Adiabatic radiation reaction to the orbits in Kerr Spacetime

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Talk Plan

1. Introduction
2. Strategy

   adiabatic approximation, radiative field

3. Background

   geometry, constants of motion, geodesics

4. Calculation of dE/dt, dL/dt and dQ/dt

   evaluation of dE/dt, dL/dt, dQ/dt, consistency check

5. Easy case

   slightly eccentric, small inclination case

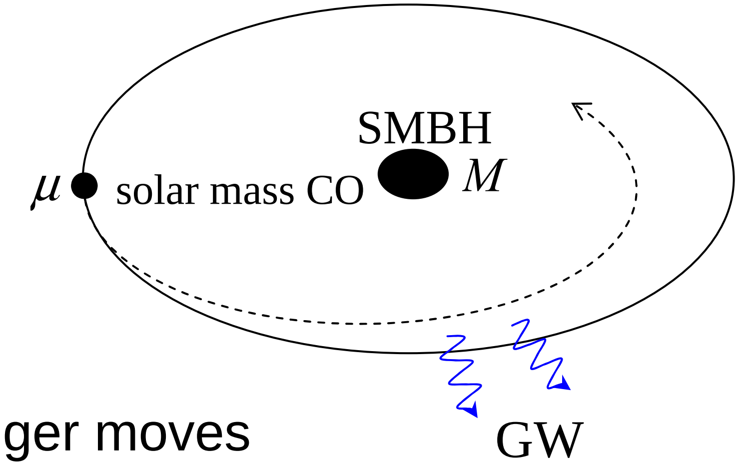
6. Summary and Future works

1. Introduction

We consider the motion of a particle orbiting the Kerr geometry.

• test particle case $\left[\ \approx O((\mu/M)^0)\ \ \right]$

A particle moves along a geodesic characterized by $E,\ L,\ Q$.



In the next order, the particle no longer moves along the geodesic because of the self-force!

Evolution of E,L,Q ⟺ Orbital evolution

2. Strategy

- Adiabatic approximation $\mathrm{T} << \tau_{\mathrm{RR}}$

  T : orbital period

  τ_{RR} : timescale of the orbital evolution

  At the lowest order, we assume that a particle moves along a geodesic.

- Evolution of constants of motion

  Use the radiative field instead of the retarded field.
  Take infinite time average.

$$\left\langle \frac{D}{D\tau} I^i \right\rangle = \frac{1}{\mu} \lim_{T\to\infty} \int_{-T}^{T} d\tau \frac{\partial I^i}{\partial u^\alpha} F^\alpha [h_{\mu\nu}^{\mathrm{rad}}]$$

$$I^i = \{E, L, Q\}, \quad h_{\mu\nu}^{\mathrm{rad}} = \left(h_{\mu\nu}^{\mathrm{ret}} - h_{\mu\nu}^{\mathrm{adv}}\right)/2$$

Key point

Radiative field is a homogeneous solution.

⇒ It has no divergence at the location of the particle.

We do not need the regularization!!

Justification

- For E and L Gal'tsov, J.Phys. A 15, 3737 (1982)

  The results are consistent with the balance argument.

- For Q Mino, PRD 67,084027 (2003)

  This estimation gives the correct long time average.

3. Background

- Kerr geometry (Boyer-Lindquest coordinate)

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2$$

$$\Sigma = r^2 + a^2\cos^2\theta \,, \quad \Delta = r^2 - 2Mr + a^2$$

- Killing vector : $\nabla_{(\mu}\xi_{\nu)} = 0$

$$\xi^{\mu}_{(t)} = (1,0,0,0) \,, \quad \xi^{\mu}_{(\phi)} = (0,0,0,1)$$

- Killing tensor : $\nabla_{(\mu}K_{\nu\lambda)} = 0$

$$K_{\mu\nu} = 2\Sigma\, l_{(\mu}n_{\nu)} + r^2 g_{\mu\nu}$$

where

$$l^{\mu} = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^{\mu} = \left(\frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right), \quad m^{\mu} = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right)$$

Constants of motion for geodesics

There are three constants of motion for Kerr geometry.

