

Aplications:

Numerical: wave extraction, tetrad, etc

Analytical: Initial Data, BC, QNM, etc

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Kerr perturbations (RW1)

A possible generalization of the Regge-Wheeler gauge conditions for spherically symmetric backgrounds, but where perturbations are not decomposed into multipoles is [7]

$$(\sin\theta)^2 h_{\theta\theta} - h_{\phi\phi} = 0, \quad (31)$$
$$h_{\theta\phi} = 0, \quad (32)$$
$$\sin\theta\, \partial_\theta(\sin\theta h_{t\theta}) + \partial_\phi h_{t\phi} = 0, \quad (33)$$
$$\sin\theta\, \partial_\theta(\sin\theta h_{r\theta}) + \partial_\phi h_{r\phi} = 0. \quad (34)$$

The first equation above leads to the condition $G^{\ell m} = 0$. The second gives then ${}^{(odd)}h_2^{\ell m} = 0$. The other two differential conditions are chosen such that they lead to ${}^{(even)}h_0^{\ell m} = 0 = {}^{(even)}h_1^{\ell m}$, but allow ${}^{(odd)}h_0^{\ell m} \neq 0$ and ${}^{(odd)}h_1^{\ell m} \neq 0$ be unconstrained.

Now we will consider the generalization of the Regge-Wheeler gauge in the Newman-Penrose formalism. In this formalism, the first two Regge-Wheeler conditions, Eqs. (31) and (32), have a simple generalization

$$h_{\mathrm{mm}} = \mathrm{m}^\mu \mathrm{m}^\nu h_{\mu\nu} = 0. \quad (35)$$

Note that requiring that the real and imaginary parts vanish contains both conditions. Obviously,

$$h_{\bar{\mathrm{m}}\bar{\mathrm{m}}} = \bar{\mathrm{m}}^\mu \bar{\mathrm{m}}^\nu h_{\mu\nu} = 0, \quad (36)$$

also holds. Note that conditions (35) and (36) are invariant under type III (spin-boosts) transformations of the background tetrad

$$\mathrm{l} \to A^2\mathrm{l}, \qquad \mathrm{n} \to A^{-2}\mathrm{n}, \qquad \mathrm{m} \to e^{2i\Theta}\mathrm{m}, \qquad \bar{\mathrm{m}} \to e^{-2i\Theta}\bar{\mathrm{m}}. \quad (37)$$

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Kerr perturbations (RW2)

To generalize the differential conditions (33) and (34) one can resort to the type III transformation properties of the δ and $\bar{\delta}$, as well as spin coefficient operators in the Kerr background acting on the metric coefficients $h_{(\mathrm{lm})}$ and $h_{(\mathrm{n\bar{m}})}$. The objects

$$(\delta - 2\bar{\alpha})h_{(\mathrm{l\bar{m}})} \to A^2(\delta - 2\bar{\alpha})h_{(\mathrm{l\bar{m}})}, \tag{38}$$

and

$$(\bar{\delta} + 2\bar{\beta})h_{(\mathrm{nm})} \to A^{-2}(\bar{\delta} + 2\bar{\beta})h_{(\mathrm{nm})}, \tag{39}$$

transform as objects of spin- 0 and boost weight $+1$ and -1 respectively under type III transformations of the background tetrad (37).

In order to reproduce the differential conditions (33) and (34) in the Schwarzschild limit, one can then require

$$\boxed{\Re\left[(\delta - 2\bar{\alpha})h_{(\mathrm{l\bar{m}})}\right] = 0,} \tag{40}$$

and

$$\boxed{\Re\left[(\bar{\delta} + 2\bar{\beta})h_{(\mathrm{nm})}\right] = 0,} \tag{41}$$

Where $\Re$ is the real part.

Ambiguity by adding τ, π and its bars.

Discussion

1. We reconstructed the metric in the RW gauge in the Schwarzschild background from Ψ_4 and Ψ_0 including source terms.

2. To Continue Kerr program:
   - by decomposing into m-modes. Still Even/odd definition applies.
   - Use GR field equations projected along the tetrad.
   - Construct the metric.

3. Chandra and inverse-Chandra transformations in the time domain.

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