- Particle's orbit and 4-velocity

$$z^\alpha(\tau) = \left(t_z(\tau), r_z(\tau), \theta_z(\tau), \phi_z(\tau)\right) \qquad \tau\text{: proper time}$$

$$u^\alpha = \frac{dz^\alpha}{d\tau}$$

- Constants of motion $\left(\cdot = \frac{d}{d\tau}\right)$

$$E = -u^\alpha \xi_\alpha^{(t)} = \left(1 - \frac{2Mr_z}{\Sigma}\right)\dot{t}_z + \frac{2Mar_z \sin^2\theta_z}{\Sigma}\dot{\phi}_z$$

$$L = u^\alpha \xi_\alpha^{(\phi)} = -\frac{2Mar_z \sin^2\theta_z}{\Sigma}\dot{t}_z + \frac{(r_z^2 + a^2)^2 - \Delta a^2 \sin^2\theta_z}{\Sigma}\sin^2\theta_z\dot{\phi}_z$$

$$Q = K_{\alpha\beta} u^\alpha u^\beta = \frac{(L - aE\sin^2\theta_z)^2}{\sin^2\theta_z} + a^2\cos^2\theta_z + \Sigma^2\dot{\theta}_z^2$$

In addition, we define another notation for the Carter constant:

$$C = Q - (aE - L)^2$$

Geodesic equations

- New parametrization (Mino '03)

$$d\lambda = d\tau / \Sigma(r_z)$$

- Equations of motion

$$\frac{dt_z}{d\lambda} = -a(aE\sin^2\theta_z - L) + \frac{r_z^2 + a^2}{\Delta} P(r_z)$$

$$\frac{d\phi_z}{d\lambda} = -\left(aE - \frac{L}{\sin^2\theta_z}\right) + \frac{a}{\Delta} P(r_z)$$

$$\left(\frac{dr_z}{d\lambda}\right)^2 = R(r_z), \quad \left(\frac{d\cos\theta_z}{d\lambda}\right)^2 = \Theta(\cos\theta_z)$$

r- and θ- component are decoupled.

$$P(r) = [E(r^2 + a^2) - aL], \quad R(r) = P(r)^2 - \Delta(r^2 + Q)$$

$$\Theta(\cos\theta) = C - [C + a^2(1 - E^2) + L^2]\cos^2\theta + a^2(1 - E^2)\cos^4\theta$$

r - and θ - components of EOM

Assume that the radial and θ motion are bounded:

$$r_1 \le r_z \le r_2, \; \theta_1 \le \theta_z \le \pi - \theta_1$$

In this case, we can expand in the Fourier series.

$$r_z(\lambda) = \sum_{n_r=-\infty}^{\infty} \tilde{r}_{n_r} \exp[i n_r \Omega_r \lambda], \quad \cos\theta_z(\lambda) = \sum_{n_\theta=-\infty}^{\infty} \tilde{z}_{n\theta} \exp[i n_\theta \Omega_\theta \lambda]$$

where

$$\Omega_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{[E(r^2+a^2)-aL]^2 - \Delta(r^2+Q)}}$$

$$\Omega_\theta = 4\int_0^{\cos\theta_1} \frac{dz}{\sqrt{C-[C+a^2(1-E^2)+L^2]z^2+a^2(1-E^2)z^4}}$$

$$z = \cos\theta$$

t - and ϕ - components of EOM

- t - component of EOM

$$\frac{dt_z}{d\lambda} = \frac{r_z^2 + a^2}{\Delta} P(r_z) - a(aE \sin^2\theta_z - L)$$

divide into oscillation term and constant term

$$t_z(\lambda) = t^{(r)} + t^{(\theta)} + \left\langle \frac{dt_z}{d\lambda} \right\rangle \lambda$$

periodic functions linear term

$$t^{(r)} = \int d\lambda \left[\frac{r_z^2 + a^2}{\Delta} P(r_z) - \left\langle \frac{r_z^2 + a^2}{\Delta} P(r_z) \right\rangle \right]$$ with period, $2\pi\Omega_r^{-1}$

$$t^{(\theta)} = \int d\lambda \left[- a(aE \sin^2\theta_z - L) + \left\langle a(aE \sin^2\theta_z - L) \right\rangle \right]$$ with period, $2\pi\Omega_\theta^{-1}$

We can obtain ϕ - motion in a similar way.

$$\phi_z(\lambda) = \phi^{(r)} + \phi^{(\theta)} + \left\langle \frac{d\phi_z}{d\lambda} \right\rangle \lambda$$

4. Evaluation of dE/dt, dL/dt and dQ/dt

- Formula of dE/dt (Gal'tsov '82)

$$\left\langle \frac{dE}{d\lambda} \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \left[-\xi^{\alpha}_{(t)} \partial_\alpha \psi \right]_{x\to z(\lambda)}$$

We can replace $-i\omega$ after mode decomposition.

where

$$\psi(x) = i \int \frac{d\omega}{2\pi\omega} \sum_{\ell,m} \phi^{(\mathrm{in})}_{\omega\ell m}(x) \int d\lambda' \overline{\phi^{(\mathrm{in})}_{\omega\ell m}(z(\lambda'))} \quad + \text{(up-field term)}$$

$$\phi_{\omega\ell m}(x) = \Sigma(r) \tilde{u}^{\mu}(r,\theta) \tilde{u}^{\nu}(r,\theta) \Pi^{\omega\ell m}_{\mu\nu}(x)$$

$\Pi^{\omega\ell m}_{\mu\nu}(x)$: mode function of metric perturbation

$$(\tilde{u}_t, \tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\varphi) := (-E, \pm\sqrt{R(r)}/\Delta, \pm\sqrt{\Theta(\cos\theta)}/\sin\theta, L).$$

Here we define the above vector field as $\lim_{x\to z} \tilde{u}^{\alpha}(x) = u^{\alpha}(\tau)$

$z^r(\lambda), z^\theta(\lambda)$: periodic function

$z^t(\lambda), z^\phi(\lambda)$: periodic function + linear term

Fourier transformation

$$\phi_{\omega\ell m}(z^\alpha(\lambda)) = \left\langle \frac{dt_z}{d\lambda} \right\rangle e^{-i\lambda(\omega\langle dt_z/d\lambda\rangle - m\langle d\phi_z/d\lambda\rangle)} \sum_{n_r, n_\theta} Z_{\ell m\omega}^{n_r, n_\theta} e^{i(n_r\Omega_r + n_\theta\Omega_\theta)\lambda}$$

After mode decomposition and λ-integration, we can obtain:

$$\left\langle \frac{dE}{dt} \right\rangle = -\int d\omega \sum_{\ell, m, n_r, n_\theta} \left| Z_{\ell m}^{n_r, n_\theta} \right|^2 \delta(\omega - \omega_m^{n_r, n_\theta})$$

$$\omega_m^{n_r, n_\theta} = \langle dt_z/d\lambda \rangle^{-1} \left(m\langle d\phi_z/d\lambda \rangle + n_r\Omega_r + n_\theta\Omega_\theta \right)$$

In a similar manner, we can obtain dL/dt:

$$\left\langle \frac{dL}{dt} \right\rangle = -\int d\omega \sum_{\ell, m, n_r, n_\theta} \frac{m}{\omega} \left| Z_{\ell m\omega}^{n_r, n_\theta} \right|^2 \delta(\omega - \omega_m^{n_r, n_\theta})$$

Simplified *dQ/dt* formula

- Formula of *dQ/dt*

$$\frac{dQ}{d\tau} = 2K^{\nu}_{\mu} u^{\mu} f^{\nu}, \quad f^{\alpha} = -\tfrac{1}{2}(g^{\alpha\beta} + u^{\alpha}u^{\beta})(2h_{\beta\gamma;\delta} - h_{\gamma\delta;\beta})u^{\gamma}u^{\delta}$$

Using the vector field, $\tilde{u}^{\mu}(x)$, we rewrite this formula.

$$\frac{dQ}{d\tau} \approx 2\left[K^{\nu}_{\mu}\tilde{u}^{\mu}\partial\frac{\psi}{\Sigma} + h_{\alpha\beta}\tilde{u}^{\alpha}\tilde{u}^{\mu}(K^{\beta}_{\mu;\nu}\tilde{u}^{\nu} - K^{\nu}_{\mu}\tilde{u}^{\beta}_{;\nu}) \right]_{x\to z(\lambda)} \qquad \psi(x) = \frac{\Sigma\tilde{u}^{\mu}\tilde{u}^{\nu}h_{\mu\nu}}{2}$$

This term vanishes in this case.

$$\left[\ \tilde{u}_{\alpha;\beta} = \tilde{u}_{\beta;\alpha}, \quad K_{(\mu\nu;\rho)} = 0\ \right]$$

$$\left\langle \frac{dQ}{d\lambda} \right\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} d\lambda \left[\left(-\frac{P(r_z)}{\Delta}\left((r_z^2 + a^2)\partial_t + a\partial_\phi\right) - \frac{dr_z}{d\lambda}\partial_r \right)\psi(x)\right]_{x\to z(\lambda)}$$

reduce it by integrating by parts

Further reduction

In general for a double-periodic function,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} d\lambda \exp\left[-i\omega_m^{n_r,n_\theta} t_z(\lambda) + im\varphi_z(\lambda)\right] f(r_z(\lambda),\theta_z(\lambda))$$
$$= \frac{\Omega_r\Omega_\theta}{(2\pi)^2}\int_0^{2\pi\Omega_r^{-1}} d\lambda_r \int_0^{2\pi\Omega_\theta^{-1}} d\lambda_\theta \exp\left[-in_r\Omega_r\lambda_r - i\omega_m^{n_r,n_\theta} t^{(r)}(\lambda_r) + im\varphi^{(r)}(\lambda_r)\right]$$
$$\times \exp\left[-in_\theta\Omega_\theta\lambda_\theta - i\omega_m^{n_r,n_\theta} t^{(\theta)}(\lambda_\theta) + im\varphi^{(\theta)}(\lambda_\theta)\right] f(r_z(\lambda_r),\theta_z(\lambda_\theta)).$$

Total derivative with respect to λ_r

$$\frac{d}{d\lambda_r}\psi(z^\alpha;\lambda_r,\lambda_\theta) = \left[\frac{\partial}{\partial\lambda_r} + \frac{dr_z}{d\lambda_r}\frac{\partial}{\partial r} + \frac{dt_z^{(r)}}{d\lambda_r}\frac{\partial}{\partial t} + \frac{d\phi_z^{(r)}}{d\lambda_r}\frac{\partial}{\partial \phi}\right]\psi(z^\alpha;\lambda_r,\lambda_\theta)$$

Using this relations, we can obtain:

$$\left\langle \frac{dQ}{dt}\right\rangle = \left\langle \frac{dt_z}{d\lambda}\right\rangle^{-1} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} d\lambda \left[\left(\left\langle \frac{(r^2+a^2)P}{\Delta}\right\rangle \frac{\partial}{\partial t} + \left\langle \frac{aP}{\Delta}\right\rangle \frac{\partial}{\partial \phi} + in_r\Omega_r\right)\psi(x)\right]_{x\to z(\lambda)}$$

$$= 2\left\langle \frac{(r^2+a^2)P}{\Delta}\right\rangle \left\langle \frac{dE}{dt}\right\rangle - 2\left\langle \frac{aP}{\Delta}\right\rangle \left\langle \frac{dL}{dt}\right\rangle$$

$$+2\int d\omega \sum_{\ell,m,n_r,n_\theta} \frac{n_r\Omega_r}{\omega}\left|Z_{\ell m\omega}^{n_r,n_\theta}\right|^2 \delta(\omega - \omega_m^{n_r,n_\theta})$$

This expression is as easy to evaluate as *dE/dt* and *dL/dt*.

Consistency check

circular orbit case: $Z^{n_r,n_\theta}_{\ell m\omega} = 0 \quad (\text{when } n_r \neq 0)$

$$\left\langle \frac{dQ}{dt} \right\rangle = 2\left\langle \frac{(r^2+a^2)P}{\Delta} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\left\langle \frac{aP}{\Delta} \right\rangle \left\langle \frac{dL}{dt} \right\rangle$$

This agrees with the circular orbit condition.

(Kennefick and Ori, PRD 53, 4319 (1996))

equatorial orbit case: $\theta = \pi/2 \text{ and } Z^{n_r,n_\theta}_{\ell m\omega} = 0 \quad (\text{when } n_\theta \neq 0)$

Rewriting dQ/dt formula in terms of $C=Q-(aE-L)^2$:

$$\left\langle \frac{dC}{dt} \right\rangle = 2\left\langle a^2 E \cos^2\theta_z \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\left\langle L \cot^2\theta_z \right\rangle \left\langle \frac{dL}{dt} \right\rangle + 2\int d\omega \sum_{\ell,m,n_r,n_\theta} \frac{n_\theta \Omega_\theta}{\omega} \left| Z^{n_r,n_\theta}_{\ell m\omega} \right|^2 \delta(\omega - \omega_m^{n_r,n_\theta})$$

$$\left\langle \frac{dC}{dt} \right\rangle = 0 \quad \left(\text{when } \theta = \frac{\pi}{2} \right)$$

An equatorial orbit does not leave the equatorial plane.

5. Easy case

We consider a slightly eccentric orbit with small inclination.

$$e << 1, \ \ y << 1$$

Eccentricity e

$$R(r_0(1+e)) = 0, \quad \left.\frac{dR}{dr}\right|_{r=r_0} = 0$$

Inclination y

$$y = \frac{C}{L^2}, \quad C = Q - (aE - L)^2$$

In this case, we obtain the geodesic motion:

$$r_z(\lambda) = \sum_{n_r=-\infty}^{\infty} \tilde{r}_{n_r} \exp[i n_r \Omega_r \lambda], \quad \cos\theta_z(\lambda) = \sum_{n_\theta=-\infty}^{\infty} \tilde{z}_{n_\theta} \exp[i n_\theta \Omega_\theta \lambda]$$

$$\tilde{r}_{n_r} = O(e^{|n_r|}), \qquad \tilde{z}_{n_\theta} = O(y^{|n_\theta/2|})$$

If we truncate at the finite order of e and y,
it is sufficient to calculate up to the finite Fourier modes.

The calculation have not been done yet but is coming soon !!

6. Summary and Future works

We considered a scheme to compute the evolution of constants of motion by using the adiabatic approximation method and the radiative field.

We found that the scheme for the Carter constant can be dramatically simplified.

$$\left\langle \frac{dQ}{dt} \right\rangle = 2\left\langle \frac{(r^2+a^2)P}{\Delta} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\left\langle \frac{aP}{\Delta} \right\rangle \left\langle \frac{dL}{dt} \right\rangle + 2\int d\omega \sum_{\ell,m,n_r,n_\theta} \frac{n_r \Omega_r}{\omega} \left| Z_{\ell m \omega}^{n_r,n_\theta} \right|^2 \delta(\omega - \omega_m^{n_r,n_\theta})$$

We checked the consistency of our formula for circular orbits and equatorial orbits.

As a demonstration, we are calculating dQ/dt for slightly eccentric orbit with small inclination, up to $O(e^2)$, $O(y^2)$, 1PN order from leading term.

Future works

- (more) Accurate estimate of dE/dt, dL/dt and dQ/dt

  - Analytic evaluation for general orbits at higher PN order
    (Ganz, Tanaka, Hikida, Nakano and NS)
  - Numerical evaluation
    Hughes, Drasco, Flanagan and Franklin (2005)
    Fujita, Tagoshi, Nakano and NS: based on Mano's method

- Evaluation of secular evolution of orbit

  - Solve EOM for given constants of motion
  - $\{\dot{E}, \dot{L}, \dot{Q}\} \Rightarrow \{\dot{r}_0, \dot{e}, (\dot{\cos i})\}$: orbital parameter

- Calculation of waveform

  - Using Fujita-Tagoshi's code
  - Parameter estimation accuracy with LISA
    (Barack & Cutler (2004), Hikida et al. under consideration)